SINGULARITIES AND QUINN SPECTRA

NILS A. BAAS AND GERD LAURES

Abstract. We introduce singularities to Quinn spectra. It enables us to talk about ads with prescribed singularities and to explicitly construct highly structured representatives for prominent spectra like Morava K-theories or for L-theory with singularities. We develop a spectral sequence for the computation of the associated bordism groups and investigate product structures in the presence of singularities.

1. Introduction

Manifolds with cone-like singularities were introduced by D. Sullivan in [Sul67]. The concept was reformulated by Baas in [Baa73a] as manifolds with a higher order (multilevel) decomposition of its boundary. Based on this definition a theory of cobordisms with singularities was developed. Many interesting homology and cohomology theories were constructed based on this theory. For example the Morava K-theories, the Johnson–Wilson theories, versions of elliptic cohomology, etc.

All these theories have played an important role in homotopy theory and algebraic topology during the last 30–40 years. However, it is surprising how many results could be obtained by just knowing their existence, not their construction. An explicit construction, however, can help in investigating the multiplicative structure of the representing spectra. Also, in order to obtain further results it seems to be important that the spectra are related to the original geometric category.

This is the goal with the present paper. With a cobordism category of manifolds in mind the theory of "ads" of [LM] is used to construct Quinn-spectra with singularities. They come with the usual exact sequences and a Bousfield-Kan spectral sequence for the computation of their coefficients. Moreover, it turns out that these spectra always give strict module spectra over the original Quinn spectra. In some cases they even have an explicit A_∞ structure (or sometimes, by a forthcoming paper, an E_∞-structure.) If the Quinn spectrum is L-theory the singularities spectrum seems to provide the natural surgery obstructions for manifolds with singularities.

This work is organized as follows: we first recall from [LM] the main results on ad theories and Quinn spectra. In the next section we introduce the singularities in the context of ads and develop new ad theories this way. Then the exact sequence for the bordism groups are constructed. It relates the ad theories among each other in case of a sequence of singularities. The next section deals with the classical example of manifolds ads. An assembly map shows that the corresponding Quinn spectrum with singularities represents the homology of manifolds with singularities of [Baa73a]. The following section is devoted to a Bousfield-Kan type spectral sequence for ads with singularities. For complex bordism such a spectral sequence

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was developed by Morava in [Mor79]. We also briefly discuss product structures and external products.

2. Ad theories and Quinn spectra

In this section we recall the basic notions of [LM] which lead to spectra of Quinn type.

Recall from [LM] that a $\mathbb{Z}$-graded category $\mathcal{A}$ is a category with involution and $\mathbb{Z}/2$-equivariant functors

$$d = dim : \mathcal{A} \longrightarrow \mathbb{Z},\ \emptyset : \mathbb{Z} \longrightarrow \mathcal{A}$$

with $d\,\emptyset = id$. Here, $\mathbb{Z}$ is regarded as a poset with trivial involution. The full subcategory of $\mathcal{A}$ of n-dimensional objects is denoted by $\mathcal{A}_n$. A k morphism between graded categories are functors which decreases the dimension by k and strictly commutes with i and $\emptyset$.

Let K be a ball complex in the sense of [BRS76]. We write $\mathcal{C}ell(K)$ for the category with objects in dimension n the oriented cells (σ, o) of K and the empty cell $\emptyset_n$. There are only identity morphisms in $\mathcal{C}ell(K)_n$ and morphisms to higher dimensional cells are given by inclusions of cells with no requirements to the orientations. The category $\mathcal{C}ell(K)$ is a graded category with the orientation reversing involution. Note that morphisms between ball complexes induce morphisms on the cellular categories. Moreover, if L is a subcomplex of K we can form the quotient category $\mathcal{C}ell(K, L)$ of $\mathcal{C}ell(K)$ by identifying the cells of L with the empty cells.

Next we recall the definition of an ad theory from [LM].

Definition 2.1. Let $\mathcal{A}$ be a category over $\mathbb{Z}$. A k-morphism from $\mathcal{C}ell(K, L)$ to $\mathcal{A}$ is called a pre (K, L)-ad of degree k. We write $\mathrm{pre}^k(K, L)$ for the set of these pre ads. An *ad theory* is an i-invariant sub functor ad^k of pre^k from ball complexes to sets for each k with the property $\mathrm{ad}^k(K, L) = \mathrm{pre}^k(K, L) \cap \mathrm{ad}^k(K)$ and which satisfies the following axioms:

(pointed) the pre ad which takes every oriented cell to $\emptyset$ is an ad for every K

(full) any pre K-ad which is isomorphic to a K-ad is a K-ad

(local) every pre K-ad which restricts to a σ-ad for each cell σ of K is a K-ad.

(gluing) for each subdivision K' of K and each K'-ad M there is a K-ad which agrees with M on each common subcomplex of K and K'.

(cylinder) there is a natural transformation

$$J : \mathrm{ad}^n(K) \longrightarrow \mathrm{ad}^n(K \times I)$$

with the property that for every K-ad M the restriction of $J(M)$ to $K \times 0$ and to $K \times 1$ coincides with M. It takes trivial ads to trivial ones.

(stable) let

$$\theta : \mathcal{C}ell(K_0, L_0) \longrightarrow \mathcal{C}ell(K_1, L_1)$$

be a k-isomorphism with the property that it preserves all incidence numbers

$$[o(\sigma), o(\sigma')] = [o(\theta\sigma), o(\theta\sigma')]$$

(see [Whi78]p.82.) Then the induced map of pre ads restricts to ads:

$$\theta^* : \mathrm{ad}^l(K_1, L_1) \longrightarrow \mathrm{ad}^{k+l}(K_0, L_0).$$

A multiplicative ad theory in a graded symmetric monoidal category $\mathcal{A}$ is equipped with a natural transformation

$$\mathrm{ad}^p(K) \wedge \mathrm{ad}^q(L) \longrightarrow \mathrm{ad}^{p+q}(K \times L)$$

and the object e in $\mathrm{ad}_0(*)$ which is associative and unital in the sense of [LM].

Example 2.2. Let R be a ring with unit. Consider R as a graded category with objects the elements of R concentrated in dimension 0, only identity morphisms and involution given by multiplication by -1. Then there is a multiplicative ad theory with K-ads M all pre K-ads with the property that for all cells $\sigma \in K$ of dimension n

$$\sum_{dim(\sigma')=n-1, \sigma' \subset \sigma} [o(\sigma), o(\sigma')] M(\sigma', o(\sigma')) = 0$$

where $[o(\sigma), o(\sigma')]$ is the incidence number.

Example 2.3. Let $\mathcal{ST}op$ be the graded category of oriented manifolds. An ad theory over $\mathcal{ST}op$ can be defined as follows: a pre K-ad M is an ad if for each $\sigma' \subset \sigma$ of one dimension lower the map $M(\sigma', o') \longrightarrow M(\sigma, o)$ factors through an orientation preserving map

$$M(\sigma', o') \longrightarrow [o, o']\partial M(\sigma, o)$$

and $\partial M(\sigma, o)$ is the colimit of M restricted to $\partial\sigma$. See [LM] for details. For instance, a decomposed (oriented) manifold in the sense of [Baa73b] is a Δ^n-ad.

Example 2.4. Let W be the standard resolution of $\mathbb{Z}$ by $\mathbb{Z}[\mathbb{Z}/2]$ modules. Define the objects of $\mathcal{A}$ to be the quasi-symmetric complexes, that is, in dimension n we have pairs $(C; \varphi)$ where C is a quasi finite complex of free abelian groups and

$$\varphi : W \to C \otimes C$$

is a $\mathbb{Z}/2$ equivariant map which raises the degree by n. The dimension increasing morphisms $f : (C; \varphi C) \to (C'; \varphi')$ are the chain maps and for equal dimension of source and target one further assumes that

$$(f \otimes f)\varphi = \varphi'.$$

The involution changes the sign of φ. The K-ads of symmetric Poincaré complexes are those (balanced) functors which

(i) are closed, that is, for each cell σ of K the map from the cellular chain complex

$$\mathrm{cl}(\sigma) \to \mathrm{Hom}(W, C \otimes C)$$

which takes (τ, o) to the composite

$$W \xrightarrow{\varphi_{(\tau,o)}} C_\tau \otimes C_\tau \to C_\sigma \otimes C_\sigma$$

is a chain map.

(ii) are well behaved, that is, each map $f_{\tau \subset \sigma}$ and

$$C_{\partial\sigma} = \mathrm{colim}_{\tau \subsetneq \sigma}\, C_\tau \to C_\sigma$$

are a cofibrations (split injective).

(iii) non degenerate, that is, the induced map

$$H^*(\mathrm{Hom}(C, \mathbb{Z})) \to H_{\dim\sigma - \deg F - *}(C_\sigma / C_{\partial\sigma}).$$

is an isomorphism.

For an ad theory the bordism groups Ω^n are obtained by identifying two *-ads of dimension n if there is a I-ad which restricts to the given ones on the ends. The main result of is

Theorem 2.5 ([LM]). *The ads form the simplexes of the spaces in a quasi Ω-spectrum $Q(ad)$ in a natural way. Its coefficients are given by the bordism groups. If the theory is multiplicative then the spectrum can be given the structure of a symmetric ring spectrum.*

Moreover, for all commutative ad theories the Quinn spectrum is homotopy equivalent to a strictly commutative symmetric ring spectrum. This is proven in an subsequent paper to [LM]. In the example of a commutative ring one obtains singular homology with R coefficients. In the example of oriented manifolds one obtains a spectrum which is homotopy equivalent to the Thom spectrum. In the example of symmetric Poincaré complexes the spectrum coincides with the symmetric L-theory spectrum.

3. Singularities

Let $\mathcal{A}$ be a symmetric monoidal graded category and suppose that we are given a multiplicative ad theory over $\mathcal{A}$.

Definition 3.1. Let $S = (P_1, P_2, \dots)$ be a sequence of $*$-ads and set

$$S_n = (P_1, \dots, P_n).$$

Let $\mathcal{A}(S_n)$ be the graded category whose objects are given by the following data:

(i) a pre σ-ad M_σ for each cell $\sigma \subset \{0, 1, \dots, n\}$ of Δ^n with

$$M_\sigma = \emptyset \quad \text{if } 0 \notin \sigma$$

For $\sigma = \Delta^n$ we sometimes simply write M for the top pread. Its dimension is $d - n$ of the object.

(ii) an isomorphism of preads for each $i \notin \sigma$

$$f_{\sigma,i} : \partial_i M_{(\sigma,i)} \xrightarrow{\cong} M_\sigma \otimes P_i.$$

Here, ∂_i denotes the restriction to the face σ and (σ, i) means $\sigma \cup \{i\}$.

We demand for each object that $\partial_0 M_\sigma = \emptyset$ and for all $i, j > 0$ the diagram

$$\begin{tikzcd}
\partial_j \partial_i M_{(\sigma,i,j)} \arrow[r] \arrow[d, "="] & \partial_j M_{(\sigma,j)} \otimes P_i \arrow[r] & M_\sigma \otimes P_j \otimes P_i \arrow[d, "1\otimes\gamma"] \\
\partial_i \partial_j M_{(\sigma,i,j)} \arrow[r] & \partial_i M_{(\sigma,i)} \otimes P_j \arrow[r] & M_\sigma \otimes P_i \otimes P_j
\end{tikzcd}$$

commutes.

Morphisms are morphisms of preads which commute with the isomorphisms $f_{\sigma,i}$.

Example 3.2. For $n = 0$ an object is determined by the value of the top cell of Δ^0 if $\emptyset$ is initial in $\mathcal{A}$. Hence we have

$$\mathcal{A}() = \mathcal{A}.$$

For $n = 1$ an object is a Δ^1-pread M and an object N of $\mathcal{A}$ such that M has faces $\emptyset$ and $N \times P_1$.

Lemma 3.3. *For a ball complex L consider $\mathcal{B} = pre_{\mathcal{A}}(L)$ as a graded category. Then there is a natural equivalence of the form*

$$pre_{\mathcal{A}}(K \times L) \cong pre_{\mathcal{B}}(K).$$

Proof. We have a natural equivalences of categories over $\mathbb{Z}$

$$\mathcal{C}ell(K) \wedge_{\mathbb{Z}/2} \mathcal{C}ell(L) \cong \mathcal{C}ell(K \times L).$$

Here, the left hand side has objects pairs of oriented cells under the obvious identification for multiplication with $\emptyset$ and for product orientations. The claim follows from the adjunction between products and functor sets. □

Proposition 3.4. *Let $ad/S_n(K)$ be the set of pre K-ads in $\mathcal{A}(S_n)$ which give $(K \times \sigma)$-ads in $\mathcal{A}$ under the adjunction of 3.3 for each cell σ of Δ^n. Then ad/S_n defines an ad theory.*

Proof. The set ad/S_n clearly is pointed and full. Suppose that we are given a pre K-ad in $\mathcal{A}(S_n)$ which restricts to an adjoint of a $\tau \times \sigma$-ad for every $\tau \in K$ then its adjoint restricts to a $K \times \sigma$-ad by locality and hence is an ad.

Next, we check the gluing property. A subdivision K' of K defines the subdivision $K' \times \sigma$ of $K \times \sigma$. Hence a $K' \times \sigma$-ad can be glued to a $K \times \sigma$-ad and the claim follows.

The cylinder J in $\mathcal{A}$ takes a $K \times \sigma$-ad to a $K \times \sigma \times I$-ad and hence defines a cylinder for ad/S_n.

Finally, we have to show the stability axiom. Every k-morphism

$$\theta : \mathcal{C}ell(K_0, L_0) \longrightarrow \mathcal{C}ell(K_1, L_1)$$

induces a k-morphism

$$\theta \times id : \mathcal{C}ell(K_0 \times \sigma, L_0 \times \sigma) \longrightarrow \mathcal{C}ell(K_1 \times \sigma, L_1 \times \sigma).$$

Hence for a (K_0, L_0)-ad$/S_n$ M the ads induced by $(\theta \times id)^*$ of its adjoints assemble to $\theta^* M$. □

Example 3.5. For $n = 0$ an object of $ad/S_0(*)$ is a Δ^0-ad which in case of manifolds corresponds to a manifold without boundary. For $n = 1$ and $S_1 = (*)$ we have a manifold with an arbitrary boundary. The picture shows a manifold with a $\mathbb{Z}/3$ singularity, that is, an element of $ad/(\mathbb{Z}/3)(*)$.

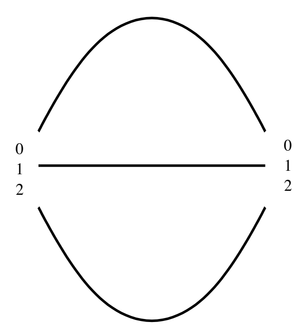

FIGURE 1

Next we investigate how the ad theories ad/S_n are related for different n. First observe that we have a map

$$\mu_{P_{n+1}} : \mathrm{ad}/S_n \longrightarrow \mathrm{ad}/S_n$$

of degree $\dim(P_{n+1})$ which multiplies the ads by P_{n+1}. Furthermore, there is a map of degree -1

$$\pi : \mathrm{ad}/S_n \longrightarrow \mathrm{ad}/S_{n+1}$$

which comes from considering an object of $\mathcal{A}(S_n)$ as an object of $A(S_{n+1})$ with $\pi M_\sigma = \emptyset$ if $n+1 \notin \sigma$. This certainly defines a pre K-ad $\pi(M)$ over S_{n+1} for each K-ad M over S_n.

Lemma 3.6. *$\pi(M)$ is an ad.*

Proof. We only check the top cell $\sigma = \Delta^n$. The other cells are similar. The adjoint of M gives a $K \times \Delta^n$-ad. The 1-morphism of graded categories

$$\mathcal{C}ell(\Delta^{n+1}) \longrightarrow \mathcal{C}ell(\Delta^{n+1}, \{n+1\} \cup \partial_{n+1}\Delta^{n+1}) \cong \mathcal{C}ell(\Delta^n)$$

can be multiplied with $\mathcal{C}ell(K)$ and hence gives the desired ad with the stability axiom. □

Finally, we have a map

$$\delta : \mathrm{ad}/S_{n+1} \longrightarrow \mathrm{ad}/S_n.$$

It takes a K-ad M over S_{n+1} to the K-ad over S_n given by the formula

$$\delta(M)(\sigma, o) = M(\sigma, o)_{|\{0,1,\ldots,n\}}.$$

Theorem 3.7. *Let $\Omega_*^{S_n}$ be the bordism group of the ad theory ad/S_n. Then the sequence*

$$\ldots \xrightarrow{\delta_*} \Omega_*^{S_n} \xrightarrow{\mu_{P_{n+1}*}} \Omega_*^{S_n} \xrightarrow{\pi_*} \Omega_*^{S_{n+1}} \xrightarrow{\delta_*} \ldots$$

is exact.

Proof. The proof is essentially the same as in [Baa73b]3.2. □

Example 3.8. Consider the sequence $S = (\emptyset, \emptyset, \ldots)$. Using the suspension axiom it is not hard to see that ad/S_n consists of $n+1$ copies of the original Quinn spectrum. Hence the above exact sequence consists of short split exact sequences.

Example 3.9. Let R be a ring and suppose $x \in R$ is a non zero divisor. Consider the ad theory of example 2.2. Then the maps of spectra induced by μ_x, π and δ corresponds to the Bockstein exact sequence in singular homology.

It is interesting to ask which multiplicative structures are inherited from an ad theory to its S_n-ad theory ad/S_n. Clearly, we have a product

$$\mathrm{ad}(K) \times \mathrm{ad}(L)/S_n \longrightarrow \mathrm{ad}(K \times L)/S_n$$

and hence we have

Corollary 3.10. *The Quinn spectrum of the ad theory with singularities is a strict module spectrum over the original Quinn spectrum.*

We will investigate further product structures in the last section.

4. Example: Manifolds with singularities and assemblies

In this section we look at the ad theory of compact manifolds. For simplicity we restrict our attention to the unoriented topological case. It will then be clear how to do other cases of bordism theories.

We fix a sequence S of closed manifolds and write $Q[X]$ for the Quinn spectrum of $\mathrm{ad}[X]/S_n$. Here, X is a topological space and $\mathrm{ad}[X]$ is defined as in example 2.3 for singular manifolds in X. In the following, we call a simplicial set without degeneracies a 'semi simplicial set' (another name in the literature is 'Δ-set'.)

Proposition 4.1. *Suppose F is a functor from semi simplicial sets to spectra with the property that*

$$X \mapsto \pi_*(F[X])$$

is a homotopy invariant. Then there is an 'assembly' map

$$F[*] \wedge X_+ \longrightarrow F[X]$$

which is the identity for X a point and which is natural up to homotopy.

Proof. This result is well known. For the reader's convenience we sketch the argument: for a semi simplicial set X we have the natural homotopy equivalence

$$\operatorname*{hocolim}_{\Delta^n \to X} F[|\Delta^n|] \longrightarrow \operatorname*{hocolim}_{\Delta^n \to X} F[*] \cong \operatorname*{colim}_{\Delta^n \to X} F[*] \wedge \Delta^n_+ \cong F[*] \wedge X_+$$

whose homotopy inverse can be composed with the map

$$\operatorname*{hocolim}_{\sigma:\Delta^n \to X} F[|\Delta^n|] \longrightarrow \operatorname*{colim}_{\sigma:\Delta^n \to X} F[|\Delta^n|] \overset{(F[|\sigma|])}{\longrightarrow} F[|X|].$$

$\square$

Theorem 4.2. *The spectrum $Q = Q[*]$ represents the homology theory of manifolds with singularities of* [Baa73b].

Proof. We first show that the bordism groups of $\mathrm{ad}[X]/S_n$ are naturally equivalent to the bordism groups of manifolds with singularities S_n in X. For that, recall that a *-ad in $\mathrm{ad}(X)/S_n$ consists of ads M_σ with $M_\sigma = \emptyset$ if $0 \notin \sigma$ and a system of compatible isomorphisms

$$\partial_i M_{(\sigma,i)} \cong M_\sigma \times P_i.$$

Hence, it defines a closed S_n-manifold and a closed S_n-manifold gives a $*$-ad. A null bordism is a family of $I \times \sigma$-ads with one end empty and the other is the bounding object. Let us illustrate the situation for $n = 1$: the null bordism takes the form

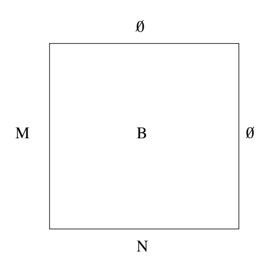

Figure 2

with $M_0 \times P_1 \cong N_0 \times P_1$. For example in the case of $\mathbb{Z}/3$-manifold considered earlier a null bordism can be pictured as follows:

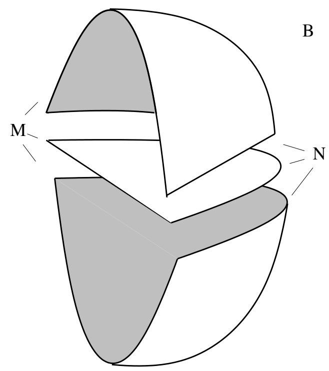

FIGURE 3

This shows that the bordism groups coincide. In particular, Q is a homotopy invariant functor and hence the assembly map is well defined by the preceding proposition.

Finally, we have to show that the assembly map is a homotopy equivalence. Using the fact that bordism of singular S_n-manifolds defines a homology theory we know that the functor

$$(X, Y) \mapsto \pi_*(Q[X], Q[Y])$$

together with the boundary operator defines a homology theory as well. Thus the assembly map defines a natural transformation between homology theories and is an isomorphism for a point. Thus the claim follows from the comparison theorem of homology theories. $\square$

5. A Bousfield-Kan spectral sequence

The exact sequences of the ad/S_n-bordism groups for different n are part of a spectral sequence of Bousfield-Kan type. In the case of classical complex bordism it first has been developed in [Mor79].

Let $S = (P_1, P_2, \dots,)$ be a sequence of *-ads and let n be fixed. Let $\mathcal{A}\langle S_n\rangle$ be the graded category with objects

(i) a Δ^n-pread M with

$$M_\sigma = \emptyset \quad \text{ if } 0 \notin \sigma$$

(ii) isomorphisms for each $i \notin \sigma$

$$f_{\sigma,i} : M_{(\sigma,i)} \xrightarrow{\cong} M_\sigma \otimes P_i$$

which are compatible with the face maps. Moreover, $\partial_0 M = \emptyset$ and for all $i, j > 0$ the diagram of 3.1(ii) commutes.

For a subset T of $\{1, \ldots, n\}$ let S_T be the subsequence of S indexed by T and let $\delta^i S_T$ be the sequence obtained from S_T with ith entry omitted. Consider the cubical diagram of graded categories with vertices $\mathcal{A}\langle S_T\rangle$ and face functors

$$\partial_k : \mathcal{A}\langle S_T\rangle \longrightarrow \mathcal{A}\left\langle \delta^k S_T\right\rangle$$

given by

$$\partial_k(M_\sigma) = M_{(\sigma,k)}.$$

There is a cubical diagram of ad theories $ad\langle S\rangle$ with this underlying graded category: the K-ads are those preads which are K-ads in $\mathcal{A}$, that is, for each cell σ of Δ^n the M_σ are K-ads in $\mathcal{A}$.

Lemma 5.1. (i) *For each vertex T there is an isomorphism*

$$ad\langle S_T\rangle \cong ad.$$

(ii) *The map*

$$\partial_{i*} : \pi_* Q(ad\langle S_T\rangle) \longrightarrow \pi_* Q(ad\left\langle \delta^i S_T\right\rangle)$$

induced by the face map is given under the above isomorphism by multiplication by P_i map on the bordism group Ω_*.

Proof. The proof is clear. □

Lemma 5.2. *Let $Q(ad\langle S\rangle)^+$ be the $n+1$-dimensional diagram indexed by subsets of $\{0, 1, \ldots, n\}$ without $\emptyset$ given by*

$$Q(ad\langle S\rangle)^+(T) = Q(ad\langle S_T\rangle)$$

if $0 \notin T$ and by $*$ else. Then we have

$$\operatorname{hocolim} Q(ad\langle S\rangle)^+ \simeq Q(ad/S).$$

Proof. The proof is an induction on the number of singularities. In the case of only one singularity the left hand side is the cokernel of the map of Quinn spectra induced by ∂_1. This map is given by the multiplication by P_1. On the other hand the map which considers an ad over $\mathcal{A}\langle\emptyset\rangle = \mathcal{A}$ as an object of $\mathcal{A}(S_1)$ can be composed with the multiplication by P_1 map. The cokernel of the induced map of spectra has the same homotopy type as $Q(ad/S)$ by the exact sequence of the previous section. Hence we get a map between the cokernels which induces an isomorphism on bordism groups. □

The homotopy colimit identification of the spectra with singularities furnishes a spectral sequence of Bousfield-Kan type [BK72] with E_2-term the homology of the chain complex

$$\cdots \longrightarrow \bigoplus_{\#T=k} \pi_* Q(ad\langle T\rangle) \xrightarrow{\partial} \bigoplus_{\#T=k-1} \pi_* Q(ad\langle T\rangle) \longrightarrow \cdots.$$

with $\partial = \sum(-1)^k \partial_k$. This gives

Theorem 5.3. *There is a spectral sequence converging to the bordism groups of ad/S_n with E_2-term the homology of the Koszul complex $K(P_1, \ldots, P_n)$, that is, the tensor product over Ω_* of the complexes*

$$0 \longrightarrow \Omega_* \xrightarrow{\cdot P_k} \Omega_* \longrightarrow 0\,.$$

6. Product structures

This section picks up the the investigation of product structures. We start with a multiplicative ad theory and finite sequences $P = (P_1, P_2, \ldots, , P_n)$ and $Q = (Q_1, Q_2, \ldots, Q_m)$. Write (P, Q) for the sequence

$$(P_1, P_2, \ldots, , P_n, Q_1, Q_2, \ldots, Q_m).$$

Proposition 6.1. *There is an external product*

$$\times : ad/P(K) \times ad/Q(L) \longrightarrow ad/(P, Q)(K \times L)$$

which is natural and associative.

Proof. Suppose M is a K ad mod P and N is an L ad mod Q. For a subset ρ of

$$\{0, 1, \ldots, n + m\}$$

set

$$\rho_0 = \rho \cap \{0, 1, \ldots n\}$$

and

$$\rho_1 = ((\rho \cap \{n + 1, n + 2, \ldots, n + m\}) - n) \cup (\rho \cap \{0\}).$$

Then the external product is given by

$$(M \times N)_\rho = M_{\rho_0} \times N_{\rho_1}.$$

The claimed properties are readily verified. □

Internal product structures are much harder to construct. In [Mor79] an internal product on the level of homotopy groups was obtained with the help of a retraction map which reduces the singularity of type (P, P) to P under suitable hypothesis. In order to rigidify the products we will proceed differently. Instead of looking for a retraction map we construct a new ad theory which comes with an internal product and is homotopy equivalent to the old one in good cases.

In this course we have to assume that the category $\mathcal{A}$ is a bipermutative category (see [May77], that is, it comes with an additive and a multiplicative permutative structure with strict two-sided units $\emptyset$ and $*$ and good distributive properties. In addition, we assume the strict left and right distributivity and that i strictly commutes with addition and multiplication. It is clear that these assumptions are too restrictive for most applications. It seems that the right way to proceed is the construction of an E_∞ operad which operates on the given spectra. This will be done somewhere else.

Definition 6.2. Let $\tau \in \Sigma_n$ be a permutation and $P = (P_1, P_2, \ldots, P_n)$ be a sequence. Let τP be the sequence $(P_{\tau_1}, \ldots, P_{\tau_n})$ and let

$$\tau^* : \mathcal{A}(P) \longrightarrow \mathcal{A}(\tau P)$$

be the map which sends an object to the object obtained by applying τ to the indexes and the orientations of the cells (the index 0 being fixed.)

Lemma 6.3. *The functor τ^* induces an isomorphism of ad theories*

$$\tau^* : ad/P \longrightarrow ad/(\tau P).$$

Proof. The proof is a consequence of the stability axiom. □

Consider the self map

$$1 + i^n \tau_{n,n} : ad/(P,P) \longrightarrow ad/(P,P)$$

where $\tau_{n,n}$ is the permutation which twists the two blocks of length n. An ad in the image of this map has the property that for each oriented cell of the ball complex we have a Δ^{2n}-ad whose kth face for $k \leq n$ coincides with the $k+n$ th face after permuting the two summands and applying i^n. Moreover, the object on the top cell is twice the original object.

Definition 6.4. Let P be an arbitrary sequence. An K-ad of $ad/(P,P)$ is said to be *close to* a K-ad/P if for each oriented cell of K the value M satisfies:

$$\partial_k M = i^n \partial_{k+n} M \text{ for all } 0 \leq k \leq n$$

and the same holds for the maps induced by the inclusions into the top cell. In other words, M is fixed under the action of $i^n \tau_{n,n}$. We write $cl(ad/P)$ for all ads in $ad/(P,P)$ which are isomorphic to ones which are close to ad/P. We say that ad/P is *well behaved* if each $cl(ad/P)$ is an ad theory.

Definition 6.5. Let $\pi : ad/P \longrightarrow ad/(P,P)$ be the inclusion map considered earlier. It comes from the map $\mathcal{A}(P) \longrightarrow \mathcal{A}(P,P)$ which fills $\emptyset$ in the faces which do not contain the last n indices. Set

$$\rho_P = (1 + i^n \tau_{n,n})\pi$$

and let $ad//P$ be the colimit of the sequence

$$ad/P \xrightarrow{\rho_P} cl(ad/P) \xrightarrow{\rho_{(P,P)}} cl(ad/(P,P)) \xrightarrow{\rho_{((P,P),(P,P))}} \cdots$$

Theorem 6.6. *Let ad/P be well behaved. Then we have*

(i) *$ad//P$ is a multiplicative ad theory.*
(ii) *the canonical map from ad to $ad//P$ respects the multiplication.*
(iii) *the canonical map from the spectrum $Q(ad/P)$ to $Q(ad//P)$ is a homotopy equivalence if 2 is inverted, P is regular and the cylinder of P admits an involution reversing isomorphism.*

Proof. The product of two ads M, N of the colimit, say in $cl(ad/(P,\dots,P))$ is given by their symmetrized exterior product $(1 + i^n \tau_{n,n})(M \times N)$. This definition is independent of n by the hypothesis on $\mathcal{A}$. Clearly, the product is compatible with the map from ad.

The last assertion is more involved. It relies on arguments which are similar to the ones given in [Mor79] for complex bordism. For simplicity we look at the case of only one singularity P of dimension m. We have short exact sequences

$$0 \longrightarrow \Omega_{*-m} \xrightarrow{P} \Omega_* \longrightarrow \Omega_*^P \longrightarrow 0$$

$$0 \longrightarrow \Omega_*^P \longrightarrow \Omega_*^{(P,P)} \longrightarrow \Omega_{*-m-1}^P \longrightarrow 0.$$

In particular, $\Omega_*^{(P,P)}$ is a free $\Omega_*^P \cong \Omega/P$-module on the generator 1 and a generator δ of dimension $m+1$. (We used here the fact that the obstruction for the vanishing of the multiplication by P map in a theory with singularities which contain P can be described by the bordism class of the mapping torus of P, see [JW75] for the classical case).

A convenient choice of δ is provided by the the suspension of P, that is the cylinder of δ on the top cell and with P as first and second face. It maps to the unit of Ω^P_*. Since the cylinder of P admits an involution reversing isomorphism we see that $i\tau_{1,1}\delta$ is isomorphic to $i\delta$. Hence, $1 + i\tau_{1,1}$ annihilates δ in the bordism group.

Hence, the map
$$Q(ad/P) \longrightarrow Q(cl(ad/P))$$
is a weak equivalence. An inverse on the level of homotopy groups is given by
$$\pi_*Q(cl(ad/P)) \longrightarrow \pi_*Q(ad/(P,P)) \longrightarrow \pi_*Q(ad/P),$$
the last map being induced by $(1 + i\tau_{1,1})/2$.

The same method apples to the other maps of the colimit. Note that the obstructions for the vanishing of the multiplication by P map vanish and hence we get short exact sequences and can proceed as before. The general case for arbitrary many singularities is analogues. □

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Fakultet for matematikk, NTNU,7491 Trondheim, Norway

Fakultät für Mathematik, Ruhr-Universität Bochum, 44780 Bochum, Germany