# COMBINATORIAL STATISTICS ON FINITE COXETER GROUPS

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TO START: EULER 1755

$$S_n = \{\pi : \{1,\dots,n\} \longrightarrow \{1,\dots,n\} \text{ BIJECTIVE}\}$$

$$\sum_{k\geq 0} k^n t^k = \frac{1}{(1-t)^{n+1}} \sum_{\pi\in S_n} t^{\mathrm{DES}\,\pi}, \quad \mathrm{DES}\,\pi = |\{i : \pi(i) > \pi(i+1)\}|$$

$$1 - x + x^2 - x^3 + \&c. = \frac{1}{1+x},$$

$$1 - 2x + 3x^2 - 4x^3 + \&c. = \frac{1}{(1+x)^2},$$

$$1 - 2^2x + 3^2x^2 - 4^2x^3 + \&c. = \frac{1-x}{(1+x)^3},$$

$$1 - 2^3x + 3^3x^2 - 4^3x^3 + \&c. = \frac{1-4x+xx}{(1+x)^4},$$

$$1 - 2^4x + 3^4x^2 - 4^4x^3 + \&c. = \frac{1-11x+11xx-x^3}{(1+x)^5},$$

$$1 - 2^5x + 3^5x^2 - 4^5x^3 + \&c. = \frac{1-26x+66xx-26x^3+x^4}{(1+x)^6},$$

$$1 - 2^6x + 3^6x^2 - 4^6x^3 \ \&c. = \frac{1-57x+302xx-302x^3+57x^4-x^5}{(1+x)^7}$$

&c.

MEM. DE L'ACADEMIE DES SCIENCES DE BERLIN, 1768

- GIVEN A CARD DECK WITH $n = 52$ CARDS

HOW GOOD IS A GIVEN SHUFFLE METHOD?

A SHUFFLE IS A PROBABILITY DISTRIBUTION ON $S_n$. ($|S_{52}| = \sim 8 \cdot 10^{67}$)

- PERFECT IF UNIFORM DISTRIBUTION
- RIFFLE SHUFFLE / OVERHAND SHUFFLE (DIACONIS 1970–TODAY)

TWO MEASURES FOR SHUFFLE METHODS

* UP-DOWN PATTERN $\{i : \pi(i) > \pi(i+1)\}$

LOCAL INFORMATION

* OUT-OF-ORDER PATTERN $\{i<j : \pi(i) > \pi(j)\}$

GLOBAL INFORMATION

# OVERVIEW

* COMBINATORIAL STATISTICS
  PERMUTATION STATISTICS
  VALUES ON RANDOM PERMUTATIONS
  LIMIT THEOREMS

* GUESSING INTERESTING BEHAVIOUR
  OF STATISTICS

* GENERALIZATIONS TO FINITE
  COXETER GROUPS

# DEFINITION

LET $S$ BE A FINITE SET.

A STATISTIC ON $S$ IS A MAP

$$st : S \longrightarrow \mathbb{N} = \{0, 1, 2, \dots\}$$

WITH ASSOCIATED DISTRIBUTION

$$G_{st}(z) = \sum_{x \in S} z^{st(x)}$$

## DEFINITION

LET $f(z) = \sum_{i \geq 0} a_i z^i \in \mathbb{N}[z]$.

THE REAL RANDOM VARIABLE $X_f$ IS GIVEN BY $\mathbb{P}(X_f = k) = \frac{[z^k]f}{f(1)} = \frac{a_k}{\sum_{i \geq 0} a_i}$

- WE STUDY $X_{st} := X_{G_{st}}$, THE STATISTIC VALUE ON A RANDOM ELEMENT

RECALL:

- $\mathbb{E}(X_{st}) = \frac{1}{|S|} \sum_{x \in S} st(x)$ MEAN VALUE
- $\mathbb{V}(X_{st}) = \mathbb{E}(X_{st}^2) - \mathbb{E}(X_{st})^2$ VARIANCE

# DESCENTS IN PERMUTATIONS

$S_n = \{\pi : \{1,\dots,n\} \to \{1,\dots,n\} \text{ BIJECTIVE}\}$

IN ONE-LINE NOTATION $\pi(1), \pi(2), \dots, \pi(n)$

- DESCENTS

  $\mathrm{DES} : S_n \to \mathbb{N}, \quad \pi \mapsto |\{i : \pi(i) > \pi(i+1)\}|$

$n = 3 : \quad 123 \quad 13|2 \quad 2|13 \quad 23|1 \quad 3|12 \quad 3|2|1$

$$G^{(3)}_{\mathrm{DES}}(z) = 1 + 4z + z^2$$

- $G^{(n)}_{\mathrm{DES}}(z)$ EULERIAN POLYNOMIAL

# THEOREM (CLASSICAL)

$$\mathbb{E}\left(X_{DES}^{(n+1)}\right) = \frac{n}{2} \qquad \mathbb{V}\left(X_{DES}^{(n+1)}\right) = \frac{1}{12}(n+2)$$

## PROOF

$$Y^{(i)} := \begin{cases} 1 \,;\ \pi(i) > \pi(i+1) \\ 0 \,;\ \pi(i) < \pi(i+1) \end{cases} \qquad 1 \leq i \leq n$$

$$\mathbb{E}(Y^{(i)}) = \frac{1}{2}, \quad X_{DES}^{(n+1)} = Y^{(1)} + \dots + Y^{(n)}$$ NOT INDEPENDENT

$$\Rightarrow \mathbb{E}\left(X_{DES}^{(n+1)}\right) = n/2$$

$$\mathbb{E}(Y^{(i)}Y^{(j)}) = \begin{cases} 1/2 \,;\ |i-j| = 0 & n \text{ SUMMANDS} \\ 1/6 \,;\ |i-j| = 1 & 2(n-1) \text{ SUMMANDS} \\ 1/4 \,;\ |i-j| > 1 & n^2 - 2(n-1) - n \text{ SUMMANDS} \end{cases}$$

$$\Rightarrow \mathbb{V}\left(X_{DES}^{(n+1)}\right) = \underbrace{\frac{n}{2} + \frac{2(n-1)}{6} + \frac{n^2 - 2(n-1) - n}{4}}_{\mathbb{E}((Y^{(1)} + \dots + Y^{(n)})^2)} - \underbrace{\frac{n^2}{4}}_{\mathbb{E}(X_{DES}^{(n+1)})^2} = \frac{n+2}{12} \qquad \square$$

# INVERSIONS IN PERMUTATIONS

- INVERSIONS

  $\mathrm{INV} : \mathfrak{S}_n \to \mathbb{N}, \quad \pi \mapsto |\{ i<j : \pi(i) > \pi(j) \}|$

$n=3$:

| 123 | 132 | 213 | 231 | 312 | 321 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 2 | 2 | 3 |

$$G_{\mathrm{INV}}^{(3)}(z) = z^3 + 2z^2 + 2z + 1$$
$$= (1+z)(1+z+z^2) = [2]_z \cdot [3]_z$$

- $G_{\mathrm{INV}}^{(n)}(z) = [2]_z \cdot [3]_z \cdot \ldots \cdot [n]_z$

MAHONIAN POLYNOMIAL 1915

## THEOREM (FOLKLORE)

LET $d_1, \dots, d_n \in \mathbb{N}_{\geq 2}$, $f(z) = [d_1]_z \cdots [d_n]_z$.

THEN $\mathbb{E}(X_f) = \frac{1}{2} \sum_{i=1}^{n} (d_i - 1)$ $\mathbb{V}(X_f) = \frac{1}{12} \sum_{i=1}^{n} (d_i^2 - 1)$

## PROOF

LET $X_d := X_{[d]_z}$ BE UNIFORM DISTRIBUTION ON $\{0, \dots, d-1\}$.

THEN $\mathbb{E}(X_d) = \frac{d-1}{2}$

$$\mathbb{V}(X_d) = \mathbb{E}(X_d^2) - \mathbb{E}(X_d)^2 = \sum_{i=1}^{n} i^2 - \frac{(d-1)^2}{4}$$

QUADRATIC PYRAMIDE

$$= \frac{1}{6}(d-1)(2d-1) - \frac{1}{4}(d-1)^2 = \frac{1}{12}(d-1)(d+1)$$

FINALLY: $X_f = \underbrace{X_{d_1} + \cdots + X_{d_n}}_{\text{INDEPENDENT}}$ $\square$

# DESCENTS & INVERSE DESCENTS

- DES+IDES : $S_n \to \mathbb{N}$ , $\pi \mapsto \mathrm{DES}(\pi) + \mathrm{DES}(\pi^{-1})$

**THEOREM** (CHATTERJEE-DIACONIS 2017)
+ OTHERS, NOT SUPER HARD

- $\mathbb{E}\left(X^{(n+1)}_{\mathrm{DES+IDES}}\right) = n$
- $\mathbb{V}\left(X^{(n+1)}_{\mathrm{DES+IDES}}\right) = \frac{n+8}{6} - \frac{1}{n+1}$

# CENTRAL LIMIT THEOREMS

## DEFINITION

LET $\{X^{(n)}\}_{n \in \mathbb{N}}$ BE SEQUENCE OF RANDOM VARS
AS ABOVE

THEN $X^{(n)}$ HAS CENTRAL LIMIT IF

$$\frac{X^{(n)} - \mathbb{E}(X^{(n)})}{\mathbb{V}(X^{(n)})^{1/2}} \xrightarrow[\text{IN DISTRIBUTION}]{n \to \infty} N(0,1)$$

STANDARD NORMAL DISTRIBUTION

**THEOREM** (FOLKLORE, FOLKLORE, C-D 2017)
HARD!

$X_{DES}$, $X_{INV}$, $X_{DES+IDES}$ HAVE CENTRAL LIMITS

# GUESSING INTERESTING BEHAVIOUR USING FINDSTAT.ORG

* FINDSTAT IS AN ONLINE DATABASE FOR COMBINATORIAL STATISTICS & MAPS.
  - CONTAINS ~ 300 PERM. STATISTICS
    ~ 1400 STATISTICS IN TOTAL
    ~ 150 MAPS

IDEA

SEARCH DATABASE FOR STATISTICS WITH POLYNOMIAL (OR RAT. FUNCTION) VARIANCE

# ANSATZ

LET $st: \mathfrak{S}_n \to \mathbb{N}$ BE GIVEN.

(1) COMPUTE $G_{st}^{(n)}(z)$, $\mathbb{E}(X_{st}^{(n)})$, $\mathbb{V}(X_{st}^{(n)})$

FOR $2 \leq n \leq N$ AND REASONABLE $N$.

(2) USE LAGRANGE INTERPOLATION

AND (POSSIBLY) FIND CONJECTURED

$\mathbb{V}(X_{st}^{(n)}) = f(n)/g(n)$ FOR NICE

$f, g \in \mathbb{Z}[n]$.

USING THIS NAIVE ANSATZ, ONE FINDS

- THE ABOVE THEOREMS FOR MEAN / VARIANCE / CENTRAL LIMITS

- ADDITIONAL UNSTUDIED DISTRIBUTIONS IN THE DATABASE WITH NICE VARIANCE

- AND ALSO THE FOLLOWING GENERALIZATIONS

(AT LEAST CONJECTURELY)

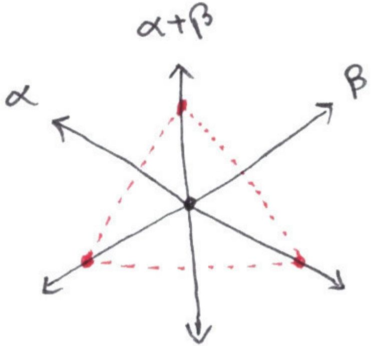
# DESCENTS & INVERSIONS IN COXETER GROUPS

LET $(W,S)$ BE FINITE COXETER SYSTEM,

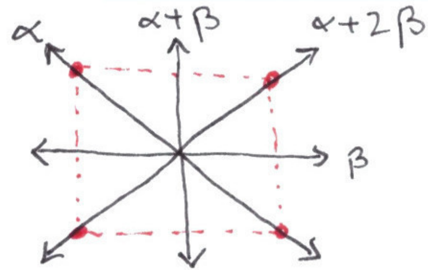
THIS IS, $W = \left\langle S : \begin{array}{ll} (st)^{m(s,t)} & = \mathbb{1} \\ s^2 & = \mathbb{1} \end{array} \right\rangle_{gp}$ FINITE.

LET $\Delta \subset \Phi^+ \subset \Phi$ ROOT SYSTEM WITH WEYL GROUP $W$.

SIMPLE POSITIVE




$\Phi = A_2$, $W(\Phi) = \mathrm{Sym}\,\triangle$
$\cong S_3$




$\Phi = B_2$, $W(\Phi) = \mathrm{Sym}\,\square$
$\cong (\mathbb{Z}/2\mathbb{Z})^2 \rtimes S_2$

- DES : $W \longrightarrow \mathbb{N}$, $w \longmapsto |\{\beta \in \Delta : w(\beta) \in -\Phi^+\}|$

W-DESCENTS

- INV : $W \longrightarrow \mathbb{N}$, $w \longmapsto |\{\beta \in \Phi^+ : w(\beta) \in -\Phi^+\}|$

W-INVERSIONS

* DEFINITION USING ROOT SYSTEM AND EQUIVALENT DEFINITION USING REFLECTION ARRANGEMENT

THEOREM (KAHLE–ST 2018$^+$)

LET $(W,S)$ IRRED. FINITE COXETER SYSTEM OF RANK $|S|=n$, COXETER NUMBER $\mathrm{ORD}(s_1\cdots s_n)=h$ INVARIANT DEGREES $d_1,\dots,d_n$.

$m=\max\{m(s,t)=\mathrm{ord}(st) : s,t\in S\}$. THEN

(1) $\mathbb{E}(X_{DES})=\frac{n}{2}$ $\quad \mathbb{V}(X_{DES})=\frac{n-2}{12}+\frac{1}{m}$

(2) $\mathbb{E}(X_{INV})=\frac{1}{2}\sum_{i=1}^{n}(d_i-1)$ $\quad \mathbb{V}(X_{INV})=\frac{1}{12}\sum_{i=1}^{n}(d_i^2-1)$

(3) $\mathbb{E}(X_{DES+IDES})=n$ $\quad \mathbb{V}(X_{DES+IDES})=2\mathbb{V}(X_{DES})+\frac{n}{h}$

# ABOUT THE PROOFS

* $X_{INV}$ AS SIMPLE AS ABOVE FOR $S_n$

* $X_{DES}$ USES COSET DECOMPOSITION OF PARABOLIC SUBGROUPS

* $X_{DES+IDES}$ CASE-BY-CASE ANALYSIS OF CARTAN-KILLING TYPES

# CENTRAL LIMIT THEOREMS

LET $W^{(1)}, W^{(2)}, \ldots$ ANY SEQUENCE OF
NO INTERACTION ASSUMED!
FINITE COXETER GROUPS OF INCREASING RANK

**THEOREM** (KAHLE-ST. 2018+)

* LET $X^{(n)} = X_{DES}$ ON $W^{(n)}$, $S_n^2 = \mathbb{V}(X^{(n)})$.

  THEN $X^{(n)}$ HAS CENTRAL LIMIT $\Leftrightarrow S_n \to \infty$.

* LET $X^{(n)} = X_{INV}$ ON $W^{(n)}$, $S_n^2 = \mathbb{V}(X^{(n)})$,

  $d_n$ = LARGEST INVARIANT DEGREE ON $W^{(n)}$

  THEN

  $X^{(n)}$ HAS CENTRAL LIMIT $\Leftrightarrow d_n / S_n \to 0$.

## REMARKS

* THE LIMIT THEOREMS DEPEND ONLY VERY MILDLY ON THE CONCRETE GROUPS INVOLVED.

* THE CONDITIONS ARE "RANK 2" CONDITIONS.

* WE DO NOT (YET) HAVE SUCH A GENERAL CENTRAL LIMIT THEOREM FOR DESTIDES

MANY THANKS

FOR YOUR ATTENTION

AND

FOR THE INVITATION