On zeta functions for E_∞ ring spectra

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Abstract

Homology and cohomology theories in topology are highly influenced by arithmetic ideas. One focus to study those prominent examples such as K-theory, bordism theories and singular (co-)homology is to concentrate on their representing ring spectra K, KO, MG and HG. John Rognes studied Galois extensions of structured ring spectra and worked towards an algebraic number theory of ring spectra. In this paper the idea of a zeta function for ring spectra is discussed.

1 The Riemann zeta function

As one knows from elementary number theory the integers contain a deep structure. Their arithmetic secrets are encoded into an analytic function, its Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p (1 - p^{-s})^{-1}$$

for $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$. Its representation as an Euler product encodes the fact that prime factorization is unique. The famous prime distribution theorem

$$\pi(x) \sim \frac{x}{\ln x}$$

first formulated Gauss principally relies on the fact that $\zeta(s) \neq 0$ for $\operatorname{Re} s = 1$. Under assumption of Riemann hypothesis that all nontrivial zeros have $Re\ s = \frac{1}{2}$ a much stronger distribution theorem would hold. Philosophically the Riemann zeta function encodes deep arithmetic secrets of the ring of integers $\mathbb{Z}$. For a number field $k/\mathbb{Q}$ let $\mathfrak{o}_k$ denote the ring of integers in k. Considering the ideals $\mathfrak{a} \subset \mathfrak{o}_k$, the Dedekind zeta function

$$\zeta_k(s) = \sum_{\mathfrak{a} \subset \mathfrak{o}_k} \frac{1}{\mathcal{N}(\mathfrak{a})^s} = \prod_{\mathfrak{p}} \frac{1}{1 - \mathcal{N}(\mathfrak{p})^{-s}}$$

with $\mathcal{N}(\mathfrak{a}) = |\mathfrak{o}_k/\mathfrak{a}|$ takes over this role.

2 Analogues of ideals

Being in the category of E_∞ ring spectra we fix a spectrum E and consider all maps of spectra $\varphi : E \to F$ up to π_*-equivalence.

$$\begin{array}{ccc} & E & \\ {}^{\varphi}\swarrow & & \searrow^{\varphi'} \\ F & \overset{\theta}{\dashrightarrow} & F' \end{array}$$

with $\pi_*\theta : \pi_*F \to \pi_*F'$ being an isomorphism of rings. Focussing on finite phenomena we suggest a formal series

$$Z_E(s) := \sum_{\varphi : E \to F} \frac{1}{|\pi_*F|^s}$$

This definition is quite close to the Dedekind zeta function, but the question of convergence is not clear at all. Therefore we first concentrate on the case of Eilenberg-MacLane spectra and then ask for its meaning for K-theory, Morava K-theory and bordism spectra MG.

3 On the functors π_0 and H

For a ring homomorphism $f : R \to R'$ one gets a morphism of Eilenberg-MacLane spectra

$$Hf : HR \to HR'.$$

In other words we get a pair of functors

$$\text{Rings} \underset{\pi_0}{\overset{H}{\rightleftarrows}} E_\infty \text{ ring spectra.}$$

One might ask whether these functors are adjoint. But having an unique ring homomorphism $\mathbb{Z} \to R$, i.e. $Ring(\mathbb{Z}, R) = *$, the situation for

$$\pi_0 E_\infty(H\mathbb{Z}, HR)$$

is more complicated (see [DS02]).

4 Targets of Eilenberg-MacLane spectra

Reconstructing the Riemann zeta function in the world of E_∞ ring spectra, one takes $E = H\mathbb{Z}$ and has to check that an E_∞ map $\varphi : E \to F$ is only possible in the case that F is also an Eilenberg-MacLane spectrum. Then the only maps are

$$\varphi_n : H\mathbb{Z} \to H\mathbb{Z}/n$$

and we recover the Riemann zeta function

$$Z_{H\mathbb{Z}}(s) = \sum_{\varphi_n : H\mathbb{Z} \to H\mathbb{Z}/n} \frac{1}{|\pi_0 H\mathbb{Z}/n|^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s).$$

5 Remark on the K-theory case

In the case of K-theory we have $\pi_* K \cong \mathbb{Z}[u^{\pm 1}]$. There are a lot of maps $\varphi : K \to F$ with $\pi_0 F$ infinite. These maps cannot be detected by our zeta function. Since spectra whose homotopy groups are torsion contribute to the zeta function, one could turn to reductions of the spectra with respect to primes and count local data. This is work in progress.

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