CHARACTERISTIC QUASI-POLYNOMIALS OF INTEGRAL HYPERPLANE ARRANGEMENTS

TAN NHAT TRAN

A typical problem in enumerative combinatorics is to count the size of a set depending upon a positive integer q. Often the result is a polynomial in q (e.g., the chromatic polynomial of a graph), and sometimes a quasi-polynomial. Generally speaking, a quasi-polynomial is a generalization of polynomials, of which the coefficients may not come from a ring but instead are periodic functions with integral periods. Another way to think of a quasi-polynomial is that it is made of a bunch of polynomials, called the constituents.

This lecture series aims at introducing the concept of *characteristic quasi-polynomials of integral hyperplane arrangements* due to Kamiya-Takemura-Terao [9], and exploring the related areas. In the simplest setting, when a finite set $\mathcal{A}$ of vectors in $\mathbb{Z}^\ell$ is given, we may naturally associate to it the integral hyperplane arrangement $\mathcal{A}(\mathbb{R})$ in the real vector space $\mathbb{R}^\ell$. We may also consider its "q-reduction" for any positive integer q and get the arrangement $\mathcal{A}(\mathbb{Z}_q)$ of subgroups in the finite cyclic group $\mathbb{Z}_q^\ell$. The central result in the theory states that the cardinality of the complement of $\mathcal{A}(\mathbb{Z}_q)$ is actually a quasi-polynomial in q. This is called the *characteristic quasi-polynomial* $\chi_{\mathcal{A}}^{\mathrm{quasi}}(q)$ of $\mathcal{A}$ as a result of the fact that its first constituent agrees with the characteristic polynomial of $\mathcal{A}(\mathbb{R})$ (e.g., [4, 2, 9]).

The lecture series consists of three main parts:

1. The constituents and arrangements over abelian groups. Apart from the first, can the other constituents be regarded as the "characteristic polynomials" of some arrangements? The set $\mathcal{A}$ also defines the toric arrangement $\mathcal{A}(\mathbb{S}^1)$ in the torus $(\mathbb{S}^1)^\ell$, which currently receives increasing attention (e.g., [6, 12, 5, 3, 7]). Can we build a general framework to study the arrangements and their characteristic (quasi-)polynomials en masse, rather than individually? We will answer these questions in this part and show that interestingly, the results are closely related [14, 11].

2. Connection to Ehrhart theory and root systems. One of the methods used in [9] for showing that $\chi_{\mathcal{A}}^{\mathrm{quasi}}(q)$ is indeed a quasi-polynomial is to write it as a sum of the Ehrhart quasi-polynomials of rational polytopes. Such an expression is certainly interesting because many results in the Ehrhart theory can be applied to further develop the theory of characteristic quasi-polynomials. One would hope for a more explicit expression if the set $\mathcal{A}$ was chosen to be a more special vector configuration. A particularly well-behaved class of the hyperplane arrangements is that of Weyl arrangements arising from root systems. We will introduce a new class of Weyl subarrangements, called (Worpitzky-)compatibility, and show that the characteristic quasi-polynomial of a compatible arrangement can be written

Date: April 26, 2021.

in terms of the Ehrhart quasi-polynomial of the fundamental alcove, through the action of shift operator by an Eulerian polynomial. Notably, the class of compatible sets contains the coconvex subsets hence contains the ideals of the root system [1]. In particular, when the root system is of type A, the compatible graphic arrangements are characterized by cocomparability graphs [13].

3. Period collapse. Determining the minimum period of a quasi-polynomial is in general a challenging problem. The most popular candidate for period of the Ehrhart quasi-polynomials is the denominator period, while that of the characteristic quasi-polynomial is the LCM period [9, 10]. Stemming from the concept of period collapse in Ehrhart theory, we say that period collapse occurs in a characteristic quasi-polynomial when the minimum period is strictly less than the LCM period. We show that in the non-central case, with respect to period collapse anything is possible: period collapse occurs in any dimension ≥ 1, occurs for any LCM period ≥ 2, and the minimum period can take arbitrary value when it is not the LCM period [8]. The question of whether period collapse happens in the central case remains open to us.

REFERENCES

[1] A. U. Ashraf, T. N. Tran, and M. Yoshinaga. Eulerian polynomials for subarrangements of Weyl arrangements. *Adv. Appl. Math.*, 120:102064, 2020. http://dx.doi.org/10.1016/j.aam.2020.102064.

[2] C. A. Athanasiadis. Characteristic polynomials of subspace arrangements and finite fields. *Adv. Math.*, 122(2):193–233, 1996.

[3] P. Brändén and L. Moci. The multivariate arithmetic Tutte polynomial. *Trans. Amer. Math. Soc.*, 366(10):5523–5540, 2014.

[4] H. Crapo and G.-C. Rota. *On the Foundations of Combinatorial Theory: Combinatorial Geometries*. Preliminary edition, MIT Press, Cambridge, MA, 1970.

[5] M. D'Adderio and L. Moci. Arithmetic matroids, the Tutte polynomial and toric arrangements. *Adv. in Math.*, 232(1):335–367, 2013.

[6] C. De Concini and C. Procesi. On the geometry of toric arrangements. *Transform. Groups*, 10(3-4):387–422, 2005.

[7] A. Fink and L. Moci. Matroids over a ring. *J. Eur. Math. Soc. (JEMS)*, 18(4):681–731, 2016.

[8] A. Higashitani and T. N. Tran. Period collapse in characteristic quasi-polynomials of non-central hyperplane arrangements. in preparation.

[9] H. Kamiya, A. Takemura, and H. Terao. Periodicity of hyperplane arrangements with integral coefficients modulo positive integers. *J. Algebr. Comb.*, 27(3):317–330, 2008.

[10] H. Kamiya, A. Takemura, and H. Terao. Periodicity of non-central integral arrangements modulo positive integers. *Ann. Comb.*, 15(3):449–464, 2011.

[11] Y. Liu, T. N. Tran, and M. Yoshinaga. G-Tutte polynomials and abelian Lie group arrangements. *Int. Math. Res. Not. IMRN*, 2021(1):150–188, 2021.

[12] L. Moci. A Tutte polynomial for toric arrangements. *Trans. Amer. Math. Soc.*, 364(2):1067–1088, 2012.

[13] T. N. Tran and A. Tsuchiya. Worpitzky-compatible subarrangements of braid arrangements and cocomparability graphs. *C. R. Math. Acad. Sci. Paris*, to appear, 2021. https://arxiv.org/abs/2007.01248.

[14] T. N. Tran and M. Yoshinaga. Combinatorics of certain abelian Lie group arrangements and chromatic quasi-polynomials. *J. Combin. Theory Ser. A*, 165:258–272, 2019.

TAN NHAT TRAN, FAKULTÄT FÜR MATHEMATIK, RUHR-UNIVERSITÄT BOCHUM, D-44780 BOCHUM, GERMANY

E-mail address: tan.tran@ruhr-uni-bochum.de