An E_∞ splitting of spin bordism

Gerd Laures

Abstract. The work determines the E_∞ structure of the $K(1)$-local spin bordism spectrum by giving a multiplicative splitting into E_∞ cells. The splitting leads to a cellular decomposition of the $\hat{A}$-map and creates new isomorphisms of Conner-Floyd type.

Introduction and statement of results

One of the most important results on spin bordism goes back to Anderson, Brown and Peterson (ABP). They showed that two spin manifolds are spin bordant if and only if all Stiefel-Whitney and KO-characteristic numbers coincide. Moreover, the spin bordism groups can be computed from the additive 2-local splitting

$$MSpin \cong \bigvee_{\substack{n(J) even \\ 1 \notin J}} ko\langle 4n(J)\rangle \vee \bigvee_{\substack{n(J) odd \\ 1 \notin J}} ko\langle 4n(J)+2\rangle \vee \bigvee_{i\in I} \Sigma^{d_i} H\mathbb{Z}/2.$$

Here, J is a finite sequence of non negative integers, $n(J)$ is the sum over all entries and I is some index set. However, the formula does not capture the ring structure of spin bordism. This problem has been around for the last thirty years because the Eilenberg Mac Lane part is very hard to track in practice. Fortunately, when the attention is restricted to the part which can be detected by K-theory this term disappears. More precisely, the localization of spin bordism with respect to mod 2 K-theory $K(1)$ is a sum of KO-theories.

The generator ζ of the -1st homotopy group of the $K(1)$-local sphere vanishes in KO and so it does in $MSpin$. These considerations lead to an E_∞ map from the E_∞ cone T_ζ over ζ to the $K(1)$-local $MSpin$. One may hope that this map is an isomorphism since the homotopy groups appear to be the same. However, when taking into account the Dyer-Lashof operations it becomes apparent that in KO-homology one gets a free θ-algebra in one generator for T_ζ and one in infinitely many generators for $MSpin$. Analyzing the action of the Adams operations and collecting all results this work shows the following multiplicative splitting:

$$MSpin \cong T_\zeta \wedge \bigwedge_{i=1}^{\infty} TS^0.$$

Here, TS^0 is the free E_∞ spectrum generated by the sphere spectrum and $\wedge$ is the coproduct in the category of E_∞ spectra. The proof of this splitting formula takes the major part of the work. The difficulty is the determination of the θ-algebra structure and a good control of the behavior of the map from T_ζ. For that, one needs to understand the ABP-splitting map in KO-homology and to conduct some 2-adic analysis.

1991 *Mathematics Subject Classification.* 55N22, 57T25, 19L64, 55N15, 55N34.

Consequences of the splitting formula. As a first immediate corollary one obtains the θ-algebra structure of the spin bordism itself:

$$\pi_* MSpin \cong \pi_* KO \otimes T\{f_1, f_2, \ldots\}$$

That is, the homotopy of $MSpin$ is the free θ-algebra over $\pi_* KO$ in infinitely many generators. Moreover, there is an algorithm for constructing the generating classes.

Despite this formula for the θ-algebra structure it turns out that $MSpin$ can not be made into a KO-algebra even in the $K(1)$-local world. (The question whether it can be made into an KO-module spectrum in the category of spectra was asked by Mahowald and answered in the negative by Stolz in [**Sto94**].)

The splitting formula allows a cellular decomposition of KO-theory when viewed as a relative E_∞ CW complex via the $\hat{A}$-map (see corollary 6.5). In particular, the $\hat{A}$-map arises from a map of E_∞ spectra. Moreover, a new proof of the formula

$$MSpin_* X \otimes_{MSpin_*} KO_* \cong KO_* X$$

is given. This result was first obtained by Hovey and Hopkins in [**HH92**] by different methods. It is not hard to see that the difficulties in proving this Conner-Floyd isomorphism appear in the $K(1)$-local setting. Thus the splitting formula can be applied and promptly furnishes the result.

In [**Lau01**] the splitting formula is used to equip the topological modular forms spectrum tmf of Hopkins with an orientation

$$W : MO\langle 8\rangle \longrightarrow tmf$$

in the $K(1)$-local E_∞ setting. This map induces the Witten genus in homotopy. Moreover, in complete analogy to real K-theory, the formula gives new isomorphisms of Conner-Floyd type for tmf.

Organization and statement of results. We first supply the technical framework for the splitting formula. We recall the basic properties of the $K(1)$-local category and then turn to E_∞ spectra. We take the classical point of view and work with operads in order to have H_∞ techniques available. However, the results can easily be translated into the brave new world of [**EKMM97**] or into the category of symmetric spectra of [**HSS00**] as well. The reader who is familiar with these concepts may skip the first section without harm.

In the second section Dyer-Lashof operations in K-theory are reviewed. The θ-algebra structure of $\pi_* K \wedge TS^0$ was first computed by McClure by using Dyer-Lashof operations in singular homology. It is a free θ-algebra in one generator and so is $\pi_* K \wedge T_\zeta$. We give a new proof of these results by looking at the representations of the symmetric groups and using results of Hodgkin and Atiyah. Hence, the right hand side of the splitting formula is completely understood in homotopy.

In the third section we construct the map φ from the cone T_ζ to $MSpin$ and determine its behavior in KO-homology. For that, we compute the image $\pi^J_* \varphi_* b$ of the generator $b \in \pi_0 KO \wedge T_\zeta$ under the ABP-splitting map

$$(\pi^J)_* : \pi_0 KO \wedge MSpin \xrightarrow{\cong} \bigoplus_{1 \notin J} \pi_0 KO \wedge KO.$$

The ring $\pi_0 KO \wedge KO$ can be identified with the ring of continuous functions on $\mathbb{Z}_2^\times / \pm 1$. It turns out

Theorem A. *The 2-adic function associated to $\pi^J_* \varphi_* b$ is the continuous homomorphism h which sends the topological generator $3 \in \mathbb{Z}_2^\times / \pm 1$ to 1 for $J = \emptyset$ and vanishes for all other J.*

Next we provide explicit polynomial generators $u_4, u_8, u_{12}, \ldots$ of the ring $\pi_0 KO \wedge BSpin_+ \cong \pi_0 KO \wedge MSpin$. In order to express $\varphi_* b$ in these classes we compute their image under the ABP-map as well.

THEOREM B. *The associated 2-adic function $\Theta_J(u_n)$ vanishes for all non empty J which do not contain 1. For $\Theta = \Theta_\emptyset$ and all odd k we have the formula*

$$\sum_n \Theta(u_n)(k)x^n = k^{-1}\frac{(1-x)^k - (1-x)^{-k}}{(1-x)-(1-x)^{-1}}$$

These two formulas show that $\varphi_* b$ is very close to u_4. A precise formulation of this statement is postponed to 5.9.

An E_∞ map induces a map of θ-algebras. Hence the map φ is determined in KO-homology once we know how θ operates in spin bordism. The fourth section is devoted to the computation of the θ-algebra structure of various bordism theories. Partial results were obtained earlier by Snaith in [**HS75**] by looking at representations of the symmetric groups. Unfortunately, all calculations of Snaith are made in characteristic 2 and hence are not of any use for us. Moreover, it seems to be difficult to generalize his calculations to higher orders of 2. Hence, we take a different approach and look at tom Dieck operations instead. A fixed point formula of Quillen and a 'change of suspension formula' help to compute the action of ψ and hence of θ for complex bordism.

THEOREM C. *The θ-algebra structure of $\pi_0 K \wedge MU$ is determined by the equation*

$$\sum_{i\geq 0} \psi(b_i)x^i = g(2)^{-1}g(1+\sqrt{1-x})g(1-\sqrt{1-x}).$$

Here, $g(x)$ is the invertible power series $\sum_{i\geq 0} b_i x^i$, $b_0 = 1$ with coefficients in the ring $\pi_0 K \wedge MU \cong \mathbb{Z}_2[b_1, b_2, \ldots]$. Note that for reasons of symmetry there is no root to take in the product: express $g(u)g(v)$ in the elementary symmetric polynomials and substitute $u + v = 2, uv = x$. With this formula we can proceed to compute the θ-algebra structure for SU and $Spin$ bordism. It is convenient to introduce a length l_2 in 4.10 which controls the modifiability of θ-algebra generators.

THEOREM D. *$\pi_0 KO \wedge MSpin$ is the free θ-algebra generated by the set of all u_{8k+4} for $k \geq 0$. Moreover, each generator u_{8k+4} can be altered by elements of strictly smaller l_2-length.*

In the fifth section we compute the action of the Adams operations to describe the spherical classes, that is, the KO-Hurewicz image $\pi_0 MSpin \subset \pi_0 KO \wedge MSpin$. In 5.6 we give an algorithm to construct spherical classes z_k for odd $k \geq 1$ with the property that each z_k coincides with u_{4k} up to elements of strictly smaller l_2-length.

MAIN THEOREM. *The E_∞ map*

$$(\varphi, z_3, z_5, \ldots) : T_\zeta \wedge \bigwedge_{i=1}^{\infty} TS^0 \longrightarrow MSpin$$

is a KO-equivalence.

This splitting theorem is a consequence of Theorem D, 5.6 and 5.9: in KO-homology the E_∞ map $(\varphi, z_3, z_5, \ldots)$ maps the free θ-algebra generators $b, z_3, z_5, \ldots$ to the free generators $u_4, u_{12}, u_{20}, \ldots$ up to smaller classes and thus is an isomorphism.

In the final section the promised applications to real K-theory are given.

Acknowledgements. The author would like to thank Mike Hopkins for posing the problem at the Oberwolfach Homotopietagung 1998 and for sharing the notes [**Hop98**] so freely. He is indebted to Matthias Kreck and Haynes Miller for their encouragement and for interesting discussions. He is also grateful to Matthew Ando, Pete Bousfield, Hans-Werner Henn and Peter May for their help in getting various points straight.

1. E_∞ spectra and localizations

This section reviews the basics of the category of E_∞ spectra and their properties under localization. We will work in the category of May, Puppe et al. which is described in [**LMS86**]. We use operads to describe E_∞ structures. This classical approach to E_∞ spectra has the advantage that every object is fibrant and hence calculations are easy to survey.

1.1. The category of spectra. In this work a topological space is a compactly generated weak Hausdorff space. We write $\mathcal{T}$ for the category of topological spaces and $\mathcal{T}_*$ for its based pendant. To talk about the category of spectra we first fix a real inner product space $\mathcal{U}$ of countable dimension, a so-called universe. A prespectrum X is a collection of pointed spaces X_V indexed by the finite dimensional subspaces $V \subset \mathcal{U}$ equipped with maps

$$\sigma_{W,V} : S^{W-V} \wedge X_V \longrightarrow X_W$$

for all $V \subset W$. Here, the space $W - V$ is the orthogonal complement of V in W and S^{W-V} denotes its one-point compactification. One requires that $\sigma_{V,V}$ is the identity and that the associativity condition $(1 \wedge \sigma_{V,U})\sigma_{W,V} = \sigma_{W,U}$ holds whenever $U \subset V \subset W$. A spectrum is a prespectrum with the additional property that the adjoint maps

$$\tau_{W,V} : X_V \longrightarrow \Omega^{W-V} X_W$$

are homeomorphisms. A map of (pre-) spectra $f : X \longrightarrow Y$ is a collection of maps $f_U : X_U \longrightarrow Y_U$ commuting with the structure maps.

A construction of Lewis shows that the forgetful functor from the category of spectra $S_\mathcal{U}$ to prespectra admits a left adjoint L, called the spectification. The spectification is easy to visualize in case that each $\sigma_{W,V}$ is a closed inclusion:

$$LX \cong (V \mapsto \operatorname*{colim}_{W \supset V} \Omega^{W-V} X_W).$$

The morphism set $\mathcal{S}_\mathcal{U}(X, Y)$ has a natural topology as subspace of the product of mapping spaces $\mathcal{T}(X_V, Y_V)$. Moreover, for each pointed space Q we may form the spectrum

$$X^Q : V \mapsto \mathcal{T}_*(Q, X_V)$$

and its adjoint

$$X \wedge Q = L(V \mapsto X_V \wedge Q).$$

These constructions are natural with respect to all variables and we have the relationship

$$T_*(Q, \mathcal{S}_\mathcal{U}(X, Y)) \cong \mathcal{S}_\mathcal{U}(X \wedge Q, Y) \cong \mathcal{S}_\mathcal{U}(X, Y^Q).$$

The category $\mathcal{S}_\mathcal{U}$ becomes a closed model category if we define the fibrations and weak equivalences spacewise. That is, a map f from X to Y is a fibration (or w.e.) if for all V the maps $f_V : X_V \longrightarrow Y_V$ are so. The resulting homotopy category is equivalent to the stable category of Adams for any infinite dimensional $\mathcal{U}$.

This coordinate free approach to spectra enables us to change universes in a continuous way. Suppose $\mathcal{U}$ and $\mathcal{V}$ are universes and let $\mathcal{L}(\mathcal{U}, \mathcal{V})$ denote the (contractible) space of linear isometries from $\mathcal{U}$ to $\mathcal{V}$. Any $f \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ defines an adjoint pair of functors:

$$f_* X = L(W \mapsto S^{W-f(U)} \wedge X_U) \qquad f^* X : V \mapsto X_{fV}$$

with $U = f^{-1}W$. Since the functors continuously depend on the isometry f this construction can be generalized as follows. A map from a space A to $\mathcal{L}(\mathcal{U}, \mathcal{V})$ gives rise to a functor $A \ltimes\ : \mathcal{S}_\mathcal{U} \longrightarrow \mathcal{S}_\mathcal{V}$ and its adjoint $F[A,\ ) : \mathcal{S}_\mathcal{V} \longrightarrow \mathcal{S}_\mathcal{U}$. The construction is natural in A and reduces to the above if A is a point. Moreover, the half smash product has the following properties:

(i) the identity map $id_{\mathcal{U}} \in \mathcal{L}(\mathcal{U},\mathcal{U})$ serves as a unit

$$id_* X = \{id_{\mathcal{U}}\} \ltimes X \cong X$$

(ii) for any $X \in \mathcal{S}_{\mathcal{U}}$ and for any $A \longrightarrow \mathcal{L}(\mathcal{U},\mathcal{V})$, $B \longrightarrow \mathcal{L}(\mathcal{V},\mathcal{W})$ the map

$$B \times A \longrightarrow \mathcal{L}(\mathcal{V},\mathcal{W}) \times \mathcal{L}(\mathcal{U},\mathcal{V}) \longrightarrow \mathcal{L}(\mathcal{U},\mathcal{W})$$

satisfies

$$(B \times A) \ltimes X \cong B \ltimes (A \ltimes X).$$

(iii) for any $X \in \mathcal{S}_{\mathcal{U}}$, $Y \in S_{\mathcal{V}}$ and for any $A \longrightarrow \mathcal{L}(\mathcal{U},\mathcal{U}')$, $B \longrightarrow \mathcal{L}(\mathcal{V},\mathcal{V}')$ the map

$$A \times B \longrightarrow \mathcal{L}(\mathcal{U},\mathcal{U}') \times \mathcal{L}(\mathcal{V},\mathcal{V}') \longrightarrow \mathcal{L}(\mathcal{U} \oplus \mathcal{V},\mathcal{U}' \oplus \mathcal{V}')$$

satisfies

$$(A \times B) \ltimes (X \wedge Y) \cong (A \ltimes X) \wedge (B \ltimes Y).$$

In [**Elm88**] Elmendorf defines a big category $\mathcal{S}$ with objects all spectra over all universes. A morphism from a spectrum X over $\mathcal{U}$ into a Y over $\mathcal{V}$ is given by a pair, an isometry $f : \mathcal{U} \longrightarrow \mathcal{V}$ and a map of spectra from $f_* X$ to Y. It is convenient to topologize $\mathcal{S}(X, Y)$ in a way that we have

$$\mathcal{T}/\mathcal{L}(\mathcal{U},\mathcal{V})(A, \mathcal{S}(X, Y)) \cong \mathcal{S}_{\mathcal{V}}(A \ltimes Y).$$

Here, $\mathcal{T}/\mathcal{L}(\mathcal{U},\mathcal{V})$ denotes the category of spaces over $\mathcal{L}(\mathcal{U},\mathcal{V})$.

These constructions are useful once it comes to smash products. If X is indexed over $\mathcal{U}$ and Y over $\mathcal{V}$ then the spectification of the (partial) prespectrum

$$U \times V \mapsto X_U \wedge Y_V$$

is indexed over $\mathcal{U} \times \mathcal{V}$. This product is associative and symmetric up to coherent equivalences and turns $\mathcal{S}$ into a symmetric monoidal category with unit $I = (S^0, 0)$. To get an internal product in $\mathcal{S}_{\mathcal{U}}$ we can choose an isometry $f : \mathcal{U} \times \mathcal{U} \longrightarrow \mathcal{U}$ and take the pushforward $f_*(X \wedge Y)$. A more canonical object is the spectrum

$$\mathcal{L}(\mathcal{U} \times \mathcal{U},\mathcal{U}) \ltimes (X \wedge Y).$$

However, it still is only associative up to coherent equivalences in the homotopy category. To talk about 'ring-like' objects in $\mathcal{S}_{\mathcal{U}}$ we need to introduce the concept of an operad.

1.2. Operads and E_∞ spectra. Let $\mathcal{M}$ be a symmetric monoidal category with product $\otimes$ and unit I. An operad in $\mathcal{M}$ is a family of objects $T_0, T_1, \ldots \in \mathcal{M}$ together with right Σ_n-actions on each T_n and $\Sigma_n \times \Sigma_{i_1} \times \ldots \Sigma_{i_n}$-equivariant structure maps

$$T_n \otimes T_{i_1} \otimes \ldots \otimes T_{i_n} \longrightarrow T_{i_1+i_2+\ldots+i_n}.$$

These maps should satisfy certain associativity axioms. A pointed operad is an operad together with a map $I \longrightarrow T_1$ which behaves like a unit when composed with structure maps.

Instead of dealing with the details here we give the most important class of examples of pointed operads: the endomorphism operad $\mathrm{End}(X)$ for each object X in $\mathcal{M}$. Its n'th object is the morphism set $\mathcal{M}(X^{\otimes n}, X)$ and the structure maps are given by compositions

$$f_n \otimes f_{i_1} \otimes \cdots \otimes f_{i_n} \mapsto f_n(f_{i_1} \otimes \cdots \otimes f_{i_n}).$$

If $\mathcal{M}$ is enriched over $\mathcal{M}'$ we obtain an operad in $\mathcal{M}'$. For instance, the linear isometry operad $\mathcal{L} = \mathrm{End}(\mathcal{U})$ is an operad of spaces and so is $\mathrm{End}(X)$ for any spectrum X indexed over $\mathcal{U}$. Moreover, the canonical projection

$$p : \mathrm{End}(X) \longrightarrow \mathcal{L}$$

is a map of pointed operads, meaning a collection of maps which is compatible with all structure maps. It associates to a map of spectra $f : X^{\wedge n} \longrightarrow X$ the underlying isometry $p(f) : \mathcal{U}^{\times n} \longrightarrow \mathcal{U}$.

An operad T over $\mathcal{L}$ is called an E_∞ operad if each of the spaces is contractible and the symmetric groups act in a free fashion. This way, the operad $\mathcal{L}$ becomes an E_∞ operad itself. An E_∞ ring spectrum is a spectrum X together with a map $T \longrightarrow \mathrm{End}(X)$ of pointed operads over $\mathcal{L}$.

There is another way to describe the action of T on X which will prove useful later. Let G be a group which acts on $\mathcal{U}$ by linear isometries and on X by a map $\mu : G \ltimes X \longrightarrow X$. Then for a G-equivariant map $A \longrightarrow \mathcal{L}(\mathcal{U}, \mathcal{V})$ let $A \ltimes_G X$ be the coequalizer of the two maps

$$(A \times G) \ltimes X \stackrel{\mu \ltimes 1}{\longrightarrow} A \ltimes X \quad \text{and} \quad (A \times G) \ltimes X \cong A \ltimes (G \ltimes X) \stackrel{1 \ltimes \mu}{\longrightarrow} A \ltimes X.$$

The spectrum

$$TX = \bigvee_{n \geq 0} T_n \ltimes_{\Sigma_n} X^{\wedge n}$$

is the free E_∞ algebra generated by X over the operad T. This way, T becomes an endofunctor of $\mathcal{S}_{\mathcal{U}}$. More precisely, T is a triple (or a 'monad') since it comes with a unit $\eta : id \longrightarrow T$ and a natural transformation $\mu : T^2 \longrightarrow T$. The latter is built from the structure maps data of the operad T. An algebra over T is a spectrum X together with a morphism $\xi : TX \longrightarrow X$ making the diagrams

$$\begin{array}{ccc} T^2X & \xrightarrow{T\xi} & TX \\ \Big\downarrow{\scriptstyle \mu_X} & & \Big\downarrow{\scriptstyle \xi} \\ TX & \xrightarrow{\xi} & X \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ & \searrow{\scriptstyle =} & \Big\downarrow{\scriptstyle \xi} \\ & & X \end{array}$$

commute. An E_∞ map between E_∞ spectra is a map of spectra which commutes with the action of T. Later we will also use the notion of an H_∞ spectrum which is defined in the same way but the diagrams need only to commute up to homotopy.

EXAMPLES 1.1. (i) The sphere spectrum is an E_∞ spectrum since it is the free object $T(*)$ generated by a point.

(ii) Let G be one of the classical groups $O, SO, Pin, Spin$ or one of their complex analogues $U, SU, Spin^c$ or Sp ect. Then the geometric bar construction [**May77**][**May72**] gives simplicial toplogical spaces $B_*(GV, S^V)$ and $B_*(GV)$ for each $V \subset \mathcal{U}$. Their geometric realizations BGV ($B(GV, S^V)$ resp.) allow multiplication maps

$$\mu : BGV \times BGW \longrightarrow B(GV \times GW) \longrightarrow BG(V \oplus W)$$

which are commutative and associative on the nose. The spectification of

$$Th : V \mapsto B(GV, S^V)/BGV$$

is called the Thom spectrum MG. For $f \in \mathcal{L}_n$ and for subspaces $V_1, V_2, \ldots, V_n$ define the structure maps

$$Th(V_1) \wedge \ldots \wedge Th(V_n) \stackrel{Th\mu}{\longrightarrow} Th(V_1 \oplus \ldots \oplus V_n) \stackrel{Th(f)}{\longrightarrow} Th(f(V_1 \oplus \ldots \oplus V_n))$$

and spectify to obtain an E_∞ structure on MG.

(iii) Other examples are less elementary. An investigation of the infinite loop spaces shows that connective real and complex K-theory ko and k are represented by E_∞ ring spaces and hence are E_∞ ring spectra. Moreover, the stable Adams operations ψ^r act as E_∞ maps after completion. Recently it was proved that the periodic theories KO and K are commutative S-algebras in [**EKMM97**] and symmetric spectra in [**Joa01**].

We write $\mathcal{S}_\mathcal{U}^T$ for the category of E_∞ spectra. The free functor T is left adjoint to the forgetful functor $U : \mathcal{S}_\mathcal{U}^T \longrightarrow \mathcal{S}_\mathcal{U}$. Hence, inverse limits in $\mathcal{S}_\mathcal{U}^T$ are inherited from $\mathcal{S}_\mathcal{U}$. Colimits are more difficult. It can be shown that for any diagram (X_α) of T-algebras the ordinary coequalizer of

$$T(\operatorname{colim} UTX_\alpha) \underset{d_1}{\overset{d_0}{\rightrightarrows}} T(\operatorname{colim} UX_\alpha)$$

admits a T-algebra structure which satisfies the universal property of a colimit in $\mathcal{S}_\mathcal{U}^T$. Here, d_0 is induced by the T-algebra structure on each X_α whereas d_1 comes from the natural transformation μ which makes T into a monad. The argument uses the fact that UT preserves reflexive coequalizers which can be checked spacewise [**Hop96**]. Alternatively, the existence of colimits follows from a general fact about categories of algebras over a triple (compare [**BW85**].) The E_∞ category $\mathcal{S}_\mathcal{U}^T$ admits a closed model category structure with the following data: a map is a fibration (resp. weak equivalence) if and only if Uf is one.

The coproduct of E_∞ spectra E and F is weakly equivalent to the product $E\wedge F$ (see [**EKMM97**].) Note that for E_∞-spectra E and F the product $E \wedge F \in \mathcal{S}_{U\times U}$ is an E_∞ spectrum in the following way: $E \wedge F$ is an algebra over the product E_∞ operad $T \times T$ via

$$\begin{aligned}(\mathcal{L}_n \times \mathcal{L}_n) \ltimes_{\Sigma_n} (E \wedge F)^n &\cong (\mathcal{L}_n \ltimes E^n \wedge \mathcal{L}_n \ltimes F^n)/\Sigma_n \\ &\longrightarrow \mathcal{L}_n \ltimes_{\Sigma_n} E^n \wedge \mathcal{L}_n \ltimes_{\Sigma_n} F^n \longrightarrow E \wedge F.\end{aligned}$$

This justifies the notation in the splitting formula. Finally we remark that

$$\bigwedge_{i=1}^{\infty} TS^0 \cong T(\bigvee_{i=1}^{\infty} S^0) \cong \operatorname*{colim}_k T(\bigvee_{i=1}^{k} TS^0).$$

This follows from the fact that T is a left adjoint and hence commutes with colimits. The middle term reveals that the above colimit of free E_∞ spectra coincides as a spectrum with the ordinary colimit in $\mathcal{S}_U$.

1.3. The Bousfield localization with respect to $K(1)$. A map of spectra $f : X \longrightarrow Y$ is called an E-equivalence if the induced map f_* in E-homology is an isomorphism. Since E does not distinguish between honest isomorphisms and E-equivalences it is useful to have a category in which E-equivalences are invertible and no other information is lost. The existence of this E-local category was proved by Bousfield in [**Bou79**]. He even showed that it can be realized as a full subcategory of the stable category $Ho\mathcal{S}_\mathcal{U}$.

In detail, a spectrum Z is E-local if each E-equivalence f induces a bijection in cohomology $f^* : Z^*Y \longrightarrow Z^*X$. The inclusion functor of the full subcategory $\mathcal{C}$ of E-local objects in $Ho\mathcal{S}_\mathcal{U}$ admits a left adjoint $L_E : Ho\mathcal{S}_\mathcal{U} \longrightarrow \mathcal{C}$, called the Bousefield localization functor. Hence, the unit of the adjunction $\eta : X \longrightarrow L_EX$ is an E-equivalence and associates an E-local spectrum to any spectrum X.

The Bousfield localization has the following elementary properties:

(i) L_E takes E-equivalences to isomorphisms.
(ii) L_E is idempotent: $L_E\, L_E \cong L_E$.
(iii) L_E preserves homotopy inverse limits.
(iv) L_E preserves cofibre sequences.
(v) If E is a ring spectrum and X is a module spectrum over E then X is E-local: $X \cong L_EX$.

Note that the localization functor L_E can be chosen to preserve E_∞ structures (see [**EKMM97**]ch.8.) We want to take E to be the first Morava K-theory. At the prime 2 the theory $K(1)$ coincides with mod 2 K-theory $K\mathbb{Z}/2$ for the following reason: the 2-typicalization of the multiplicative formal group law $\hat{G}_m$ is the Honda formal group law since the 2-series are the same. Hence, the localization with respect to $K(1)$ can be obtained as a process in two stages:

$$L_{K(1)} \cong L_{S\mathbb{Z}/2} L_{K_{(2)}}.$$

Here, $S\mathbb{Z}/2$ is the $\mathbb{Z}/2$-Moore spectrum. Note that for the localization it makes no difference to work with complex or real K-theory [**Mei79**]. The latter was investigated by Adams, Baird and Ravenel. They showed that the KO-local sphere is closely related to the image of J spectrum: Let $J_{(2)}$ be the fibre of

$$\psi^3 - 1 : KO\mathbb{Z}_{(2)} \longrightarrow KO\mathbb{Z}_{(2)}.$$

Here, ψ^3 is the third stable Adams operation. Then there is a fibration of the form

$$L_{KO\mathbb{Z}_{(2)}} S \longrightarrow J_{(2)} \longrightarrow \Sigma^{-1} S\mathbb{Q}.$$

The rational part vanishes once we localize with respect to the Moore spectrum $S\mathbb{Z}/2$: for all X the localization $L_{S\mathbb{Z}/2}X$ is the function spectrum $F(\Sigma^{-1}S\mathbb{Z}/2^\infty, X)$. The localization with respect to $S\mathbb{Z}/2$ acts as a completion on homotopy groups since there is an exact sequence [**Bou79**]

$$0 \longrightarrow \mathrm{Ext}(\mathbb{Z}/2^\infty, \pi_* X) \longrightarrow \pi_* L_{S\mathbb{Z}/2} X \longrightarrow \mathrm{Hom}(\mathbb{Z}/2^\infty, \pi_{*-1} X) \longrightarrow 0.$$

Summarizing, we see that the $K(1)$-local sphere coincides with the completed image of J-spectrum $L_{S\mathbb{Z}/2} J_{(2)}$. Moreover, we get the fibration

$$L_{K(1)} S \longrightarrow L_{K(1)} KO \stackrel{\psi^3-1}{\longrightarrow} L_{K(1)} KO.$$

which enables us to calculate the homotopy groups of the K(1)-local sphere. In addition, it turns out that $KO_{(2)}$-theory is smashing (see [**Rav84**]8.12.) This means that for all X the spectrum $L_{KO_{(2)}} S \wedge X$ is a $KO_{(2)}$-localization of X. Hence, smashing the sequence above with X gives the fibre sequence

$$X \longrightarrow KO \wedge X \stackrel{\psi^3-1}{\longrightarrow} KO \wedge X$$

in the $K(1)$-local category.

2. θ-Algebras

This section is devoted to the algebraic objects which come up as the homotopy of $K(1)$-local E_∞ ring spectra. A θ-algebra is an algebra together with a single operation θ which satisfies certain properties. The classical theory of such θ-algebras goes back to Grothendieck and Atiyah who investigated the exterior power operations in the representation theory and in K-cohomology rings. These power operations were later generalized by McClure, Ando and Hopkins to Dyer-Lashof operations for arbitrary $K(1)$-local E_∞ spectra. Their general properties were recently studied by Bousefield in [**Bou99**][**Bou96b**][**Bou96a**] from an axiomatic point of view.

We first review Atiyah's and Hodgkin's work on power operations. Then we give a new proof of McClure's result on the structure of TS^0. Finally, we turn to the spectrum T_ζ and work out the computations of Hopkins. There is hardly any new result in this section. However, the new treatments and proofs provide a convenient framework for things to come.

We work in the category $\mathcal{C}$ of $K(1)$-local spectra and omit the localization functor from the notation.

2.1. The K-homology ring of TS^0. Since Atiyah's work on power operations we know that there is a close relationship between operations in K-theory and the K-homology ring of

$$TS^0 \cong \bigvee_{n=0}^{\infty} B\Sigma_{n+}.$$

The latter can be computed.

THEOREM 2.1. *There is an isomorphism of rings*

$$\pi_*K \wedge TS^0 \cong \pi_*K[\theta_1, \theta_2, \theta_4, \ldots]$$

The ring of the right hand side is to be understood as an object in the category of 2-complete π_*K-algebras. More precisely, it is the free commutative 2-complete algebra on the indicated generators. For instance, the power series $\sum_n \theta_{2^n} 2^n$ is a valid class. The proof of the theorem uses the following result of Hodgkin which can be found in [**Hod72**]:

PROPOSITION 2.2. *Let $R\Sigma_n$ be the representation ring and $R\Sigma^\wedge$ its completion with respect to the augmentation ideal $I\Sigma_n$. Equip $R\Sigma_n \subset R\Sigma_n^\wedge$ with the $I\Sigma_n$-adic topology. Then we have*

$$\begin{aligned} \pi_*K\mathbb{Z}/2^r \wedge B\Sigma_{n+} &\cong Hom_{cts}(R\Sigma_n, \pi_*K\mathbb{Z}/2^r) \\ \pi_*K\mathbb{Z}/2^r \wedge TS^0 &\cong \pi_*K\mathbb{Z}/2^r[\theta_1, \theta_2, \theta_4, \ldots]. \end{aligned}$$

Here, Hom_{cts} *denotes the group of continuous homomorphisms.*

Hence, the theorem follows from the following lemma. (Recall that the $K(1)$-localization functor is omitted from the notation.)

LEMMA 2.3.

$$\begin{aligned} \pi_*K \wedge B\Sigma_{n+} &\cong \lim_r \pi_*K\mathbb{Z}/2^r \wedge B\Sigma_{n+} \cong \mathrm{Hom}_{cts}(R\Sigma_n, \pi_*K) \\ \pi_*K \wedge TS^0 &\cong \lim_r \pi_*K\mathbb{Z}/2^r \wedge TS^0 \end{aligned}$$

PROOF. For all spectra X the $K(1)$-local $K \wedge X$ can be written as the homotopy inverse limit of the sequence

$$K\mathbb{Z}/2 \wedge X \longleftarrow K\mathbb{Z}/4 \wedge X \longleftarrow K\mathbb{Z}/8 \wedge X \longleftarrow \ldots$$

Hence, there is a short exact sequence

$$0 \to \lim_r{}^1 \pi_{*+1} K\mathbb{Z}/2^r \wedge X \to \pi_*K \wedge X \to \lim_r \pi_*K\mathbb{Z}/2^r \wedge X \to 0$$

and it suffices to show that the $\lim^1$-term vanishes. This is obvious for the case $X = TS^0$ from Hodgkin's result and follows for the classifying spaces since they are direct summands of TS^0. □

We are going to describe the elements θ_k in more detail. The class θ_1 comes from the unit of the operad T

$$S^0 \longrightarrow T_1 S^0 \longrightarrow TS^0 \cong S^0 \wedge TS^0 \longrightarrow K \wedge TS^0.$$

For the others we consider the dual representation ring $\mathrm{Hom}(R\Sigma_n, \mathbb{Z}_2)$ of all homomorphisms. It admits an interpretation as the group of elements of degree n in the ring of symmetric polynomials in indeterminants t_i of degree 1: let

$$\Delta_{k,n} \in \mathbb{Z}_2[t_1, t_2, \ldots, t_k]_n^{\Sigma_k} \otimes R\Sigma_n$$

be the function

$$\Delta_{k,n}(t_1, t_2, \ldots, t_k, g) = \mathrm{Trace}(gT^{\otimes n}).$$

In this notation we regard $R\Sigma_n$ as the character ring and g lies in Σ_n. The letter T denotes here the diagonal matrix $(t_1, t_2, \ldots, t_k)$ acting on $\mathbb{C}^k$. Atiyah showed in [**Ati66**] that the map

$$\Delta : \bigoplus_{n \geq 0} \mathrm{Hom}(R\Sigma_n, \mathbb{Z}_2) \longrightarrow \bigoplus_{n \geq 0} \lim_k \mathbb{Z}_2[t_1, t_2, \ldots, t_k]_n^{\Sigma_k}$$

given by $\sum f_n \mapsto \sum (1 \otimes f_n)\Delta_{k,n}$ is a ring isomorphism.

Example 2.4. Σ_2 admits two irreducible representations: the trivial 1 and the sign representation σ. One readily verifies

$$\Delta_{k,2} = e_2 \otimes \sigma + (e_1^2 - e_2) \otimes 1$$

for all $k \geq 2$. Here, the e_i's denote the elementary symmetric polynomials. The augmentation ideal of $R\Sigma_2 = \mathbb{Z}[\sigma]/\sigma^2 - 1$ is generated by $1 - \sigma$. We claim that all homomorphisms are continuous. Obviously e_1^2 is so. Hence it suffices to check e_2:

$$e_2(1-\sigma)^n = -\sum_{k\, odd} \binom{n}{k} = -2^{n-1}$$

In these terms, we can inductively define the θ_i's by declaring the powers sums $\sigma_{2^k} = \sum t_i^{2^k}$ to be the k-th Witt polynomial in the θ_i's

$$\sigma_{2^k} = \theta_1^{2^k} + \ldots + 2^k \theta_{2^k}$$

or, equivalently,

$$\prod_{i=0}^{\infty} (1 - t_i x) = \prod_{n=0}^{\infty} (1 - \theta_n x^n).$$

A proof for the continuity of θ_n can be found in [**Hod72**]. Atiyah explained how such an element θ_k leads to an operation

$$\theta^k : K(X) \stackrel{()^{\otimes n}}{\longrightarrow} K_{\Sigma_n}(X) \cong K(X) \otimes \mathrm{Rep}\Sigma_n \stackrel{1\otimes\theta_k}{\longrightarrow} K(X)$$

For instance, the power sums σ_k give rise to the Adams operation ψ^k. Hence, the formula for the first Witt polynomial may be interpreted as

$$\psi^2(x) = x^2 + 2\theta^2(x)$$

for all $x \in K(X)$. This was the first topological example of what is called a θ-algebra. Later McClure [**BMMS86**] showed how the θ_i come up by operations of Dyer-Lashof type.

2.2. The category of θ-algebras. We now turn to the algebraic picture and work in the category of 2-complete groups.

Definition 2.5. *(compare* [**Bou96b**][**Bou96a**][**Bou99**]*) A θ-algebra is a commutative algebra A over a ring R with unit together with a function $\theta : A \longrightarrow A$ such that*

$$\begin{aligned} \theta(1) &= 0 \\ \theta(a+b) &= \theta(a) + \theta(b) - a\,b \\ \theta(a\,b) &= \theta(a)b^2 + a^2\,\theta(b) + 2\theta(a)\theta(b). \end{aligned}$$

For a θ-algebra A we define the operation $\psi : A \longrightarrow A$ by the equation $\psi(x) = x^2 + 2\theta(x)$. One easily checks the

Proposition 2.6. *ψ is a ring homomorphism and commutes with θ.*

Examples 2.7.

(i) $\mathbb{Z}_2$ is a θ-algebra via

$$\theta(x) = \frac{x - x^2}{2} \quad \text{and} \quad \psi(x) = x.$$

Similarly, the ring $C = \mathcal{T}(\mathbb{Z}_2, \mathbb{Z}_2)$ of continuous functions on the 2-adics is a θ-algebra with $\psi(f) = f$.

(ii) There is not any θ-algebra of characteristic 2: when setting $(a, b) = (1, 1)$ (and $(1, 0)$ resp.) in the addition formula we see that 1 equals 0 for such algebras.

(iii) For all spaces X the ring KX is a θ-algebra via the operation θ^2 as explained in 2.1. The properties of θ immediately follow from the naturality and the fact that the Adams operation $\psi^2 = \psi$ is a ring homomorphism. In particular, the ring of unstable operations in K-theory $Op(K)$ is a θ-algebra by

$$\theta(a)(x) = \theta^2(a(x)) \text{ for } a \in Op(K),\, x \in K(X).$$

It turns out that the subring

$$\pi_0 K \wedge TS^0 \cong \bigoplus_{n\geq 0} \mathrm{Hom}_{cts}(R\Sigma_n, \mathbb{Z}_2) \subset \bigoplus_{n\geq 0} \mathrm{Hom}(R\Sigma_n, \mathbb{Z}_2) \cong Op(K)$$

is a θ-subalgebra. But instead of dealing with continuity questions here we define a second θ-algebra structure on $\pi_0 K \wedge TS^0$ in the next paragraph and show in 2.12 that they coincide.

The following observation carries the name Wilkerson criterion [**Bou96a**]:

PROPOSITION 2.8. *Let A be torsion free and ψ an algebra endomorphism of A with the property that $\psi(a) = a^2$ mod 2. Then A has a unique θ-algebra structure with $\psi x = x^2 + 2\theta x$ for all $x \in A$.*

The forgetful functor from θ-algebras $\mathcal{F}$ to 2-complete modules admits a left adjoint T: if M is free on one generator x we define

$$TM = R[x, x_1, x_2, \ldots]$$

and set $\theta(x_i) = x_{i+1}$, $\theta(x) = x_1$. This algebra will be denoted by $R \otimes T\{x\}$ in the sequel. If M is free on generators $\{x_i\}_{i\in I}$ we set

$$TM = R \otimes \bigotimes_{i\in I} T\{x_i\} \cong R \otimes T\{x_i\}_{i\in I}.$$

For the general case we first observe that $\mathcal{F}$ has coequalizers and tensor products and thus colimits: if $f, g : A \longrightarrow B$ are θ-maps, then the ideal generated by the set

$$\{f(a) - g(a)\,;\, a \in A\}$$

is closed under the operation of θ:

$$\theta(f(a) - g(a)) = (f(\theta a) - g(\theta a)) + (f(a) - g(a))g(a).$$

Hence, the quotient ring is a coequalizer in $\mathcal{F}$. Tensor products of θ-algebras are obtained by choosing free representations and taking cokernels successively. Similarly, the free functor T of a general module M is obtained by presenting M as the cokernel of a map of free modules.

The free algebra $T\{x\}$ has another basis which is constructed as follows. In each θ-algebra A there is family of natural operations θ_n which satisfies

$$\psi^n a = (\theta_0 a)^{2^n} + 2(\theta_1 a)^{2^{n-1}} + \cdots + 2^n \theta_n a$$

Here, ψ^n is the iteration of ψ. These operations can be inductively defined by the equations (compare [**Bou96a**])

$$\begin{aligned} \theta_n a &= \theta^{\sim n}(\theta_0 a) + \theta^{\sim n-1}(\theta_1 a) + \cdots + \theta^{\sim 1}(\theta_{n-1} a). \\ \theta^{\sim n} a &= \sum_{i=1}^{2^{n-1}} (-1)^{i+1} 2^{i-n} \binom{2^{n-1}}{i} (\psi a)^{2^{n-1}-i} (\theta a)^i. \end{aligned}$$

For instance, for our θ-ring $K(X)$ the operations θ_n coincide with Hodgkin's operations θ_{2^n} considered earlier.

LEMMA 2.9. *Let A be the free θ-algebra on x and set $\theta_i = \theta_i(x)$. Then we have $\theta(\theta_k) = \theta_{k+1} + \epsilon$ with a polynomial ϵ depending only on $\theta_0, \theta_1, \dots, \theta_k$. In particular, there is an isomorphism of rings*

$$A \cong \mathbb{Z}_2[\theta_0, \theta_1, \theta_2, \dots]$$

PROOF. Compute

$$\sum_{i=0}^{k+1} 2^i \theta_i^{2^{k+1-i}} \quad = \quad \psi(\sum_{i=0}^{k} 2^i \theta_i^{2^{k-i}}) = \sum_{i=0}^{k} 2^i (\psi\theta_i)^{2^{k-i}} = \sum_{i=0}^{k} 2^i (\theta_i^2 + 2\theta(\theta_i))^{2^{k-i}}$$

□

2.3. Dyer-Lashof Operations for $K(1)$-local E_∞ spectra. In 2.7(iii) we claimed that the ring $\pi_0 K \wedge TS^0$ carries a θ-algebra structure by interpreting its elements as operations in K-theory. There is a more intrinsic description of the θ-algebra structure which works for arbitrary $K(1)$-local E_∞ spectra E.

An E_∞ structure ξ on E determines a power operation

$$P : E^0 X \longrightarrow E^0 T_n X$$

by setting

$$P(x) : T_n X \longrightarrow TX \xrightarrow{Tx} TE \xrightarrow{T\xi} E$$

for each $x \in E^0 X$. For $X = S^0$ and $n = 2$ this gives a map $P(x) : B\Sigma_{2+} \longrightarrow E$ for each $x \in \pi_0 E$. The classifying space $B\Sigma_{2+}$ reduces to two copies of S^0 in the $K(1)$-local world. To see this, consider the map

$$(\epsilon, Tr) : B\Sigma_{2+} \longrightarrow pt_+ \vee (E\Sigma_2)_+ \cong S^0 \vee S^0$$

which consists of the constant map $\epsilon = const_+$ and the transfer Tr. It is a weak equivalence in $\mathcal{C}$: the transfer of a one dimensional trivial bundle is the bundle corresponding to the representation in which Σ_2 acts on $\mathbb{C}^2$ by permuting coordinates. Since this bundle comes from $1 + \sigma$ we obtain the isomorphism

$$(1 \wedge \epsilon, 1 \wedge Tr)_* : \pi_0 K \wedge B\Sigma_{2+} \cong \mathbb{Z}_2\sigma_2 \oplus \mathbb{Z}_2 e_2 \xrightarrow{1} \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

as one easily checks.

We follow the lines of [**Hop98**] and define maps

$$\theta, \psi : S^0 \longrightarrow B\Sigma_{2+}$$

by requiring

$$\begin{pmatrix} Tr \\ \epsilon \end{pmatrix} \begin{pmatrix} \theta & \psi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $e : S^0 \cong Be_+ \longrightarrow B\Sigma_{2+}$ we have $\epsilon e = 1$, $Tr\, e = 2$ and thus

$$e = \psi - 2\theta.$$

With $\theta(x) = P(x)\,\theta$ and $\psi(x) = P(x)\,\psi$ the last equation gives

$$\psi(x) - 2\theta(x) = P(x)\, e = x^2$$

which is Atiyah's equation. This also justifies the sign in the definition of θ. It is clear that θ is natural with respect to E_∞ maps. In fact the H_∞ property suffices in this context.

PROPOSITION 2.10. *The operation θ turns $\pi_0 E$ into a θ-algebra.*

PROOF. First assume that $\pi_0 E$ is torsion free. Then it suffices to check that the operation ψ is a ring homomorphism. Recall from [**BMMS86**]p.253 the formulae

$$\begin{aligned} P(x+y) &= Px + Py + Tr^*(x\,y) \\ P(x\,y) &= Px\,Py. \end{aligned}$$

With these it suffices to show that the stable map ψ induces a ring map in E-cohomology. That is, ψ should commute with the diagonal map

$$\Delta_+\psi = (\psi \wedge \psi)\Delta_+ \ \in \pi_0 B\Sigma_{2+} \wedge B\Sigma_{2+}.$$

Since ϵ commutes with Δ_+ we have

$$(\epsilon \wedge \epsilon)\Delta_+\psi = \Delta_+\epsilon\psi = \Delta_+ = (\epsilon \wedge \epsilon)(\psi \wedge \psi)\Delta_+.$$

Moreover, the map $f = (Tr \wedge Tr)\Delta_+\psi$ is null: in K-theory we have

$$f^*1 = \psi^*(Tr(1)^2) = \psi^*((1+\sigma)^2) = \psi^*(2(1+\sigma)) = 2\psi^* Tr(1) = 0.$$

Also the composite $\Delta_+ Tr\,\psi$ vanishes for trivial reasons. Since (ϵ, Tr) is a $K(1)$-equivalence and E is local we have established the commutativity of ψ and Δ_+. This finishes the proof for the torsion free case.

For general E let x, y be classes in $\pi_0 E$. Consider the E_∞ map

$$(x, y) : T(S^0 \vee S^0) \cong TS^0 \wedge TS^0 \longrightarrow E.$$

In order to establish the addition and multiplication laws it suffices to show that $\pi_0 T(S^0 \vee S^0)$ is torsion free. There are many ways to see that the Hurewicz map

$$\pi_0 T(S^0 \vee S^0) \longrightarrow \pi_0 K \wedge T(S^0 \vee S^0) \cong \pi_0 K \otimes T\{x, y\}$$

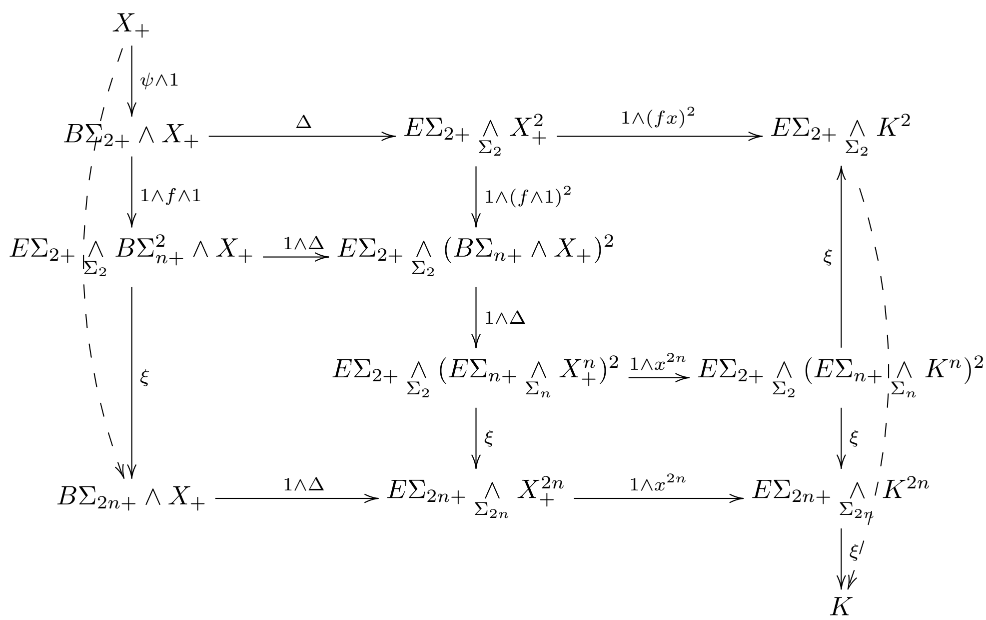
is injective. For instance, the computation in the Adams-Novikov spectral sequence based on K at the end of this section gives an argument. □

Example 2.11. For a space X consider the function spectrum K^{X_+}. Since it is an E_∞ ring spectrum we have a θ-algebra structure on $\pi_0 K^{X_+} = K(X)$. From the definition of the power operations we see that the operations ψ and θ coincide with the second Adams operation ψ^2 and Atiyah's operation θ^2.

The ring of all operations contains the subring $\pi_0 K \wedge TS^0$ as explained earlier. Its θ-algebra structure is determined by the

Theorem 2.12. *In $\pi_0 K \wedge TS^0$ we have the equality $\psi(\sigma_{2^k}) = \sigma_{2^{k+1}}$. In particular, $\pi_* K \wedge TS^0$ is the free θ-algebra on θ_1.*

Proof. The equation looks like the relation among Adams operations $\psi^2\psi^{2^k} = \psi^{2^{k+1}}$. Indeed the formula will follow once we have established $(\psi f)(x) = \psi^2(f(x))$ for all $x \in K(X)$ and $f \in \pi_0 K \wedge B\Sigma_{n+}$. In this context f is regarded as an operation as explained earlier. The equality is part of the following commutative diagram



Here, we assumed that f lies in the K-Hurewicz image which is allowed by induction. Then the composite over the left curved side of the outer square is $(\psi f)(x)$ whereas the right side gives $\psi^2(f(x))$ by 2.11. The last statement is immediate from 2.1 and 2.9. □

An alternative proof of the theorem is given in [**BMMS86**]ch.9. McClure used the ordinary Dyer-Lashof operations in singular homology instead of representation theory to obtain the result.

COROLLARY 2.13. *For every K(1)-local E_∞ spectrum E, $\pi_* E \wedge TS^0$ is the free θ-algebra on the generator θ_1.*

Before proving the corollary we need a lemma which is easily checked.

LEMMA 2.14. *Suppose E, F are E_∞ ring spectra and $x \in \pi_0 E \wedge F$ lies in the Hurewicz image of $\pi_0 E$. Then we have*

$$\theta_E x = \theta_{E\wedge F} x \in \pi_0 E \wedge F.$$

PROOF OF 2.13: The case $E = K$ of the corollary was shown in the theorem. For $E = S^0$ we consider the Adams-Novikov spectral sequences

$$\begin{aligned} \mathrm{Ext}_{K_*K}(\pi_*K, \pi_*K) &\implies \pi_*S^0 \\ \mathrm{Ext}_{K_*K}(\pi_*K, \pi_*K \wedge TS^0) &\implies \pi_*TS^0 \end{aligned}$$

They converge by the theorem 6.10 of [**Bou79**]. To compute the E_2-term of the second observe that for the isomorphism

$$\pi_*K \wedge TS^0 \cong \pi_*K \otimes T\{\theta_1\}$$

all classes θ_n are spherical by the lemma. Hence, the spectral sequence for π_*TS^0 takes the form

$$\mathrm{Ext}_{K_*K}(\pi_*K, \pi_*K) \otimes T\{\theta_1\} \Longrightarrow \pi_*S^0 \otimes T\{\theta_1\}$$

and we are done.

Finally, the general statement follows from the Kuenneth isomorphism:

$$\pi_*E \wedge TS^0 \cong \pi_*E \otimes_{\pi_*S^0} (\pi_*S^0 \otimes T\{\theta_1\}) \cong \pi_*E \otimes T\{\theta_1\}$$

for arbitrary $K(1)$-local E_∞ spectra E. □

The corollary can be found without proof in [**Hop98**].

2.4. The spectrum T_ζ. In the last paragraph we computed the θ-algebra structure associated to the homotopy of the free spectrum generated by the sphere. Now we proceed with our basic calculations and investigate the sphere spectrum with one more E_∞ cell attached. The cone is taken over a class in the homotopy of S^0 which comes up in other contexts in topology as well.

Recall from 1.3 that we have a cofibre sequence

$$S^0 \longrightarrow KO \overset{\psi^3-1}{\longrightarrow} KO.$$

Since ψ^3 acts trivially in $\pi_0 KO$ the element $1 \in \pi_0 KO \cong \mathbb{Z}_2$ gives rise to a non trivial class $\zeta \in \pi_{-1}S^0$. Obviously, ζ topologically generates $\pi_{-1}S^0 \cong \mathbb{Z}_2$.

REMARK 2.15. The class $\zeta = \zeta_1$ belongs to a family of homotopy classes

$$\zeta_n : S^{-1} \longrightarrow L_{K(n)}S^0$$

which play an important role in the reassembling of spectra from their monochromatic parts, that is, in Hopkins' chromatic splitting conjecture. The higher ζ_n correspond to the determinant map on the Morava stabilizer group S_n under the homotopy fixed point spectral sequence

$$E_2 = H^{*,*}(S_n; E_{n*})^{\mathbb{Z}/n} \Longrightarrow \pi_* L_{K(n)}S^0.$$

The interested reader is refered to [**Hov95**].

DEFINITION 2.16. *We define T_ζ to be the homotopy pushout of the diagram*

$$\begin{array}{ccc} TS^{-1} & \xrightarrow{T*} & T* = S^0 \\ \downarrow{\scriptstyle \zeta} & & \downarrow \\ S^0 & \longrightarrow & T_\zeta \end{array}$$

in the category $\mathcal{C}^T$ of $K(1)$-local E_∞ ring spectra. In particular, the class ζ is regarded as an E_∞ ring map here.

It is convenient to think of T_ζ as the pushout in $\mathcal{C}^T$ of

$$S^0 \xleftarrow{\zeta} TS^{-1} \longrightarrow T(S^{-1} \wedge I).$$

Hence, T_ζ corepresents the functor which associates to an object E of $\mathcal{C}^T$ the set of all null homotopies of ζ in E. This set is non empty for KO-theory. A choice of a null homotopy ι defines a map of cofibre sequences

$$\begin{array}{ccccc} S^0 & \xrightarrow{\gamma} & C_\zeta & \xrightarrow{\delta} & S^0 \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle 1} \\ S^0 & \xrightarrow{1} & KO & \xrightarrow{\psi^3 - 1} & KO \end{array}$$

and a splitting (ι_*, ι'_*) of the exact sequence

$$0 \longrightarrow \pi_0 KO \xrightarrow{\gamma_*} \pi_0 KO \wedge C_\zeta \xrightarrow{\delta_*} \pi_0 KO \longrightarrow 0.$$

Let b be the image of $1 \in \pi_0 KO$ under the composite

$$\pi_0 KO \xrightarrow{\iota'_*} \pi_0 KO \wedge C_\zeta \xrightarrow{\eta_{C_\zeta *}} \pi_0 KO \wedge TC_\zeta \xrightarrow{\xi_{S^0 *}} \pi_0 KO \wedge T_\zeta$$

COROLLARY 2.17. *(compare* [**Hop98**]*) The KO-linear extension of b*

$$b_* : \pi_* KO \wedge TS^0 \longrightarrow \pi_* KO \wedge T_\zeta$$

is an isomorphism of θ-algebras. Thus $\pi_* KO \wedge T_\zeta$ is the free θ-algebra on the generator b.

PROOF. Our choice of a null homotopy defines an E_∞ map from T_ζ to $KO \wedge TS^0$. Its KO-linear extension is the inverse of b_* as one easily checks. □

The class $f = \psi(b) - b$ is fixed under ψ^3 since

$$\begin{aligned} \psi^3(\psi(b) - b)) &= \psi^3\psi(b) - \psi^3(b) = \psi\psi^3(b) - \psi^3(b) \\ &= \psi(b+1) - (b+1) = \psi(b) - b. \end{aligned}$$

In fact, it turns out that f is represented by a unique spherical class and we have

THEOREM 2.18. *(compare* [**Hop98**]*) There is an isomorphism of θ-algebras*

$$\pi_* T_\zeta \cong \pi_* KO \otimes T\{f\}.$$

Moreover, $\pi_0 KO \wedge T_\zeta$ is free as $\pi_0 T_\zeta$-module.

This result will not be used for the splitting theorem. For the proof the reader is referred to the work of Hopkins. He also shows in [**Hop98**] how the Bott class behaves under the KO-Hurewicz map

$$i : \pi_* KO \otimes T\{f\} \cong \pi_* T_\zeta \longrightarrow \pi_* KO \wedge T_\zeta \cong \pi_* KO \otimes T\{b\}.$$

We have

$$i(v^4) = v^4 9^{-2b} = v^4 \sum_{n=0}^{\infty} \binom{-2b}{n} 2^{3n} \in T\{b\}.$$

In particular,

$$i(v^4) \equiv v^4 \mod 2$$

Note that the image of η is clear since it is spherical.

3. The ABP-splitting and 2-adic functions

In this section we construct an E_∞ map from the cone T_ζ to the $K(1)$-local spin cobordism theory $MSpin$ and investigate its behavior in KO-homology. Using the ABP-splitting we determine the 2-adic functions associated to the class b and to generators of the KO-homology ring of $MSpin$. The established formulae will play an important role in the proof of the multiplicative structure of $MSpin$.

3.1. The 2-adic functions associated to the class b. As a start we remind ourselves of the KO-Pontryagin classes π^j. They are defined by setting

$$\pi^0(L) = 1,\ \pi^1(L) = L - 2,\ \pi^j(L) = 0 \text{ for } j \geq 2$$

for complex line bundles L and then requiring naturality and

$$\pi_s(\xi + \eta) = \pi_s(\xi)\pi_s(\eta)$$

where $\pi_s = \sum_j \pi^j s^j$. Here ξ and η are oriented bundles. In fact, these properties determine π_s because the group $KO(BSO(m))$ injects into $K(B\mathbb{T}^{[\frac{m}{2}]})$ under the complexification of the map which is induced by the restriction to the maximal torus $\mathbb{T}^{[\frac{m}{2}]}$ (compare [**ABP66**]).

It is possible to express $\pi^j(\xi)$ in the exterior powers of ξ. Explicitly, the equation

$$\pi_s(\xi) = \sum_{i=0}^{\infty} \Lambda^i(\xi - \dim\xi)t^i = (1+t)^{-dim\,\xi} \sum_{i=0}^{\infty} (\Lambda^i\xi)t^i$$

is easily verified for the new generator t of $KO(X)[\![s]\!]$ which is given by the equation $s = t/(1+t)^2$.

We write $\pi^j \in KO(BSpin)$ for the jth KO-characteristic class of the universal stable spin bundle. Without changing the notation we also consider the same class as an object of $KO(MSpin)$ via the Thom isomorphism. For all non ordered sequences of positive numbers (partitions) $J = (j_1, \dots, j_n)$ we set

$$\pi^J = \pi^{j_1} \cdots \pi^{j_n} : MSpin \longrightarrow KO.$$

In these terms the ABP-splitting says

THEOREM 3.1. *(compare [**ABP66**]) There is a countable set I of cohomology classes $x_i \in H^*(MSpin, \mathbb{Z}/2)$ such that the map*

$$(\pi^J, x_i) : MSpin \longrightarrow \bigvee_{\substack{n(J)even \\ 1\notin J}} ko\,\langle 4n(J)\rangle \vee \bigvee_{\substack{n(J)odd \\ 1\notin J}} ko\,\langle 4n(J)+2\rangle \vee \bigvee_{i\in I} \Sigma^{|x_i|} H\mathbb{Z}/2$$

is a 2-local homotopy equivalence.

COROLLARY 3.2. *The map $(\pi^J) : MSpin \xrightarrow{\cong} \bigvee_{1\notin J} KO$ is a $K(1)$-equivalence.*

PROOF. This is an immediate consequence of the the unlocalized ABP-splitting, the vanishing of the group $K(1)_*H\mathbb{Z}/2$ and the fact that the cover $ko\,\langle n\rangle \longrightarrow KO$ is a $K(1)$-equivalence for all $n \geq 0$ (see [**HH92**].) □

COROLLARY 3.3. *ζ is null in $\pi_{-1}MSpin$.*

PROOF. 2-locally the unlocalized ABP-splitting gives a map from ko to $MSpin$ which induces an isomorphism in π_0. Hence it suffices to show that ζ vanishes in ko. The latter coincides with the periodic KO in the $K(1)$-local category. In KO the class ζ vanishes by its definition. □

In the last subsection we have chosen a null homotopy ι of ζ in KO. Hence the corollary supplies us with an E_∞ map

$$\varphi : T_\zeta \longrightarrow MSpin$$

which will be the object of study for the rest of this subsection. We are interested in the image of the class b of 2.17 under the induced map

$$\pi_* KO \wedge T_\zeta \xrightarrow{\varphi_*} \pi_* KO \wedge MSpin \cong \bigoplus_{1 \notin J} \pi_* KO \wedge KO.$$

To describe each component of its image we have the

Proposition 3.4. [**Hop98**] *Let Φ be the map*

$$\pi_* K \wedge K \longrightarrow \mathcal{T}(\mathbb{Z}_2^\times, \pi_* K)$$

which associates to a class $f \in \pi_k K \wedge K$ the continuous 2-adic function

$$\lambda \mapsto (f(\lambda) : S^k \xrightarrow{f} K \wedge K \xrightarrow{1\wedge\psi^\lambda} K \wedge K \xrightarrow{\mu} K).$$

Then Φ is an isomorphism. Analogously, there are isomorphisms

$$\begin{array}{rcl} \pi_* K \wedge KO & \xrightarrow{\cong} & \mathcal{T}(\mathbb{Z}_2^\times / \pm 1, \pi_* K) \\ \pi_* KO \wedge KO & \xrightarrow{\cong} & \mathcal{T}(\mathbb{Z}_2^\times / \pm 1, \pi_* KO). \end{array}$$

which are denoted by the same letter in the sequel.

Proof. The first statement was proved in [**Rav84**]7.10. For the second one observe that the fibration [**And64**] $\Sigma KO \xrightarrow{\eta} KO \longrightarrow K$ induces a short exact sequence in complex K-theory. Moreover, one readily verifies by rationalizing that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_* K \wedge KO & \longrightarrow & \pi_* K \wedge K & \longrightarrow & \pi_{*-2} K \wedge KO & \longrightarrow & 0 \\ & & \downarrow \Phi & & \cong \downarrow \Phi & & \downarrow \Phi & & \\ 0 & \longrightarrow & \mathcal{T}(\mathbb{Z}_2^\times / \pm 1, \pi_* K) & \xrightarrow{\alpha} & \mathcal{T}(\mathbb{Z}_2^\times, \pi_* K) & \xrightarrow{\beta} & \mathcal{T}(\mathbb{Z}_2^\times / \pm 1, \pi_{*-2} K) & \longrightarrow & 0 \end{array}$$

commutes if α is the obvious inclusion and β sends a function f to the even function

$$\lambda \mapsto \lambda^{-1}(f(\lambda) - f(-\lambda)).$$

Since the bottom row is exact as well the result follows. For the last statement recall from [**AHS71**] that the action map

$$\pi_* KO \otimes \pi_0 KO \wedge KO \longrightarrow \pi_* KO \wedge KO$$

is an isomorphism. In particular, the result holds for $* = 3, 5, 6, 7 \bmod 8$. In the remaining dimensions an inspection of the cone over η fibration in KO homology and the 5 lemma finish the proof. $\square$

Remark 3.5. The map Φ even becomes an isomorphism of θ-algebras if we set $\psi = id$ for the ring of continuous functions as we did in 2.7(i) .

We are now able to prove the first theorem.

Proof of Theorem A: The commutative diagram

$$\begin{array}{ccccccc} & & C_\zeta & \xrightarrow{\iota} & KO & & \\ & \swarrow & \downarrow & & \downarrow & \searrow^{1} & \\ TC_\zeta & \longrightarrow & T_\zeta & \xrightarrow{\varphi} & MSpin & \xrightarrow{\pi^\emptyset} & KO \end{array}$$

tells us that all $\pi^J\varphi_* b$ must vanish except for $J \neq \emptyset$. For $J = \emptyset$ we compute with $a = \iota'_*(1)$ with the notation of 2.4

$$\begin{aligned}\Phi(\pi^{\emptyset}_*\varphi_* b)(3^n) &= \Phi((1\wedge\iota)a)(3^n) = \mu(1\wedge\psi^{3^n})(1\wedge\iota)a \\ &= \left\langle \psi^{3^n}\iota, a\right\rangle = \langle \iota + n(1\,\delta), a\rangle = n\end{aligned}$$

□

Since φ induces a map of θ-algebras the last result completely determines its behaviour in KO-homology. Unfortunately, the ABP-splitting does not tell us anything about the θ-algebra structure of the spin bordism. Hence other methods are required in things to come.

3.2. The KO-homology ring of $BSpin$. We now provide polynomial generators of the real and complex K-homology of $MSpin$. We first need the Thom isomorphism for homology.

Lemma 3.6. *(compare* [**MR81**] *et al.) Let E be one of the theories $ko, k, KO, K, K\mathbb{Z}/2^r$. Then there is the Thom isomorphism*

$$\tau_* : \pi_* E \wedge MSpin \xrightarrow{\cong} \pi_* E \wedge BSpin_+.$$

Hence, it suffices to look at the classifying space $BSpin$. A further simplification is given by the following result of Snaith.

Lemma 3.7. *(compare* [**HS75**]*8.11) The canonical map from $BSpin$ to BSO is a $K(1)$-equivalence.*

Let L be the canonical line bundle over $\mathbb{C}P^\infty$. Equip K-theory with the complex orientation in the way that the Euler class of a line bundle L is given by $x = v^{-1}(1 - L^*)$ As usual, define additive generators $\beta_i \in \pi_{2i} K \wedge \mathbb{C}P^\infty_+$ as the dual of the classes x^i.

Lemma 3.8. *Let $f : S^1 \longrightarrow S^1$ be the map which sends a complex number z to its square. Then $Bf : BS^1 \longrightarrow BS^1$ has the following impact on the generators:*

$$Bf_*(\beta_k) = \sum_j (-1)^{k-j}\binom{j}{2j-k} 2^{2j-k}\beta_j.$$

Here, we omitted the Bott class from the notation.

Proof.

$$\left\langle Bf_*\beta_k, x^j\right\rangle = \left\langle \beta_k, Bf^* x^j\right\rangle = \left\langle \beta_k, [2](x)^j\right\rangle = (-1)^{k-j}\binom{j}{2j-k} 2^{2j-k}$$

□

Proposition 3.9. *Let S^1 be the maximal torus of $Spin(2)$ and let u_j be the image of β_j in $\pi_0 K \wedge BSpin$. Then*

$$\pi_* K \wedge BSpin_+ \cong \pi_* K[u_4, u_8, u_{12}, \ldots].$$

Moreover, let $b_k \in \pi_0 K \wedge BSO_+ \cong \pi_0 K \wedge BSpin_+$ be the image of the class $\beta_k \in \pi_0 K \wedge BSO(2)_+$. Then we have

$$u_k = \sum_j (-1)^{k-j}\binom{j}{2j-k} 2^{2j-k} b_j$$

Hence, $\pi_* K \wedge BSpin_+$ is a polynomial algebra in $b_2, b_4, b_6 \ldots$. Finally, since

$$\pi_0 KO \wedge BSpin_+ \cong \pi_0 K \wedge BSpin_+$$

the same classes also freely generate $\pi_* KO \wedge BSpin_+$.

Before proving 3.9 we need some preparation. A 2-adic number λ can be written in its 2-adic expansion

$$\lambda = \sum_k \alpha_k(\lambda)\, 2^k.$$

This way, each α_k becomes a continuous function with values in $\{0,1\} \subset \mathbb{Z}_2$. It turns out

LEMMA 3.10. *(compare* [**Hop98**]*) The map*

$$\mathbb{Z}_2[\alpha_0, \alpha_1, \ldots]/(\alpha_k^2 - \alpha_k) \longrightarrow \mathcal{T}(\mathbb{Z}_2, \mathbb{Z}_2)$$

is an isomorphism of rings.

PROOF. Since [**Hop98**] is not published yet we repeat the proof here: it suffices to show that for all m, n the map

$$\mathbb{Z}/2^n[\alpha_0, \alpha_1, \ldots, \alpha_{m-1}]/(\alpha_k^2 - \alpha_k) \longrightarrow \mathcal{T}(\mathbb{Z}/2^m, \mathbb{Z}/2^n)$$

is an isomorphism. For finite, discrete sets S and T the natural map

$$\mathcal{T}(S, \mathbb{Z}/2^n) \otimes \mathcal{T}(T, \mathbb{Z}/2^n) \longrightarrow \mathcal{T}(S \times T, \mathbb{Z}/2^n)$$

is an isomorphism. Hence the claim follows from the isomorphism of sets

$$(\alpha_i)_{i<m} : \mathbb{Z}/2^m \longrightarrow \prod_{i<m} \mathbb{Z}/2.$$

□

Hence one readily verifies with 3.2 and 3.4 the

LEMMA 3.11. (i) *$\pi_* K \wedge MSpin \cong \lim_r \pi_* K\mathbb{Z}/2^r \wedge MSpin$.*

(ii) *The Bockstein sequences*

$$\pi_* K\mathbb{Z}/2^i \wedge MSpin \rightarrowtail \pi_* K\mathbb{Z}/2^{i+1} \wedge MSpin \twoheadrightarrow \pi_* K\mathbb{Z}/2^i \wedge MSpin\ .$$

are short exact for all $i \geq 1$.

PROOF OF 3.9. By the first part of the lemma it suffices to check the corresponding result mod 2^i for all $i > 0$. Using the second part of the lemma and the 5-lemma we only need to show the statement for mod 2 K-theory. This is a result of Snaith [**HS75**]8.5.

The second statement follows from 3.8 and the fact that the diagram

$$\begin{array}{ccc} Spin(2) \supset S^1 & \xrightarrow{z^2} & S^1 \cong SO(2) \\ \downarrow & & \downarrow \\ Spin & \longrightarrow & SO \end{array}$$

commutes. □

3.3. Stable cannibalistic classes in KO. In order to express the class $\varphi_* b$ in terms of generators of $\pi_0 KO \wedge MSpin$ we are going to compute the 2-adic functions associated to the ABP-image of each generator explicitly. The calculation involves the cannibalistic classes $\theta^k(\xi) \in K(X)$ which we are recalling from [**Bot69**]: these classes are defined for complex vector bundles ξ over compact spaces X and are characterized by the following two properties

(i) $\theta^k(L) = 1 + L^* + \cdots + (L^*)^{k-1}$ for all line bundles L

(ii) $\theta^k(\xi \times \xi') = \theta^k(\xi)\, \theta^k(\xi')$ for all complex bundles ξ, ξ'.

In particular, we have the equality

$$\theta^k(\xi + n) = k^n \theta^k(\xi).$$

Assume in the following that k is an odd number. Then we can turn each θ^k into a stable operation by setting

$$\hat{\theta}^k(\xi) \stackrel{def}{=} \frac{\theta^k(\xi)}{k^{dim_{\mathbb{C}}\xi}} \in K(X)$$

The formulae for line bundles and sums of vector bundles stay the same. The complex classes θ^k do have a real counterpart for spin bundles ξ which we also denote by $\theta^k(\xi) \in KO(X)$. If the underlying spin bundle admits a reduction to the special unitary group then the complexification of these real classes coincide with the complex classes [**Bot69**]p.87f.

In the following we write $\hat{\theta}^k \in \widetilde{KO}(BSpin)$ for the universal cannibalistic classes. We will see in a moment that they come up in the ABP-splitting map.

LEMMA 3.12. *The diagram*

$$\begin{array}{ccc} \pi_0 KO \wedge MSpin & \xrightarrow{\Xi} & \mathrm{Hom}_{cts}(KO(MSpin), \mathbb{Z}_2) \\ {\scriptstyle (1\wedge\pi^J)_*}\downarrow & & \downarrow{\scriptstyle \pi^{J*}} \\ \pi_0 KO \wedge KO & \xrightarrow{\Phi} & \mathcal{T}(\mathbb{Z}_2^\times/\pm 1, \mathbb{Z}_2) \end{array}$$

commutes. Here, the upper horizontal arrow is the duality map. The right vertical arrow takes a homomorphism $\alpha : KO(MSpin) \longrightarrow \mathbb{Z}_2$ to the map

$$\lambda \mapsto \alpha(MSpin \xrightarrow{\pi^J} KO \xrightarrow{\psi^\lambda} KO).$$

The lemma is easily checked. We now consider the J-component of the ABP-splitting map after composing it with the Thom isomorphism

$$\Theta_J \stackrel{def}{=} \pi^{J^*}(\tau^*)^{-1}\Xi = \Phi(1 \wedge \pi^J)_* \tau_*^{-1} : \pi_0 KO \wedge BSpin_+ \longrightarrow \mathcal{T}(\mathbb{Z}_2^\times/\pm 1, \mathbb{Z}_2).$$

PROPOSITION 3.13. *For all $a \in \pi_0 KO \wedge BSpin_+$ we have*

$$\Theta_J(a)(k) = \left\langle a, \hat{\theta}^k \psi^k(\pi^J) \right\rangle.$$

PROOF. Let ξ_{8n} be the universal spin bundle over $BSpin(8n)$. Then we have the relation

$$\psi^k(z_n) = \theta^k(\xi_{8n}) z_n$$

between the Adams operations and the cannibalistic classes (compare [**Bot69**]p.89.) Here, $z_n \in \widetilde{KO}(MSpin(8n))$ is the Thom class. Let $\beta \in \pi_8 KO$ be the Bott class and for $a \in \pi_0 BSpin(8n)$ set $g = {\tau^*}^{-1}\Xi(a)$. Then compute

$$\begin{aligned} \Theta_J(a)(k) &= \pi^{J^*}(g)(k) = g(\psi^k(\beta^{-n} z_n \pi^J(\xi_{8n}))) \\ &= g(k^{-4n}\beta^{-n}\theta^k(\xi_{8n}) z_n \psi^k(\pi^J(\xi_{8n}))) \\ &= \tau^* g(\hat{\theta}^k(\xi_{8n})\psi^k(\pi^J(\xi_{8n})) = \left\langle a, \hat{\theta}^k(\xi_{8n})\psi^k(\pi^J(\xi_{8n}))\right\rangle. \end{aligned}$$

Thus the claim follows after stabilization. □

LEMMA 3.14. *Let L be the canonical line bundle over $\mathbb{C}P^\infty$. Then the complexified real cannibalistic classes satisfy with $x = 1 - L^*$*

$$k\,\hat{\theta}^k(L^2 - 1) \otimes \mathbb{C} = \frac{(1-x)^k - (1-x)^{-k}}{(1-x) - (1-x)^{-1}} \in K(\mathbb{C}P^\infty).$$

PROOF. We decompose the spin bundle $1-L^2$ into a sum of bundles which admit reductions to the special unitary group by writing

$$(1-L^2) = (L-\bar{L}) + 1 - L^2 = (1-L)(1-\bar{L}) - (1-L)^2$$

To determine the real cannibalistic classes of the latter we compute the complex cannibalistic classes in $K(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong \mathbb{Z}_2[\![x,y]\!]$

$$\hat{\theta}^k((1-L_1)(1-L_2)) = \frac{(1-L_1^*)(1-L_2^*)(1-(L_1^*L_2^*)^k)}{(1-L_1^{*k})(1-L_2^{*k})(1-L_1^*L_2^*)} = \frac{q_k(x +_{\hat{G}_m} y)}{q_k(x)q_k(y)}$$

Here, $q_k(x)$ is the polynomial

$$q_k(x) = \frac{1-(1-x)^k}{x} = (1-x) + (1-x)^2 + \cdots + (1-x)^{k-1}$$

Thus we obtain

$$\hat{\theta}^k(1-L^2)\otimes\mathbb{C} = \frac{k}{q_k(x)q_k(-_{\hat{G}_m}x)}\left(\frac{q_k([2](x))}{q_k(x)}\right)^{-1} = k\frac{q_x(x)}{q_k(-_{\hat{G}_m}x)q_k([2](x))}.$$

An elementary calculation finishes the proof. □

We are now well prepared to compute the 2-adic functions which correspond to the generators u_n defined in the last subsection.

PROOF OF THEOREM B: The proposition gives

$$\Theta_J(u_n)(k) = \left\langle u_n, \hat{\theta}^k\psi^k(\pi^J)\otimes\mathbb{C}\right\rangle = \left\langle \beta_n, \hat{\theta}^k(L^2-1)\psi^k(\pi^J(L^2-1))\otimes\mathbb{C}\right\rangle.$$

Hence, Θ_J vanishes for all non empty J which do not contain 1. Moreover, for $J=\emptyset$ we get with the lemma

$$\sum_n \Theta(u_n)(k)x^n = \sum_n \left\langle \beta_n, \hat{\theta}^k(L^2-1)\otimes\mathbb{C}\right\rangle x^n = k^{-1}\frac{(1-x)^k-(1-x)^{-k}}{(1-x)-(1-x)^{-1}}.$$

□

REMARK 3.15. Similarly, we can determine the 2-adic functions which correspond to other generators. For instance, consider the map

$$f : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow BSU \longrightarrow BSpin$$

which classifies the product $(1-L_1)(1-L_2)$. Then for the generators (compare [**Lau02**]) $a_{ij} = f_*(\beta_i\otimes\beta_j)$ in $\pi_{2(i+j)}K\wedge BSpin_+$ we have

$$\sum_{i,j}\Theta(a_{ij})(k)x^iy^j = \frac{q_k(x +_{\hat{G}_m} y)}{q_k(x)q_k(y)}.$$

The computations of the KO-homology of the ABP-map enable us to compare the generators to the class φ_*b. At this point, the reader can easily verify that φ_*b corresponds to u_4 and to $a_{1,2}$ modulo 2. We will not go through the calculation here since we will work out a closer relationship later.

4. The θ-algebra structures of bordism theories

In this section we determine the θ-algebra structure of the unitary, special unitary and spin bordism theories. This problem was partially answered by Snaith in [**HS75**] who used group theoretical methods to compute the action of θ modulo 2 for the classifying spaces. However, in order to give a complete description of the θ-algebra structure we need integral information: each time when applying θ we loose a power of 2. That is, the image of θ on a mod 2^n class is only well defined modulo 2^{n-1}. Hence, in order to show that a class generates a free summand we must know θ integrally. Unfortunately, Snaith's (and Priddy's [**Pri75**]) method does not generalize that easily to the integral situation. So we use a completely different approach here.

4.1. The θ-algebra structure of $\pi_0 K \wedge MU$. We investigate the H_∞ structure of the function spectrum $(K \wedge MU)^{BS^1_+}$ by computing the value of $\tilde{x}_{MU} = v\, x_{MU}$ under the operation

$$P : (K \wedge MU)^0(BS^1) \longrightarrow (K \wedge MU)^0(B\Sigma_2 \times BS^1).$$

described in 2.3. Here, $x_{MU} \in MU^2(BS^1)$ denotes the MU-Euler class and $v \in \pi_2 K$ is the Bott class. In this subsection it is important not to suppress the Bott class from the notation.

Our strategy is to calculate the operation on each factor of the product $\tilde{x}_{MU}$ separately. This means the following: the complex bordism theory and the K-theory admit H^2_∞-structures or, equivalently, H_∞ structures on the wedges $\bigvee_i \Sigma^{2i} MU$ and $\bigvee_i \Sigma^{2i} K$ respectively. Hence so do the corresponding function spectra. The associated operation P is the tom Dieck-Steenrod operation

$$P : MU^n BS^1 \longrightarrow MU^{2n} B\Sigma_2 \times BS^1.$$

for complex bordism and the Atiyah power operation for K-theory (see [**BMMS86**] p.272ff.) Hence, we know how to compute the operation on each factor and only have to relate their product to the class $P\tilde{x}_{MU}$. To state the result, we use the standard notation and write DX for the gadget $E\Sigma_2 \wedge_{\Sigma_2} X^2$. We also write

$$\delta : D(X \wedge Y) \longrightarrow DX \wedge DY$$

for the diagonal map. Then we have the following "change of suspension" formula:

Lemma 4.1. *Suppose E is a H^d_∞ ring spectrum and F is a H_∞ ring spectrum. Then for all based spaces X, $\alpha : X \longrightarrow \Sigma^d E$ and $\beta : \Sigma^d X \longrightarrow F$ we have*

$$y_E\, P(\alpha\, (\Sigma^{-d}\beta)) = P(\alpha)\, \Sigma^{-d} P(\beta) : X \wedge B\Sigma_{2+} \longrightarrow \Sigma^d E \wedge F$$

with $y_E = P(\Sigma^d 1) \in E^d B\Sigma_{2+}$.

Proof. It is easy to check that the diagram

$$\begin{array}{ccccc} D\Sigma^d X & \xleftarrow{\Delta_X} & B\Sigma_{2+} \wedge \Sigma^d X & \xrightarrow{\Sigma^d \Delta} & B\Sigma_{2+} \wedge X \wedge B\Sigma_{2+} \wedge \Sigma^d X \\ \downarrow{\scriptstyle D\Sigma^d \Delta_X} & & & & \downarrow{\scriptstyle \Delta\wedge\Delta} \\ D(X \wedge \Sigma^d X) & & \xrightarrow{\qquad\delta\qquad} & & DX \wedge D\Sigma^d X \end{array}$$

commutes. Moreover, when we write $\mathcal{P}$ for the external Steenrod operation and give $E \wedge F$ the H^d_∞ ring structure which is induced from the isomorphism

$$\textstyle\bigvee_i \Sigma^{d\,i}(E \wedge F) \cong (\bigvee_i \Sigma^{d\,i} E) \wedge F.$$

then we get

$$\begin{aligned} y_E\, P(\alpha\, \Sigma^{-d}\beta) &= P(\Sigma^d(\alpha\, \Sigma^{-d}\beta)) = P(\Sigma^d \Delta_X^*(\alpha \wedge \Sigma^{-d}\beta)) \\ &= \Delta^* \mathcal{P}((\Sigma^d \Delta_X)^*(\alpha \wedge \beta)) = \Delta^*(D\Sigma^d \Delta_X)^* \mathcal{P}(\alpha \wedge \beta) \\ &= \Delta^*(D\Sigma^d \Delta_X)^* \delta^*(\mathcal{P}(\alpha) \wedge \mathcal{P}(\beta)) \\ &= (\Sigma^d \Delta_{B\Sigma_{2+} \wedge X})^* (\Delta \wedge \Delta)^* (\mathcal{P}(\alpha) \wedge \mathcal{P}(\beta)) = P(\alpha)\, \Sigma^{-d} P(\beta) \end{aligned}$$

Here, we used [**BMMS86**] p.250f. □

Lemma 4.2. *Let E be a H^2_∞-ring spectrum which is complex oriented by a H^2_∞-map $f : MU \longrightarrow E$. Then $P(\Sigma^2 1)$ is the Euler class y_E of the sign representation.*

Proof. This immediately follows from [**BMMS86**]p.257. □

We next show how the operation works in complex bordism.

LEMMA 4.3. *For all $n > r > 0$ we have the formula*

$$y_{MU}^{n-1}P(x_{MU}) = y_{MU}^{n-1}x_{MU}(x_{MU} +_{\hat{G}_u} y_{MU}) \quad \in MU^{2n+2}(B\Sigma_2 \times \mathbb{C}P^r).$$

Here, $\hat{G}_u$ is the universal formal group law.

PROOF. For the sake of simplicity we omit the index MU from the notation. Choose s arbitrary and recall from [**Qui71**]3.17) the formula

$$y^{n-1}P(x) = \sum_{l(\alpha)\le n} y^{n-l(\alpha)}a(y)^\alpha s_\alpha(x) \quad \in MU^{2n+2}(\mathbb{R}P^s \times \mathbb{C}P^r)$$

which relates the tom Dieck-Steenrod operation to the Landweber-Novikov operations s_α. Here, α is a sequence of non negative integers and $l(\alpha) = \sum \alpha_i$. The power series $a^\alpha = a_1^{\alpha_1}a_2^{\alpha_2}\cdots$ are defined by the equation

$$x +_{\hat{G}_u} y = x + \sum_{j\ge 1} a_j(x)\, y^j.$$

Let $i : \mathbb{C}P^{r-1} \hookrightarrow \mathbb{C}P^r$ be the inclusion and c_t be the total Conner-Floyd Chern class. Then we have for the Euler class $z = e(L^*) = [-1](x)$

$$s_t(z) = s_t(i_!(1)) = i_!c_t(L^*) = i_!(\sum_{j\ge 1} t_j z^j) = \sum_{j\ge 1} t_j z^{j+1}.$$

Since the Landweber-Novikov operations are natural the same formula holds for x instead of z. Hence we get

$$y^{n-1}\, P(x) = y^n\, x + y^{n-1}\sum_{j\ge 1} a_j(y)x^{j+1} = y^{n-1}x\,(x +_{\hat{G}_u} y).$$

The claim follows by passing to the limit

$$MU^*(\mathbb{C}P^r \times B\Sigma_2) = \lim_s MU^*(\mathbb{C}P^r \times \mathbb{R}P^s).$$

□

In the following let $g(x)$ be the invertible power series

$$g(x) = \sum_{i=0}^{\infty} b_i x^i; \quad b_0 = 1$$

with coefficients in $\pi_0 K\wedge MU \cong \mathbb{Z}_2[b_1, b_2, \ldots]$. It is known (compare [**Ada74**]p.60ff) that the two Euler classes $\tilde{x}_{MU}$ and $x = \tilde{x}_K$ are related in $(K \wedge MU)^0(BS^1) \cong \pi_0 K \wedge MU[\![x]\!]$ by the formula

$$\tilde{x}_{MU} = x\, g(x).$$

A similar relation holds for $\tilde{y}_{MU}$ and $y = \tilde{y}_K$ in $(K \wedge MU)^0 B\Sigma_2$. Observe that in the notation of 2.4, the class y corresponds to $1-\sigma$ under the isomorphism

$$K^0(B\Sigma_2) \cong R\Sigma_2^\wedge \cong (\mathbb{Z}_2[\sigma]/\sigma^2 - 1)^\wedge \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2(1-\sigma).$$

LEMMA 4.4. *The power sum $\sigma_2 = \sum t_i^2 \in \pi_0 K \wedge B\Sigma_{2+}$ satisfies*

$$\langle \sigma_2, y^n\rangle = 2^n.$$

PROOF. Since $y^{n+1} = 2^n y$ the formula follows from

$$\langle \sigma_2, y\rangle = \langle e_1^2 - e_2, 1-\sigma\rangle - \langle e_2, 1-\sigma\rangle = 2$$

Here we used the explicit description of $\Delta_{k,2}$ given in 2.4. □

Proof of Theorem C: We are going to show that

$$\sum_{i\geq 0}\psi(b_i)x^i(2-x)^i=\psi(g(x))=\frac{g(x)g(2-x)}{g(2)}.$$

The first equation is clear since for K-theory the operation ψ coincides with the second Adams operation and

$$\psi^2x=[2]_{\hat{G}_m}(x)=2x-x^2.$$

To do the second consider the curve $b(x)=x\,g(v\,x)$ and regard $K\wedge MU$ as a H^2_∞-ring spectrum via the equivalence $K\wedge\bigvee_i\Sigma^{2i}MU\cong\bigvee_i\Sigma^{2i}K\wedge MU$. Then we have the formula

Sublemma 4.5. $\psi(x_{MU})=b(x_K)\,b(2v^{-1}-x_K)$.

Proof. First note that $b(x_K)$ is the MU-Euler class. Moreover, the K-Hurewicz map $\pi_*MU\longrightarrow\pi_*K\wedge MU$ classifies the formal group law

$${}^b\hat{G}_m(x,y)=b(\hat{G}_m(b^{-1}(x),b^{-1}(y))).$$

Hence when pairing the equality of 4.3 with σ_2 we get with 4.4

$$\begin{aligned} b(2v^{-1})^{n-1}\,\psi(x_{MU}) &= \left\langle y_{MU}^{n-1}\,P(x_{MU}),\sigma_2\right\rangle=\left\langle y_{MU}^{n-1}x_{MU}(x_{MU}+_{{}^b\hat{G}_m}y_{MU}),\sigma_2\right\rangle\\ &= b(2v^{-1})^{n-1}x_{MU}(x_{MU}+_{{}^b\hat{G}_m}b(2v^{-1}))=b(2v^{-1})^{n-1}b(x_K)\,b(2v^{-1}-x_K).\end{aligned}$$

Since the coefficient ring $\pi_*K\wedge MU$ is an integral domain and the module $(K\wedge MU)^*(B\Sigma_2\wedge\mathbb{C}P^r)$ is free we may cancel the term $b(2v^{-1})^{n-1}$ on both sides. The claim follows by passing to the limit. □

Our change of suspension formula 4.1 reads for $X=BS^1_+$, $\alpha=x_{MU}$ and $\beta=\Sigma^2v$

$$y_{MU}P(v\,x_{MU})=P(x_{MU})\,P(\Sigma^2v)=P(x_{MU})\,y_KP(v)=P(x_{MU})\,y_K\,v^2$$

In the last two equations we used [**BMMS86**] p.274f. Hence we obtain with the sublemma

$$\begin{aligned}\psi(x)\psi(g(x)) &= \psi(\tilde{x}_{MU})=\psi(x_{MU})\left\langle\frac{y_K}{y_{MU}}v^2,\sigma_2\right\rangle\\ &= b(x_K)b(2v^{-1}-x_K)\frac{2v^{-1}}{b(2v^{-1})}v^2=\psi(x)\frac{g(x)g(2-x)}{g(2)}.\end{aligned}$$

The result follows by canceling $\psi(x)$ on both sides. □

Corollary 4.6. *In $\pi_0K\wedge MU$ we have the formula mod 2*

$$\theta(b_r)=(1+b_1)b_r^2+\sum_{i=0}^{r}b_i(b_{2r-i}+b_{2r-i+1})$$

In particular, for $r>0$ this gives modulo 2 and decomposables

$$\theta(b_r)=b_{2r}+b_{2r+1}$$

Proof. Since $\pi_0K\wedge MU$ is torsion free and $2\theta(x)=\psi(x)-x^2$ it is enough to compute the action of ψ on the generators. Let D_k be the operator $(d^k/k!\,dx^k)_{|x=0}$. Then we get mod 4

$$(-1)^r\psi(b_r)=D_{2r}\psi(g(x))=(1+2b_1)(\sum_{i+j=2r}(-1)^jb_ib_j+2\sum_{i+j-1=2r}j\,b_ib_j)$$

The result follows after some elementary transformations which are left to the reader. □

REMARK 4.7. Since the Thom isomorphism is compatible with the action of θ (see [**HS75**]) we also determined its action on $\pi_0 K \wedge BU_+$. Hence the last formula is equivalent modulo 2 and decomposables to the one of Snaith in [**HS75**] 6.3.6.

4.2. The θ-algebra structure of $\pi_0 K \wedge MSU$. We next turn to the special unitary bordism theory. The result will not be needed for the proof of the splitting theorem. Once more let

$$f : \mathbb{C}P^\infty_+ \wedge \mathbb{C}P^\infty_+ \longrightarrow BSU_+ \longrightarrow K \wedge BSU_+$$

be the map which classifies $(1 - L_1)(1 - L_2)$ and

$$f(x, y) = \sum_{i,j} a_{ij} x^i y^j$$

be the associated power series.

THEOREM 4.8. *The θ-algebra structure of $\pi_0(K \wedge MSU_+)$ is determined by the equations*

$$\sum_{i,j} \psi(a_{ij})(x(2-x))^i (y(2-y))^j = \psi f(x, y) = \frac{f(x, y)\, f(2-x, y)}{f(2, y)}.$$

PROOF. The first equation is clear. To see the second, observe that the decomposition $(1 - L_1)(1 - L_2) = (L_1L_2 - 1) + (1 - L_1) + (1 - L_2)$ implies

$$\iota_*\, f(x, y) = \frac{g(x)\, g(y)}{g(x +_{\hat{G}_m} y)}.$$

Here the map $g : \mathbb{C}P^\infty_+ \longrightarrow K \wedge BU_+$ corresponds to $1 - L$ and $\iota : BSU \longrightarrow BU$ is the inclusion. Since ι_* is an injection it may as well be omitted from the notation. Using the naturality of ψ we compute with Theorem C

$$\psi(g(x +_{\hat{G}_m} y)) = \mu^* \psi(g(x)) = \frac{g(x +_{\hat{G}_m} y)\, g(2 - (x +_{\hat{G}_m} y))}{g(2)}$$

and hence

$$\psi f(x, y) = \frac{\psi g(x)\, \psi g(y)}{\psi g(x +_{\hat{G}_m} y)} = \frac{g(x) g(2-x) g(y) g(2-y)}{g(2) g(x +_{\hat{G}_m} y) g(2 - (x +_{\hat{G}_m} y))} = \frac{f(x, y)\, f(2-x, y)}{f(2, y)}.$$

Here we used the identities

$$2 - (x +_{\hat{G}_m} y)) = (2 - x) +_{\hat{G}_m} y; \qquad 2 +_{\hat{G}_m} y = 2 - y$$

which are easily checked. □

COROLLARY 4.9. *In $\pi_0 K \wedge MSU$ we have modulo 2 and decomposables of lower index*

$$\theta(a_{ij}) = a_{2i,2j} + a_{2i+1,2j}.$$

PROOF. In view of the theorem it is clear how to proceed. □

It is not hard to give an explicit formula for the action of θ on the nose but we will not go through the tedious calculations here.

4.3. The θ-algebra structure of $\pi_0 KO \wedge MSpin$. The θ-algebra structure of $\pi_0 KO \wedge MSpin$ is determined by 4.8 and the surjective map of θ-algebras

$$\pi_0 K \wedge MSU \longrightarrow \pi_0 K \wedge MSpin \cong \pi_0 KO \wedge MSpin.$$

Alternatively, we may use the Thom isomorphism and look at the surjective θ-algebra map

$$\pi_0 K \wedge BU_+ \longrightarrow \pi_0 K \wedge BSO_+ \cong \pi_0 K \wedge BSpin_+ \cong \pi_0 K \wedge MSpin.$$

In each case the analysis of the spin θ-algebra structure is ultimately based on the θ-algebra $\pi_0 K \wedge MU$. In Theorem C we found an implicit formula for the action of θ on the free generators b_i. To identify free θ-algebra summands we also need to know the action of the powers θ^j which we do next.

Definition 4.10. *For a monomial $m = b_{i_1} \cdots b_{i_k}$ in $\pi_0 K \wedge MU$ we define its length $l(m)$ to be the maximum of the set of indices $\{i_1, \ldots, i_k\}$. For a general element x of the form $\sum_{s\in S} m_s 2^{i_s}$ with $m_s \neq m_{s'}$ whenever $i_s = i_{s'}$ and $s \neq s'$ we define lengths*

$$\begin{aligned} l_1(x) &= sup\{l(m_s) - i_s |\ s \in S\} \\ l_2(x) &= sup\{l(m_s)2^{-i_s} |\ s \in S\}. \end{aligned}$$

Lemma 4.11. *The two lengths l_k , $k = 1, 2$ have the following properties*

(i) $max\{l_s(a\, b), l_s(a + b)\} \leq \ max\{l_s(a), l_s(b)\}$.
(ii) $l_2(a) \leq l_1(a)$ *if* $l_1(a) > 0$

Proof. The easy proof is left to the reader. □

Next we consider the action of θ and ψ on the generators. In 4.6 we have seen that the l_1-length of $\theta(b_i)$ (and $\psi(b_i)$) is at least $2i + 1$ (and $2i$ respectively.) In fact, the following result shows the equality.

Lemma 4.12. *For each i we have*

(i) $l_1(\psi(b_i)) = 2i$
(ii) $l_1(\theta(b_i)) = 2i + 1$
(iii) $l_2(\theta(x)) < 2i$ *if* $l_2(x) < i$.

Proof. Since $l(\psi(b_0)) = l(1) = 0$ we may assume that the equality is true for all numbers lower than i. Then we have Theorem C (or, more convenient, with its equivalent statement on page 24)

$$l_1(2^i\psi(b_i)) = l_1(g(2)\sum_{k+l=i}\binom{l}{k}2^{l-k}\psi(b_l)) = l_1(\sum_{k+l=i,j}\binom{j}{k}2^{j-k}b_lb_j) = i$$

The second estimation follows from the first

$$l_1(2\theta(b_i)) = l_1(b_i^2 + 2\theta(b_i)) = l_1(\psi(b_i)) = 2i.$$

To show the last statement let $x = \sum_s m_s 2^{i_s}$ be an element with $l_2(x) < i$. Then there is a $N > 0$ with the property that the length of each m_s is strictly smaller than $2^{i_s}(i - 2^{-N})$. Hence we have with the multiplication formula 2.2.5 for θ

$$l_2(\theta(m_s)) \leq l_1(\theta(m_s)) \leq 2^{i_s+1}(i - 2^{-N}).$$

Using $\theta(2a) = 2\theta(a) - a^2$ we conclude

$$\begin{aligned} l_2(\theta(x)) &\leq \mathrm{sup}_s l_2(\theta(m_s 2^{i_s})) \leq \mathrm{sup}_s \mathrm{max}\{l_2(2^{i_s}\theta(m_s)), 2i - 2^{1-N}\} \\ &= \max\{\mathrm{sup}_s 2^{-i_s} l_2(\theta(m_s)), 2i - 2^{1-N}\} = 2i - 2^{1-N} < 2i. \end{aligned}$$

□

It is convenient to work with Landau symbols. We let $o_i(k)$ represent classes whose l_i-length is strictly smaller than k.

Lemma 4.13. *For all $i > 0$ we have the formula*

$$\theta(b_i + o_2(i)) = b_{2i+1} + (1 + b_1)b_{2i} + o_2(2i).$$

Proof. By 4.6 the class

$$a = \theta(b_i) - (b_{2i+1} + (1 + b_1)b_{2i})$$

is a sum of monomials of length at most $2i-1$ modulo 2. Moreover, by the previous lemma we have

$$l_2(a) = l_2(o_2(2i) + 2o_1(2i - 1)) < \max\{2i, \frac{2i-1}{2}\} = 2i.$$

Thus the claim follows from the third part of the lemma. □

The last formula is particularly nice when it comes to the spin groups. Before proving Theorem D we work out the relations that appear in the passage from the unitary to the special orthogonal groups.

LEMMA 4.14. *Let $i : \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$ be the map which classifies the conjugate tautological bundle $\bar{L}$. Then we have the formula*

$$i_*\beta_s = \sum_{t=1}^{s} (-1)^t \binom{s-1}{t-1} \beta_t.$$

PROOF. Compute

$$\left\langle i_*\beta_s, x^t \right\rangle = \left\langle \beta_s, (i^*x)^t \right\rangle = \left\langle \beta_s, x^t(x-1)^{-t} \right\rangle = (-1)^t \binom{s-1}{s-t}$$

□

LEMMA 4.15. *In $\pi_0 K \wedge BSO_+$ we have for all k*

$$b_{2k+1} = k\, b_{2k} + \textit{terms with lower index} \in \pi_0 K \wedge BSO_+.$$

PROOF. Since L and $\bar{L}$ are isomorphic as stable oriented real bundles the claim follows from

$$i_*\beta_{2k+2} - \beta_{2k+2} = \sum_{j=1}^{2k+1} (-1)^j \binom{2k+1}{j-1} \beta_j = -(2k+1)\beta_{2k+1} + (2k+1)k\beta_{2k}$$

modulo terms with lower index. □

We have seen in 3.9 that the map

$$\mathbb{Z}_2[b_2, b_4, \ldots] \longrightarrow \pi_0 K \wedge BU_+ \longrightarrow \pi_0 K \wedge BSO_+$$

is an isomorphism. Hence we can define $l_i(x)$ for all $x \in \pi_0 K \wedge BSO_+$ to be the l_i-length of its preimage in $\mathbb{Z}_2[b_2, b_4, \ldots]$.

PROPOSITION 4.16. *In $\pi_0 K \wedge BSO_+$ we have the formula for all $i > 0$*

$$\theta^j(b_{2i} + o_2(2i)) = b_{2^{j+1}i} + o_2(2^{j+1}i).$$

PROOF. Since b_1 vanishes in $\pi_0 K \wedge BSO_+$ the formula inductively follows from 4.15 and 4.13. □

PROOF OF THEOREM D: It is enough to show that the ring homomorphism

$$\mathbb{Z}_2[\theta^j u_{8k+4} |\, k, j \geq 0] \longrightarrow \mathbb{Z}_2[b_2, b_4, b_6, \ldots]$$

is an isomorphism modulo 2. We know from 3.9 that u_{4k} coincides with b_{2k} up to a class of l_2-length strictly smaller than $2k$. Hence, the proposition 4.16 gives

$$\theta^j u_{8k+4} = b_{2^{j+1}(2k+1)} + o_2(2^{j+1}(2k+1)).$$

Since each even number can uniquely be written in the form $2 \cdot 2^j(2k+1)$ for some j, k there is an obvious correspondence of the highest terms of the generators. This finishes the proof of the theorem. □

REMARK 4.17. We could have proved the theorem without using the Thom isomorphism. For that we merely observe that the class $b_{2k} \in \pi_0 K \wedge MSpin$ can be lifted to a class of the form $b_{2k} + x \in \pi_0 K \wedge MSU$ with $l_2(x) < 2k$ and proceed as above.

5. The proof of the splitting theorem

In the previous sections we computed the θ-algebra $\pi_0 KO \wedge MSpin$ and determined the values of the free generators under the ABP-map. Now we show that all except of one generator can be chosen to be spherical. The only missing generator is hit by the class b under the map coming from the cone. These results lead us to the proof of the splitting theorem.

5.1. Spherical classes. The spherical classes can be identified with the elements of $\pi_0 KO \wedge MSpin$ which are invariant under the action of the Adams operation with the help of the exact sequence

$$0 \longrightarrow \pi_0 MSpin \longrightarrow \pi_0 KO \wedge MSpin \stackrel{\psi^3 - 1}{\longrightarrow} \pi_0 KO \wedge MSpin \longrightarrow 0.$$

Note that it is enough to look for classes which are invariant under ψ^g for any given topological generators g of $\mathbb{Z}_2^\times / \pm 1 \cong \mathbb{Z}_2$.

Unlike the θ-operation the action of ψ^g is not compatible with the Thom isomorphism. We denote the operation on the base $\pi_0 KO \wedge BSpin_+$ by ψ^g_B and the one on the Thom spectrum $\pi_0 KO \wedge MSpin$ by ψ^g_M in the sequel. Before describing these we need the

Lemma 5.1. *For all $k \in \mathbb{Z}_2^\times / \pm 1$ the two self maps $\psi^k \wedge 1$ and $k^{4n}\,(1 \wedge \psi^{k^{-1}})$ of $\pi_{8n} KO \wedge KO$ coincide.*

Proof. It is enough to check the corresponding statement for complex K-theory. Since $\pi_{2n} K \wedge K$ is torsion free we even may rationalize. A general element of $\pi_{2n} K \wedge K \otimes \mathbb{Q}$ takes the form $a = \sum_s a_s u^s v^{n-s}$ if u, v denote the left and right Bott classes. Hence we compute

$$(\psi^k \wedge 1)(a) = \textstyle\sum_s a_s (k\,u)^s v^{n-s} = k^n \sum_s a_s u^s (k^{-1} v)^{n-s} = k^n\,(1 \wedge \psi^{k^{-1}})(a).$$

□

Lemma 5.2. *The operation $\psi^{3^{-1}}_B$ is given by the formula*

$$\psi^{3^{-1}}_B u_i = \sum_{j=0}^{i} ((-1)^{i-j} \sum_{s+t=i-j} \binom{j}{s} \binom{s}{t} 3^{j-t}) u_j.$$

Proof. It suffices to show the equation in $\pi_0 K \wedge BS^1_+$ after replacing the classes u_k with β_k. The previous lemma tells us that for all i, j the equality

$$\left\langle \psi^{3^{-1}} \beta_i, x^j \right\rangle = \langle \beta_i, \psi^3 x^j \rangle$$

holds. Hence we obtain

$$\left\langle \psi^{3^{-1}} \beta_i, x^j \right\rangle = \langle \beta_i, (1 - (1-x)^3)^j \rangle = (-1)^{i-j} \sum_{s+t=i-j} \binom{j}{s} \binom{s}{t} 3^{j-t}$$

□

Proposition 5.3. *We have the formula*

$$\psi^{3^{-1}}_M u_i = \sum_{j=0}^{i} a_j \psi^{3^{-1}}_B u_{i-j}.$$

Here the numbers a_j are determined by

$$\sum_{j=0}^{\infty} a_j x^j = \frac{1}{3} \frac{(1-x)^3 - (1-x)^{-3}}{(1-x) - (1-x)^{-1}}.$$

PROOF. Since the duality map of 3.12 is injective it suffices to show the equality after pairing each side with an arbitrary class $a = \tau^* b \in KO(MSpin)$. Let $f : BS^1 \longrightarrow BSpin$ be the inclusion of the maximal torus of $Spin(2)$. Then we compute with 3.14 and 5.1:

$$\begin{aligned}
\left\langle \psi_M^{3^{-1}} u_i, a \right\rangle &= \langle u_i, \psi_M^3(\tau^* b)\rangle = \left\langle u_i, \hat{\theta}^3\, \tau\, \psi_B^3(b)\right\rangle \\
&= \left\langle \beta_i, (\hat{\theta}^3(L^2-1)\otimes \mathbb{C}) f^*(\psi_B^3(b))\right\rangle \\
&= \sum_{j=0}^{\infty} a_j \left\langle \beta_{i-j}, f^*(\psi_B^3(b))\right\rangle = \left\langle \sum_{j=0}^{\infty} a_j \psi_B^{3^{-1}} u_{i-j}, b \right\rangle
\end{aligned}$$

□

It will prove useful to introduce another measure for the monomials of the ring

$$\mathbb{Z}/2[u_{4k};\ k \geq 1] \cong \mathbb{Z}/2[b_{2k};\ k \geq 1].$$

DEFINITION 5.4. *Let the degree d of a monomial $u_{4i_1} \cdot u_{4i_2} \cdots u_{4i_k}$ be $\sum_k i_k$ and let the degree of a sum of such be the maximum degree of the monomials. We will write $o(n)$ for terms of degree strictly smaller than n.*

PROPOSITION 5.5. *We have modulo 2 for all i*

(i) $\psi_B^{3^{-1}} u_{4i} = u_{4i} + o(i-1)$
(ii) $\psi_M^{3^{-1}} u_{4i} = u_{4i} + u_{4i-4} + o(i-1)$

PROOF. Consider first the case that $i = 2k$ is even. Then we obtain modulo 2 and $o(i-1)$ with 3.9 and 4.15

$$\begin{aligned}
\psi_B^{3^{-1}} u_{8k} &= u_{8k} + \sum_{s+t=2} \binom{8k-2}{s}\binom{s}{t} u_{8k-2} + \sum_{s+t=4} \binom{8k-4}{s}\binom{s}{t} u_{8k-4} \\
&= u_{8k} + u_{8k-2} + u_{8k-4} = u_{8k}.
\end{aligned}$$

Similarly, for $i = 2k+1$ we get mod 2 and $o(i-1)$

$$\begin{aligned}
\psi_B^{3^{-1}} u_{8k+4} &= u_{8k+4} + \sum_{s+t=2} \binom{8k+2}{s}\binom{s}{t} u_{8k+2} + \sum_{s+t=4} \binom{8k}{s}\binom{s}{t} u_{8k} \\
&= u_{8k+4} + u_{8k+2} = u_{8k+4}.
\end{aligned}$$

To see the second statement observe that $\sum_{j=0}^{\infty} a_j x^j = 1 + x^4 + \ldots$ and hence with the previous lemma

$$\psi_M^{3^{-1}} u_{4i} = \psi_B^{3^{-1}} u_{4i} + \psi_B^{3^{-1}} u_{4i-4} + o(i-1) = u_{4i} + u_{4i-4} + o(i-1).$$

□

Now we are well prepared to show the

THEOREM 5.6. *For each odd $k > 1$ there exists a $z_k \in \pi_0 KO \wedge MSpin$ which is invariant under the action of the Adams operations and which coincides with u_{4k} modulo elements of strictly smaller l_2-length.*

PROOF. We first construct the class z_k modulo 2. When we write Δ for the homomorphism $\psi_M^{3^{-1}} - 1$ then the previous lemma reads

$$\Delta u_{4i} = u_{4i-4} + o(i-1).$$

Moreover, we have for all s, t

$$\begin{aligned}
\Delta(u_{4s}\, u_{4t}) &= \Delta(u_{4s})\Delta(u_{4t}) + \Delta(u_{4s}) u_{4t} + u_{4s}\, \Delta(u_{4t}) \\
&= u_{4s-4} u_{4t} + u_{4s}\, u_{4t-4} + o(s+t-1).
\end{aligned}$$

In particular, we obtain modulo terms which are the Δ-image of classes with degree at most $n+m+1$ and with length at most $2\max(n, m+j)$

$$u_{4n}u_{4m} = u_{4n-4}u_{4m+4} + o(n+m) = \cdots = u_{4(n-j)}u_{4(m+j)} + o(n+m).$$

Setting $n = k-1, m = 0$ and $j = (k-1)/2$ we thus find a class x of length strictly smaller than $2k$ with the property that

$$\Delta(u_{4k} + x) = u_{4j}^2 + o(k-1).$$

We can get rid of the highest term by adding u_{4j+4}^2:

$$\Delta(u_{4k} + u_{4j+4}^2 + x) = u_{4j}^2 + (\Delta u_{4j+4})^2 + o(k-1) = o(k-1).$$

Now we have won since the remaining terms are of the form $u_{4n}u_{4m}$ with degree strictly smaller than $k-1$. They can be removed in the same fashion as above: Setting $j = n+1$ we see inductively that $u_{4n}u_{4m}$ lies in the Δ-image of classes with length strictly smaller than $2k$.

Actually we have shown a bit more. Let S_r be the set of pairs (i,j) with $i+j < k+r+2$. Then modulo 2 we can choose z_k to be of the form

$$z_k = \sum_{(i,j)\in I_0} u_{4i}u_{4j} = u_{4k} + o_2(2k)$$

for some $I_0 \subset S_0$. In the general situation it suffices to inductively construct sets $I_s \subset S_s$ such that

$$z_k^{(s)} = \sum_{r=0}^{s} 2^r \sum_{(i,j)\in I_r} u_{4i}u_{4j}$$

is invariant modulo 2^{s+1}. Suppose that we have already found $I_0, \ldots, I_{s-1}$. Then $\Delta z_k^{(s-1)}$ is a sum of terms of the form $2^t u_n u_m$ with $n+m < 4k+4t+3$. The lemma 5.8 below tells us that we may assume that n and m are multiples of 4. Since the monomials in the generators u_{4i} are linearly independent and $\Delta z_k^{(s-1)}$ vanishes modulo 2^s we are left with terms of the form $2^s u_{4i}u_{4j}$ with $i+j < k+s+1$. These can be removed with the method above. □

Lemma 5.7. *Let R be a complete, local ring with maximal ideal m and $a \in m$ be given. Let $v_k = \sum_{s\geq 0} n_s^{(k)} w_{k+s}$ be a convergent series in $R[w_1, w_2, \ldots]$ with $n_0^{(k)} \in R^\times$ and $a^s \mid n_s^{(k)}$ for all s, k. Then there are $m_t^{(k)}$ such that $w_k = \sum_{s\geq 0} m_s^{(k)} v_{k+s}$ and a^s divides $m_s^{(k)}$ for all s, t.*

Proof. Suppose the elements $m_s^{(k)}$ are already constructed modulo m^j and let $n_s^{(k)} = a\, l_s^{(k)}$ with $a^{s-1} \mid l_s^{(k}$ for all $s > 0, k \geq 0$. Then we have modulo m^{j+1}

$$n_0^{(k)} w_k = v_k - a\sum_{s>0} l_s^{(k)} w_{k+s} = v_k - \sum_{r>0}(a \sum_{s+t=r, s>0} l_s^{(k)} m_t^{(k+s)})v_{k+r}.$$

Since the coefficients $m_s^{(k)}$ are unique the claim follows. □

Lemma 5.8. *Each u_n can be written as a convergent series of the form $\sum_s a_s u_{4s}$ with $2^{4s-n} \mid a_s$ for $4s \geq n$.*

Proof. We know from 3.9 and 4.15 that we can write u_n in the form $\sum_j k_j b_{2j}$ with $2^{4j-n} \mid k_j$ for $4j \geq n$. Hence, the previous lemma gives with $a = 16$, $v_k = u_{4k}$ and $w_k = b_{2k}$

$$u_n = \sum_s (\sum_{j+t=s} k_j m_t^{(j)}) u_{4s} \qquad \text{and } 2^{4t} \mid m_t^{(j)}.$$

□

Note that the proof of the theorem created an algorithm which produces the spherical classes z_k. The first two ones can be chosen as follows modulo 2:

$$\begin{aligned} z_3 &= u_{12} + u_8^2 + u_8 u_4 + u_8 + u_4 \\ z_5 &= u_{20} + u_{12}^2 + u_{12} u_8 + u_{16} u_4 + u_{12} u_4 + u_8 u_4 + u_4^2 \end{aligned}$$

It is not possible to alter the class u_4 by terms of strictly smaller l_2-length in a way that it becomes a spherical class. However, the sum $u_4 + u_4^2$ happens to be invariant modulo 2. Hence u_4 behaves in the same way as the class b which was defined earlier. A closer relationship between the two classes is established in the next section.

5.2. Some 2-adic analysis. In section 3 we constructed an E_∞ map $\varphi : T_\zeta \longrightarrow MSpin$ and investigated its behavior in KO-theory. In Theorem A we calculated the image of the θ-algebra generator $b \in \pi_0 KO \wedge T_\zeta$ under the image of $\Phi\pi_*^J \varphi_*$ for all $1 \notin J$. The resulting continuous functions determine $\varphi_* b$ since the map

$$\pi_0 KO \wedge MSpin \overset{\Phi(1\wedge\pi^J)_*}{\longrightarrow} \bigoplus_{1 \notin J} \mathcal{T}(\mathbb{Z}_2^\times / \pm 1, \mathbb{Z}_2)$$

is an isomorphism. In the same section we also calculated the 2-adic functions which correspond to the algebra generators u_{4k}. In this section we compare the 2-adic functions and prove the

THEOREM 5.9. $\varphi_* b = u_4 + o_2(2)$.

A weaker statement is shown in the following

LEMMA 5.10. *Mod 4 the class $\varphi_* b$ coincides with $-u_4$.*

PROOF. With $y = -2x + x^2$ the formula of Theorem B reads

$$k \sum_n \Theta(u_n)(k) x^n = (1+y)^{(1-k)/2} \frac{(1+y)^k - 1}{y} = \sum_{s,t} \binom{(1-k)/2}{s} \binom{k}{t+1} y^{s+t}.$$

Moreover, observe that for all integers $n = \sum_s \alpha_s 2^s$ we have mod 16

$$3^n = 1 + 2\alpha_0 + 8\alpha_1$$

as one easily verifies. Thus we obtain for $k = 3^n$ mod 4

$$\Theta(u_4)(k) = k^{-1} \sum_{s+t=2} \binom{(1-k)/2}{s} \binom{k}{t+1} = -\alpha_0 + 2\alpha_1 = -n$$

The result now follows from Theorem A. □

It is clear that $\varphi_* b$ is some convergent series in the u_{4k}-monomials. One might hope to get along with the indecomposable classes u_{4k} itself. For this purpose, we mention that the group of continuous functions has a simple basis which is given by the binomial functions (compare Mahler [**Sch84**]p.149ff.) For all continuous $f : \mathbb{Z}_2 \longrightarrow \mathbb{Q}_2$ there is a convergent series of the form

$$f(x) = \sum_{n=0}^\infty a_n \binom{x}{n}.$$

Moreover, the null sequence $a_n \in \mathbb{Q}_2$ is unique. Another basis is given by the family $x \mapsto \binom{2x}{2n}$ since it coincides with the binomial basis modulo 2.

PROPOSITION 5.11. *Let $f : \mathbb{Z}_2^\times / \pm 1 \longrightarrow \mathbb{Z}_2$ be an even continuous function. Then f admits an expansion of the form*

$$f = \sum_{n=0}^\infty a_n \Theta(u_{4n})$$

for some null sequence $a_n \in \mathbb{Z}_2$. *Moreover, the expansion is unique.*

PROOF. The continuous function $g(x) = f(2x-1)(2x-1) : \mathbb{Z}_2 \longrightarrow \mathbb{Q}_2$ admits an expansion

$$g(x) = \sum_m a_m \binom{2x}{2m} = \sum_m a_m (\binom{2x-1}{2m} + \binom{2x-1}{2m-1}).$$

Hence, there is a null sequence a'_m such that for all $k \in \mathbb{Z}_2^\times$

$$f(k) = \frac{f(k)+f(-k)}{2} = \sum_m a'_m k^{-1}(\binom{k}{m} - \binom{-k}{m}).$$

Moreover, for each m the function

$$\varphi_m(k) = k^{-1}(\binom{k}{m} - \binom{-k}{m})$$

can be expressed in terms of the $\Theta(u_{4n})$ with the help of Theorem B:

$$((1-x)-(1-x)^{-1})\sum_j \Theta(u_j)(k)x^j = k^{-1}((1-x)^k - (1-x)^{-k}) = \sum_m \varphi_m(k)x^m.$$

Using 5.8 we hence constructed an expansion of f with a null sequence $a_n \in \mathbb{Q}_2$. Its integrality and uniqueness follows from the bijectivity of the ABP-map $(\Theta_J)_{1 \notin J}$. □

The proof of the proposition provides an algorithm for the coefficients in the expansion of the function associated to b. Note that we may write the latter in the form

$$b(x) = \frac{\log(x)}{\log(3)} : \mathbb{Z}_2^\times / \pm 1 \longrightarrow \mathbb{Z}_2.$$

Here, the 2-adic logarithm is given by the formula

$$\log(1+x) = \sum_{n=1}^\infty (-1)^{n+1}\frac{x^n}{n} \text{ for } |x| < 1.$$

It is elementary to check that the logarithm always is divisible by 4 and that $\log(3) = 4$ modulo 8. Hence, the quotient $\log(x)/\log(3)$ is well defined. By Theorem A it coincides with $b(x)$ since the 2-adic logarithm satisfies the usual properties. Before carrying out the program of expanding b we observe

LEMMA 5.12. $l_1(\varphi_s) \leq [\frac{s}{2}] - 1$ *for all* $s \geq 1$.

PROOF. It is easy to see with Theorem A that

$$\varphi_s = -2\Theta(u_{s-1}) - \sum_{j=0}^{s-2}\Theta(u_j)$$

Hence the assertion follows from 3.9 and 5.8. □

PROOF OF 5.9. Let α be the linear operator which takes a continuous function f on $\mathbb{Z}_2^\times$ to the even function

$$\alpha(f) : \mathbb{Z}_2^\times / \pm 1 \longrightarrow \mathbb{Z}_2;\ k \mapsto k^{-1}(f(k) - f(-k)).$$

Then we have for all $k = 2x - 1 \in \mathbb{Z}_2^\times$

$$\log(k) = 2^{-1}\alpha(k\log(k)) = \sum_{n\geq 1}(-1)^{n+1}\frac{2^{n-1}}{n}\sum_{s=0}^{n}(-1)^{n-s}\binom{n}{s}(2\alpha(x^{s+1}) - \alpha(x^s)).$$

It is known that

$$x^n = \sum_{m\leq n} a_{mn}\binom{x}{m} \text{ with } a_{mn} = S(m,n)m!.$$

Here, $S(m,n)$ is the Stirling number of the second kind. Furthermore, the expansion

$$\binom{2x}{2m} = \sum_{l=0}^{\infty} 2^{2l}\binom{m+l}{2l}\binom{x}{m+l}$$

shows with 5.7, Pascal's equality and the lemma that

$$l_1(\alpha(\binom{x}{m})) \leq l_1(\alpha(\binom{2x}{2m})) = l_1(\varphi_{2m}+\varphi_{2m-1}) \leq m-1.$$

The number $m!$ is divisible by $2^{m-\sigma(m)}$. Here, $\sigma(m)$ is the sum of the number of digits in the 2-adic decomposition of m. This gives

$$l_2(a_{mn}\alpha(\binom{x}{m})) \leq 2^{-m+\sigma(m)}(m-1) \leq 1$$

for all m. Since for $n \geq 5$ the number $2^{n-1}/n$ is divisible by 4 all summands in the expansion have l_2-length at most $1/4$ or l_1-length at most 2. Hence b is an expression in terms with l_2-length strictly smaller than 2 and terms with l_1-length at most 4. Thus the assertion follows from 5.10. □

6. The relations of spin bordism to real K-theory

6.1. The θ-algebra structure of $\pi_* MSpin$. A first consequence of the splitting formula is the

COROLLARY 6.1. *Let $f \in \pi_0 MSpin$ be the image of $f \in \pi_0 T_\zeta$. Then we have an isomorphism of θ-algebras*

$$\pi_* MSpin \cong \pi_* KO \otimes T\{f, z_3, z_5, z_7, \ldots\}.$$

PROOF. This immediately follows from the main theorem and 2.18. □

REMARK 6.2. The formula for the homotopy ring of spin bordism evokes the hope that $MSpin$ can be made into a KO-algebra spectrum. We have seen earlier that $MSpin$ splits into a sum of KOs and hence is a KO-module spectrum (in contrary to the unlocalized $MSpin$ [**Sto94**].) However, there does not exist any map of ring spectra from KO to $MSpin$ even in the $K(1)$-local world: any such would give a self map of KO when composed with the $\hat{A}$-map $\pi^\emptyset$. The induced map in KO-homology factorizes over the free ring $\pi_0 KO \wedge MSpin$ and thus coincides with the augmentation

$$\epsilon_* : \pi_0 KO \wedge KO \longrightarrow \pi_0 KO \subset \pi_0 KO \wedge KO$$

by 3.4 and 3.10. Even rationally, there is no self ring map of KO which induces the augmentation map in KO-homology.

6.2. The E_∞ cellular structure of the $\hat{A}$-map. Note that the map of spectra $\pi^\emptyset : MSpin \longrightarrow KO$ is the Atiyah-Bott-Shapiro orientation. It is denoted by $\hat{A}$ in the sequel. We assume that we have chosen the classes $z_k \in \pi_0 MSpin$ of the splitting map in a way that $\hat{A}z_k \in \pi_0 KO \cong \mathbb{Z}_2$ is null. This is possible since the addition of constants does not change the l_2-length of each z_k. By 3.3 we may assume that the null homotopy of ζ in $MSpin$ has the following property: when restricted to the cone C_ζ and composed with $\hat{A}$ the resulting E_∞ map $\varphi : T_\zeta \longrightarrow MSpin$ gives the null homotopy $\iota : C_\zeta \longrightarrow KO$ up to homotopy. The induced E_∞ map $T_\zeta \longrightarrow KO$ is denoted by the same letter.

COROLLARY 6.3. *The composite of E_∞-maps*

$$MSpin \cong T_\zeta \wedge \bigwedge TS^0 \xrightarrow{(\iota,*)} KO$$

coincides with the $\hat{A}$-map.

The proof uses the

Lemma 6.4. *A map from $MSpin$ to KO is determined by its behavior in KO-homology. That is, the map*

$$KO(MSpin) \longrightarrow \mathrm{Hom}_{cts}(\pi_0 KO \wedge MSpin, \pi_0 KO \wedge KO)$$

is injective.

Proof. Since the real and complex completed representation rings of each group $Spin(8k)$ coincide (compare [**And64**]) we have that $KO(MSpin) \cong K(MSpin)$. Hence it suffices to prove the claim for complex K-theory. The pairing

$$\pi_0(K \wedge MSpin, \mathbb{Z}_{2^\infty}) \otimes K(MSpin) \longrightarrow \mathbb{Z}_{2^\infty}$$

induces an isomorphism (compare 2.3 of [**Bou99**]) from $K(MSpin)$ to the Pontyagin dual of the 2-profinite abelian group $\pi_0(K \wedge MSpin, \mathbb{Z}_{2^\infty})$. This map factorizes over $\mathrm{Hom}_{cts}(\pi_0 K \wedge MSpin, \pi_0 K \wedge K)$. □

Proof of 6.3: It is well known that $\hat{A}$ is an H_∞^8 map (see [**BMMS86**] p.280ff) and hence induces a map of θ-algebras in KO-homology. When restricted to $C_\zeta \vee \bigvee TS^0$ this map coincides with $(\iota, *)$ by our choices above. Thus the θ-algebra generators $b, z_3, z_5, \ldots$ are identically mapped and the claim follows from the lemma. □

The corollary 6.1 suggests how to obtain KO-theory by attaching E_∞ cells to $MSpin$.

Corollary 6.5. *The diagram*

$$\begin{array}{ccc} \bigwedge_{i=1}^{\infty} TS^0 & \xrightarrow{\ *\ } & T* \\ {\scriptstyle (f,z_3,z_5,\ldots)}\downarrow & & \downarrow \\ MSpin & \xrightarrow{\ \hat{A}\ } & KO \end{array}$$

is a homotopy pushout of $K(1)$-local E_∞ ring spectra.

Proof. In [**Hop98**] it is shown that the right square of the diagram

$$\begin{array}{ccccc} TS^0 \wedge \bigwedge TS^0 & \xrightarrow{1\wedge *} & TS^0 \wedge T* & \xrightarrow{\ *\ } & T* \\ {\scriptstyle (f,(z_3,z_5,\ldots))}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow \\ MSpin & \longrightarrow & T_\zeta & \longrightarrow & KO \end{array}$$

is a homotopy pushout. The splitting theorem gives the homotopy pushout property of the left square and hence furnishes the result. □

6.3. Another additive splitting and the Conner-Floyd isomorphism. We have seen earlier that $MSpin$ additively splits into a sum of KO-theories. Using the multiplicative splitting theorem we are now able to write down an additive splitting which recovers more structure.

Corollary 6.6. *In the category of $K(1)$-local spectra there is a natural isomorphism of $\pi_0 MSpin$-modules*

$$\pi_* MSpin \wedge X \cong \pi_* KO \wedge X \otimes T\{f, z_3, z_5, \ldots\}.$$

Here, the module structure of the right hand side is given by the isomorphism of 6.1.

PROOF. Choose a projection pr of the free $T\{f\}$-module $T\{b\}$ onto the summand $T\{f\}$. Then the composite

$$\begin{aligned}\pi_*T_\zeta \wedge X &\longrightarrow \pi_*KO \wedge T_\zeta \wedge X \cong \pi_*KO \wedge T_\zeta \otimes_{\pi_*KO} \pi_*KO \wedge X \\ &\cong \pi_*KO \wedge X \otimes T\{b\} \xrightarrow{1\otimes pr} \pi_*KO \wedge X \otimes T\{f\}\end{aligned}$$

is a natural transformation between cohomology theories. Hence the result follows from the splitting theorem. □

This result immediately implies the

COROLLARY 6.7. *(Hopkins, Hovey* [**HH92**]*)*
In the category of $K(1)$-local spectra the natural map

$$\pi_*MSpin \wedge X \otimes_{\pi_*MSpin} \pi_*KO \longrightarrow \pi_*KO \wedge X$$

induced by the $\hat{A}$-orientation is an isomorphism.

REMARK 6.8. Hopkins and Hovey prove the general Conner-Floyd isomorphism for $MSpin$ and KO by localizing at each prime. The essential work is done at the prime 2 since for odd primes the original method of Conner and Floyd applies. Let $\beta \in \pi_8 MSpin$ correspond to the Bott class of the first ko-summand in the ABP-splitting. Then they show that $\beta^{-1}MSpin$ is K-local. Hence there is a natural isomorphism

$$\begin{aligned}&(MSpin \wedge R)_*X \otimes_{(MSpin\wedge R)_*} (KO \wedge R)_* \\ &\cong \quad (L_K MSpin \wedge R)_*X \otimes_{(L_K MSpin\wedge R)_*} (KO \wedge R)_*\end{aligned}$$

for all ring spectra R. When setting $R = S\mathbb{Z}/2^k$ we get with 6.6

$$\begin{aligned}&(L_K MSpin \wedge S\mathbb{Z}/2^k)_*X \otimes_{(L_K MSpin\wedge S\mathbb{Z}/2^k)_*} (KO \wedge S\mathbb{Z}/2^k)_* \\ \cong\quad & \pi_*L_{K(1)}MSpin \wedge S\mathbb{Z}/2^k \wedge X \otimes_{\pi_*L_{K(1)}MSpin\wedge S\mathbb{Z}/2^k} \pi_*L_{K(1)}KO \wedge S\mathbb{Z}/2^k \\ \cong\quad & \pi_*KO \wedge S\mathbb{Z}/2^k \wedge X\end{aligned}$$

The general statement now can be finished as in section 6 of [**HH92**].

References

[ABP66] D.W. Anderson, Edgar H.jun. Brown, and Franklin P. Peterson, *Spin cobordism*, Bull. Am. Math. Soc. **72** (1966), 256–260.

[Ada74] J.F. Adams, *Stable homotopy and generalised homology.*, Chicago Lectures in Mathematics. Chicago - London: The University of Chicago Press. X, 373, 1974.

[AHS71] J.F. Adams, A.S. Harris, and R.M. Switzer, *Hopf algebras of cooperations for real and complex K-theory.*, Proc. Lond. Math. Soc., III. Ser. **23** (1971), 385–408.

[And64] D.W. Anderson, *The real K-theory of classifying spaces*, Proc. Natl. Acad. Sci. USA **51** (1964), 634–636 (English).

[Ati66] Michael F. Atiyah, *Power operations in K-theory*, Q. J. Math., Oxf. II. Ser. **17** (1966), 165–193.

[BMMS86] R.R. Bruner, J.P. May, J.E. McClure, and M. Steinberger, *H_∞ ring spectra and their applications.*, Lecture Notes in Mathematics, 1176. Berlin etc.: Springer-Verlag. VII, 388 p., 1986.

[Bot69] Raoul Bott, *Lectures on $K(X)$*, New York-Amsterdam: W.A. Benjamin, Inc. X, 203 p., 1969.

[Bou79] A.K. Bousfield, *The localization of spectra with respect to homology.*, Topology **18** (1979), 257–281.

[Bou96a] ———, *On λ-rings and the K-theory of infinite loop spaces.*, K-Theory **10** (1996), no. No.1, 1–30.

[Bou96b] ———, *On p-adic λ-rings and the K-theory of H-spaces.*, Math. Z. **223** (1996), no. 3, 483–519.

[Bou99] ———, *The K-theory localizations and v_1-periodic homotopy groups of H-spaces.*, Topology **38** (1999), no. 6, 1239–1264.

[BW85] Michael Barr and Charles Wells, *Toposes, triples and theories.*, Grundlehren der Mathematischen Wissenschaften, 278. New York etc.: Springer-Verlag. XIII, 345, 1985.

[EKMM97] A.D. Elmendorf, I. Kriz, M.A. Mandell, and J.P. May, *Rings, modules, and algebras in stable homotopy theory. With an appendix by M. Cole.*, Mathematical Surveys and Monographs. 47. Providence, RI: American Mathematical Society. xi, 249 p. [ISBN 0-8218-0638-6/hbk], 1997.

[Elm88] A.D. Elmendorf, *The Grassmannian geometry of spectra.*, J. Pure Appl. Algebra **54** (1988), 37–94.

[HH92] Michael J. Hopkins and Mark A. Hovey, *Spin cobordism determines real K-theory.*, Math. Z. **210** (1992), no. 2, 181–196.

[Hod72] Luke Hodgkin, *The K-theory of some well-known spaces. I: QS^0.*, Topology **11** (1972), 371–375.

[Hop96] M.J. Hopkins, *Course note for elliptic cohomology*, unpublished notes, 1996.

[Hop98] ———, *$K(1)$-local E_∞ ring spectra*, unpublished notes, 1998.

[Hov95] Mark Hovey, *Bousfield localization functors and Hopkins' chromatic splitting conjecture.*, Cenkl, Mila (ed.) et al., The Cech centennial. A conference on homotopy theory dedicated to Eduard Cech on the occasion of his 100th birthday, June 22-26, 1993, Northeastern University, Boston, MA, USA. Providence, RI: American Mathematical Society. Contemp. Math. 181, 225-250 [ISBN 0-8218-0296-8/pbk], 1995.

[HS75] L. Hodgkin and V. Snaith, *Topics in K-theory. L. Hodgkin: The equivariant Kuenneth theorem in K- theory. V. Snaith: Dyer-Lashof operations in K-theory.*, Lecture Notes in Mathematics. 496. Berlin - Heidelberg - New York: Springer-Verlag. 294 p., 1975.

[HSS00] Mark Hovey, Brooke Shipley, and Jeff Smith, *Symmetric spectra.*, J. Am. Math. Soc. **13** (2000), no. 1, 149–208.

[Joa01] Michael Joachim, *A symmetric ring spectrum representing KO-theory.*, Topology **40** (2001), 299–308.

[Lau01] Gerd Laures, *An E_∞-splitting of spin bordism with applications to real K-theory and topological modular forms.*, Habilitationsschrift Universität Heidelberg., 2001.

[Lau02] ———, *Characteristic numbers from 2-structures on formal groups.*, J. Pure Appl. Algebra **172** (2002), no. 2-3, 239–256.

[LMS86] L.G.jun. Lewis, J.P. May, and M. Steinberger, *Equivariant stable homotopy theory. With contributions by J. E. McClure.*, Lecture Notes in Mathematics, 1213. Berlin etc.: Springer-Verlag. IX, 538 p., 1986.

[May72] J.P. May, *The geometry of iterated loop spaces.*, Lecture Notes in Mathematics. 271. Berlin-Heidelberg-New York: Springer- Verlag. IX, 175 p., 1972.

[May77] J.Peter May, *E_∞ ring spaces and E_∞ ring spectra. With contributions by Frank Quinn, Nigel Ray, and Jorgen Tornehave.*, Lecture Notes in Mathematics. 577. Berlin - Heidelberg - New York: Springer-Verlag. 268 p., 1977.

[Mei79] W. Meier, *Complex and real K-theory and localization.*, J. Pure Appl. Algebra **14** (1979), 59–71.

[MR81] Mark Mahowald and Nigel Ray, *A note on the Thom isomorphism.*, Proc. Am. Math. Soc. **82** (1981), 307–308.

[Pri75] Stewart Priddy, *Dyer-Lashof operations for the classifying spaces of certain matrix groups.*, Q. J. Math., Oxf. II. Ser. **26** (1975), 179–193.

[Qui71] D. Quillen, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Adv. Math. **7** (1971), 29–56.

[Rav84] Douglas C. Ravenel, *Localization with respect to certain periodic homology theories.*, Am. J. Math. **106** (1984), 351–414.

[Sch84] W.H. Schikhof, *Ultrametric calculus. An introduction to p-adic analysis.*, Cambridge Studies in Advanced Mathematics, 4. Cambridge etc.: Cambridge University Press. XI, 306 p., 1984.

[Sto94] Stephan Stolz, *Splitting certain $MSpin$-module spectra.*, Topology **33** (1994), no. 1, 159–180.

Mathematisches Institut der Universität Heidelberg, Im Neuenheimer Feld 288, D-69120 Heidelberg, Germany

E-mail address: `gerd@laures.de`