Artin-Schreier classes for higher bordism theories

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Abstract

The construction of Artin-Schreier classes plays an essential role in the construction of direct summands of higher bordism spectra such as MSU and $MString^{\mathbb{C}}$. Making use of Chern classes and linear algebra techniques a suitable rational class is constructed. It is shown that this class can also be represented by an associated manifold.

In the $K(1)$-local world at the prime $p = 2$, we take the fiber sequence $S \to KO \overset{\psi^3-1}{\to} KO$ and look at the homotopy long exact sequence

$$... \to \pi_0 S^0 \longrightarrow KO_0 \overset{\psi^3-1}{\longrightarrow} KO_0 \longrightarrow \pi_{-1} S^0 \to ...$$

Since $KO_0 \cong \mathbb{Z}_2$ are the 2-adic integers and ψ^3 is a ring homomorphism, $\psi^3 - 1$ is the zero map on KO_0. Thus $KO_0 \to \pi_{-1} S^0$ is injective and the image of 1 is a non-trivial element $\zeta \in \pi_{-1} S^0 \cong \mathbb{Z}_2$. Now we are attaching a 0-cell along ζ and take the homotopy pushout in the category of E_∞ spectra:

$$\begin{array}{ccc} S^{-1} & \overset{\zeta}{\longrightarrow} & S^0 \\ \downarrow{\scriptstyle *} & & \\ D^0 & & \end{array} \qquad\qquad \begin{array}{ccc} TS^{-1} & \overset{\zeta}{\longrightarrow} & S^0 \\ \downarrow{\scriptstyle T*} & & \downarrow \\ TD^0 = S^0 & \longrightarrow & T_\zeta \end{array}$$

This E_∞ spectrum T_ζ will be an E_∞ summand in MSU. For this

$$\begin{array}{cccc} TS^{-1} & \overset{\zeta}{\longrightarrow} & S^0 & \\ \downarrow{\scriptstyle T*} & & \downarrow & \searrow \\ TD^0 = S^0 & \longrightarrow & T_\zeta & \\ & \searrow & & \searrow \\ & & & MSU \end{array}$$

we have to show that $\zeta \in \pi_{-1} MSU$ vanishes. Considering the diagram

$$\begin{array}{ccccc} & & KO_0 S^0 & \longrightarrow & \pi_{-1} S^0 \\ & & \downarrow & & \downarrow \\ KO_0 MSU & \overset{\psi^3-1}{\longrightarrow} & KO_0 MSU & \longrightarrow & \pi_{-1} MSU \end{array}$$

it is sufficient to find an element $b \in KO_0 MSU$ mapping to 1, because on the one hand the element $1 \in KO_0 S^0$ maps to $1 \in KO_0 MSU$ going to $0 \in \pi_{-1} MSU$ due to the long exact sequence, and on the other hand the element $1 \in KO_0 S^0$ maps to $\zeta \in \pi_{-1} S^0$, which has to vanish in $\pi_{-1} MSU$ because the diagram commutes.

Definition 1 *An Artin-Schreier class is a class $b \in KO_0 MSU$ with $\psi^3 b = b + 1$.*

In the following part we construct such a class rationally and then give a construction of an SU-manifold which realizes this class.

1 The image of $MSU_* \to MU_*$

In MSU_* every torsion is 2-torsion which is the kernel of $MSU_* \to MU_*$ concentrated in dimensions $8k+1$ and $8k+2$ for $k \geq 0$; in these cases $MSU_{8k+1} \cong MSU_{8k+2}$ is an $\mathbb{F}_2$ vector space whose dimension is the number of partitions of k (compare [CF66b]). Due to a theorem by Thom, complex bordism is rationally represented by complex projective spaces:

Theorem 1 (Thom)

$$MU_* \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{CP}^n | n \geq 1].$$

The obstruction for a U-manifold to be an SU-manifold is the first Chern class c_1 of the tangent bundle. Hence a manifold $M \in MSU_4$ is rationally a linear combination

$$M = A \cdot \mathbb{CP}^1 \times \mathbb{CP}^1 + B \cdot \mathbb{CP}^2 \quad \text{with} \quad c_1^2[M] = 0;$$

in the above notation we always mean their bordism classes and have omitted the brackets for brevity. An example of an SU-manifold is the Kummer surface

$$\mathcal{K} = K3 = \{z \in \mathbb{CP}^3 | z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}$$

which is U-bordant to $K3 \sim_U 18(\mathbb{CP}^1)^2 - 16\mathbb{CP}^2$. Indeed $MSU_4 = \mathbb{Z}\langle K3 \rangle$ since the Todd-genus ($\hat{A}$-genus respectively) of an SU-manifold is even and $Td(K3) = 2$. It turns out that we cannot construct an Artin-Schreier class out of a class in MSU_4 since we need an SU-manifold with $\hat{A} = 1$. Therefore we are interested in the image of $MSU_8 \to MU_8$. Rationally this is a linear combination

$$M = A \cdot \mathbb{CP}^4 + B \cdot \mathbb{CP}^1 \times \mathbb{CP}^3 + C \cdot (\mathbb{CP}^2)^{\times 2} + D \cdot (\mathbb{CP}^1)^{\times 4} + E \cdot (\mathbb{CP}^1)^{\times 2} \times \mathbb{CP}^2;$$

requiring the first Chern class to vanish implies the conditions $c_1^4[M] = c_1 c_3[M] = c_1^2 c_2[M] = 0$ in the Chern numbers. To express them as linear equations in the coefficients we first have to calculate the total Chern classes of the complex projective spaces and their products:

$$
\begin{aligned}
c(T\mathbb{CP}^4) &= c(1\oplus T\mathbb{CP}^4) = c(5L^*) = (1+x)^5 = 1+5x+10x^2+10x^3+5x^4\\
c(T(\mathbb{CP}^1\times\mathbb{CP}^3)) &= pr_1^*c(T\mathbb{CP}^1)\cdot pr_2^*c(T\mathbb{CP}^3) = (1+x_1)^2(1+x_2)^4\\
&= (1+2x_1)(1+4x_2+6x_2^2+4x_2^3)\\
&= 1+(2x_1+4x_2)+(8x_1x_2+6x_2^2)+(12x_1x_2^2+4x_2^3)+8x_1x_2^3\\
c(T(\mathbb{CP}^2\times\mathbb{CP}^2)) &= pr_1^*c(T\mathbb{CP}^2)\cdot pr_2^*c(T\mathbb{CP}^2) = (1+x_1)^3(1+x_2)^3\\
&= (1+3x_1+3x_1^2)(1+3x_2+3x_2^2)\\
&= 1+(3x_1+3x_2)+(3x_1^2+9x_1x_2+3x_2^2)+(9x_1^2x_2+9x_1x_2^2)+9x_1^2x_2^2\\
c(T(\mathbb{CP}^1)^{\times 4}) &= (1+x_1)^2(1+x_2)^2(1+x_3^2)(1+x_4)^2\\
&= (1+2x_1)(1+2x_2)(1+2x_3)(1+2x_4)\\
&= 1+2(x_1+x_2+x_3+x_4)\\
&\quad +4(x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4)\\
&\quad +8(x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4)+16x_1x_2x_3x_4\\
c(T((\mathbb{CP}^1)^2\times\mathbb{CP}^2)) &= (1+x_1)^2(1+x_2)^2(1+x_3)^3 = (1+2x_1)(1+2x_2)(1+3x_3+3x_3^2)\\
&= 1+(2x_1+2x_2+3x_3)+(4x_1x_2+6x_1x_3+6x_2x_3+3x_3^2)\\
&\quad +(6x_1x_3^2+6x_2x_3^2+12x_1x_2x_3)+12x_1x_2x_3^2
\end{aligned}
$$

Now we calculate the Chern numbers $c_1^4(TM)[M]$, $c_1c_3(TM)[M]$ and $c_1^2c_2(TM)[M]$ by evaluating them on the complex projective spaces:

$$
\begin{aligned}
c_1^4(T\mathbb{CP}^4)[\mathbb{CP}^4] &= (5x)^4[\mathbb{CP}^4] = 625\\
c_1^4[\mathbb{CP}^1\times\mathbb{CP}^3] &= (2x_1+4x_2)^4[\mathbb{CP}^1\times\mathbb{CP}^3] = 512x_1x_2^3[\mathbb{CP}^1\times\mathbb{CP}^3] = 512\\
c_1^4[\mathbb{CP}^2\times\mathbb{CP}^2] &= 3^4(x_1+x_2)^4[\mathbb{CP}^2\times\mathbb{CP}^2] = 486x_1^2x_2^2[\mathbb{CP}^2\times\mathbb{CP}^2] = 486\\
c_1^4[(\mathbb{CP}^1)^{\times 4}] &= 2^4(x_1+x_2+x_3+x_4)^4[(\mathbb{CP}^1)^{\times 4}]\\
&= 2^4\cdot 4!\cdot x_1x_2x_3x_4[(\mathbb{CP}^1)^{\times 4}] = 384\\
c_1^4[(\mathbb{CP}^1)^{\times 2}\times\mathbb{CP}^2)] &= (2x_1+2x_2+3x_3)^4[(\mathbb{CP}^1)^{\times 2}\times\mathbb{CP}^2)]\\
&= 432x_1x_2x_3^2[(\mathbb{CP}^1)^{\times 2}\times\mathbb{CP}^2)] = 432
\end{aligned}
$$

gives the equation

$$c_1^4[M] = 0 = 625A+512B+486C+384D+432E,$$

evaluation of $c_1c_3(TM)[M]$

$$
\begin{aligned}
c_1c_3(T\mathbb{CP}^4)[\mathbb{CP}^4] &= 5x\cdot 10x^3[\mathbb{CP}^4] = 50\\
c_1c_3[\mathbb{CP}^1\times\mathbb{CP}^3] &= (2x_1+4x_2)(12x_1x_2^2+4x_2^3)[\mathbb{CP}^1\times\mathbb{CP}^3] = 56x_1x_2^3[\mathbb{CP}^1\times\mathbb{CP}^3] = 56\\
c_1c_3[\mathbb{CP}^2\times\mathbb{CP}^2] &= (3x_1+3x_2)(x_1^3+9x_1^2x_2+9x_1x_2^2+x_2^3)[\mathbb{CP}^2\times\mathbb{CP}^2]\\
&= 54x_1^2x_2^2[\mathbb{CP}^2\times\mathbb{CP}^2] = 54\\
c_1c_3[(\mathbb{CP}^1)^{\times 4}] &= 16(x_1+x_2+x_3+x_4)(x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4)[(\mathbb{CP}^1)^{\times 4}]\\
&= 64x_1x_2x_3x_4[(\mathbb{CP}^1)^{\times 4}] = 64\\
c_1c_3[(\mathbb{CP}^1)^{\times 2}\times\mathbb{CP}^2)] &= (2x_1+2x_2+3x_3)(6x_1x_3^2+6x_2x_3^2+12x_1x_2x_3)[(\mathbb{CP}^1)^{\times 2}\times\mathbb{CP}^2)]\\
&= 60x_1x_2x_3^2[(\mathbb{CP}^1)^{\times 2}\times\mathbb{CP}^2)] = 60
\end{aligned}
$$

gives the equation

$$c_1c_3[M] = 0 = 50A + 56B + 54C + 64D + 60E,$$

and evaluation of $c_1^2c_2(TM)[M]$

$$\begin{aligned}
c_1^2c_2(T\mathbb{CP}^4)[\mathbb{CP}^4] &= (5x)^2 \cdot 10x^2[\mathbb{CP}^4] = 250x^4[\mathbb{CP}^4] = 250 \\
c_1^2c_2[\mathbb{CP}^1 \times \mathbb{CP}^3] &= (2x_1 + 4x_2)^2(x_1^2 + 8x_1x_2 + 6x_2^2)[\mathbb{CP}^1 \times \mathbb{CP}^3] \\
&= 224x_1x_2^3[\mathbb{CP}^1 \times \mathbb{CP}^3] = 224 \\
c_1^2c_2[\mathbb{CP}^2 \times \mathbb{CP}^2] &= (3x_1 + 3x_2)^2(3x_1^2 + 9x_1x_2 + 3x_2^2)[\mathbb{CP}^2 \times \mathbb{CP}^2] \\
&= 216x_1^2x_2^2[\mathbb{CP}^2 \times \mathbb{CP}^2] = 216 \\
c_1^2c_2[(\mathbb{CP}^1)^{\times 4}] &= 16(x_1 + x_2 + x_3 + x_4)^2 \times \\
&\quad (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)[(\mathbb{CP}^1)^{\times 4}] \\
&= 192x_1x_2x_3x_4[(\mathbb{CP}^1)^{\times 4}] = 192 \\
c_1^2c_2[(\mathbb{CP}^1)^{\times 2} \times \mathbb{CP}^2)] &= (2x_1 + 2x_2 + 3x_3)^2 \times \\
&\quad (4x_1x_2 + 6x_1x_3 + 6x_2x_3 + 3x_3^2)[(\mathbb{CP}^1)^{\times 2} \times \mathbb{CP}^2)] \\
&= 204x_1x_2x_3^2[(\mathbb{CP}^1)^{\times 2} \times \mathbb{CP}^2)] = 204
\end{aligned}$$

gives the equation

$$c_1^2c_2[M] = 0 = 250A + 224B + 216C + 192D + 204E.$$

Hence we consider the system of linear equations

$$\begin{array}{rrrrrrrrrrr}
c_1^4[M]= & 0= & 625\text{A} & + & 512\text{B} & + & 486\text{C} & + & 384\text{D} & + & 432\text{E} \\
c_1c_3[M]= & 0= & 50\text{A} & + & 56\text{B} & + & 54\text{C} & + & 64\text{D} & + & 60\text{E} \\
c_1^2c_2[M]= & 0= & 250\text{A} & + & 224\text{B} & + & 216\text{C} & + & 192\text{D} & + & 204\text{E}
\end{array}$$

which is integrally equivalent to the following system of homogeneous linear equations:

$$\begin{array}{rrrrrrrrrr}
0= & 25\text{A} & + & 8\text{B} & & & & & & \\
0= & & + & 4\text{B} & & & + & 16\text{D} & + & 9\text{E} \\
0= & & & & - & 27\text{C} & + & 48\text{D} & + & 15\text{E}
\end{array}$$

The space of solutions is 2-dimensional. We know one solution $K3 \times K3$, i.e. the square of the Kummer surface, having the parameter representation

$$(A, B, C, D, E) = (0, 0, 256, 324, -576)$$

or

$$\mathcal{K}^2 = K3 \times K3 \sim_U 256\mathbb{CP}^2 \times \mathbb{CP}^2 + 324(\mathbb{CP}^1)^{\times 4} - 576(\mathbb{CP}^1)^{\times 2} \times \mathbb{CP}^2.$$

Another independent solution is given in parameter representation as $(A, B, C, D, E) = (8, -25, -12, -23, 52)$ or as

$$N := 8\mathbb{CP}^4 - 25\mathbb{CP}^1 \times \mathbb{CP}^3 - 12\mathbb{CP}^2 \times \mathbb{CP}^2 - 23(\mathbb{CP}^1)^{\times 4} + 52(\mathbb{CP}^1)^{\times 2} \times \mathbb{CP}^2.$$

Hence we can rationally describe bordism classes of SU-manifolds under the injection $MSU_8 \to MU_8$ via

$$M = k \cdot (K3)^2 + l \cdot N$$

with $k, l \in \mathbb{Q}$. In the next section we take the values $(k, l) = (\frac{1}{4}, 12)$ and study its K-theory class under the map $MU_* \to K_*MU$ using Miscenkos formula which gives us an Artin-Schreier class.

2 Formal group laws and Miscenkos formula

Formal group laws

In the following part we briefly recall the notions of the theory of formal group laws which we use to construct the morphism $MU_* \to K_*MU$. We restrict to commutative, one-dimensional formal group laws.

Definition 2 *Let R be a commutative ring with unit. A formal group law over R is a power series $F(x,y) \in R[\![x,y]\!]$ satisfying*

1. $F(x,0) = x = F(0,x)$
2. $F(x,y) = F(y,x)$
3. $F(x,F(y,z)) = F(F(x,y),z)$.

These axioms correspond to the existence of a neutral element, commutativity and associativity in the group case. Obviously we can write $F(x,y) = x + y + \sum_{i,j\geq 1} a_{ij}x^iy^j$ with $a_{ij} = a_{ji}$, and in terms of the power series it is clear that there exists an inverse, i.e. a formal power series $\iota(x) \in R[\![x]\!]$ such that $F(x,\iota(x)) = 0$. Formal group laws are naturally related to complex oriented theories in the following way: The Euler class of a tensor product of line bundles defines a formal group law

$$\widehat{G_E}(x,y) = e(L_1 \otimes L_2) \in E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \pi_*E[\![x,y]\!]$$

with $x = e(L_1)$ and $y = e(L_2)$.

Example 1 *The additive formal group law $\mathbb{G}_a(x,y) = x+y$ arises as an orientation of singular cohomology. The multiplicative formal group law $\mathbb{G}_m(x,y) = x+y-xy$ comes up as an orientation of complex K-theory. In the following we will encounter the universal formal group law F_u via complex cobordism (MU-theory).*

Definition 3 *Let F and G be formal group laws. A homomorphism $f : F \to G$ is a power series $f(x) \in R[\![x]\!]$ with constant term 0 such that $f(F(x,y)) = G(f(x),f(y))$. It is an isomorphism if it is invertible, i.e. if $f'(0)$ (the coefficient of x) is a unit in R, and a strict isomorphism if $f'(0) = 1$. A strict isomorphism from F to the additive formal group law $\mathbb{G}_a$ is called a logarithm for F, denoted $\log_F(x)$. Its inverse power series is called exponential, denoted $\exp_F(x)$.*

Example 2 *Over a $\mathbb{Q}$-algebra every formal group law is isomorphic to the additive formal group law. Especially the logarithm of the universal formal group law is given by*

$$\log_{MU}(x) = \sum_{n\geq 0} \frac{[\mathbb{CP}^n]}{n+1} x^{n+1}.$$

Proposition 2 *If x_1, x_2 are two complex orientations for $E^*(-)$, then their associated formal group laws F_1 and F_2 are isomorphic.*

In the context of formal group laws let F_{MU} denote the universal formal group law

$$F_{MU}(x, y) = x + y + \sum_{i,j\geq 1} a_{ij} x^i y^j$$

with the coefficients $a_{ij} \in L$ in the Lazard ring with degree $|a_{ij}| = 2 - 2(i + j)$. Let

$$F_K(x, y) = x + y + vxy$$

denote the multiplicative formal group law corresponding to the K-theory spectrum with v the inverse Bott element with $|v| = -2$. Now we are going to construct a morphism

$$f : MU_* \to K_*MU$$

such that the induced formal group law

$$f^*F_{MU}(x, y) := x + y + \sum_{i,j\geq 1} f(a_{ij}) x^i y^j$$

is the formal group law F_K twisted by the invertible power series

$$g(x) = \sum_{i\geq 0} b_i x^{i+1}$$

(with $b_0 = 1$) defined by

$${}^gF_K(x, y) := g(F_K(g^{-1}(x), g^{-1}(y))) = g(g^{-1}(x) + g^{-1}(y) + vg^{-1}(x)g^{-1}(y))$$

with $g^{-1}(g(x)) = x$ the inverse function.

Boardman homomorphism

The element $a_{ij} \in \pi_{2(i+j-1)}$ can be represented by a weakly almost complex manifold. To ask for the (normal) characteristic numbers of this manifold is (essentially) equivalent to asking for the image of a_{ij} under the Hurewicz homomorphism

$$\pi_*MU \to H_*MU.$$

We introduce the Boardman homomorphism, which is (slightly) more general than the Hurewicz homomorphism. Let E be a (commutative) ring spectrum, then for any (space or spectrum) Y we consider the map

$$Y \cong S^0 \wedge Y \stackrel{i\wedge 1}{\to} E \wedge Y.$$

Composing a map $X \to Y$ with this map induces a homomorphism

$$B : [X, Y]_* \to [X, E \wedge Y]_*$$

called the Boardman homomorphism. The Hurewicz homomorphism is recovered by setting $X = S^0$ and $E = H$ (the Eilenberg-MacLane spectrum representing singular homology).

Since $E \wedge Y$ is at least a module spectrum over the ring spectrum E, we may obtain information about $[X, E \wedge Y]_r = (E \wedge Y)^{-r}(X)$ from $E_*(X)$, for example there is a universal coefficient theorem

$$\begin{array}{ccc} [X,Y]_* & \xrightarrow{\quad B \quad} & [X, E\wedge Y]_* \\ & \searrow^{\alpha} \quad \swarrow^{p} & \\ & \mathrm{Hom}_{\pi_* E}(E_*X, E_*Y) & \end{array}$$

where $\alpha(f) = f_* : E_*X \to E_*Y$ is the induced map in E-homology and p is defined by $(p(h))(k) = \langle h, k \rangle \in E_*Y$ using the Kronecker pairing

$$(E \wedge Y)^*(X) \otimes E_*X \to E_*Y$$

with

$$h \otimes k \mapsto \langle h, k \rangle : S \to E \wedge X \overset{1\wedge h}{\to} E \wedge E \wedge Y \overset{\mu\wedge 1}{\to} E \wedge Y.$$

Miscenkos formula

We recall that power series of the form $g(x) = x + b_1x^2 + b_2x^3 + \ldots$ are strict isomorphisms

$$g : F \xrightarrow{\cong} {}^gF = g(F(g^{-1}x, g^{-1}y))$$

and want to give the explicit coefficients of the inverse power series $g^{-1}(x) = \sum_{i\geq 0} c_ix^{i+1}$. We calculate the first coefficients taking everything modulo x^6 and using the identity

$$\begin{aligned} x &\equiv g^{-1}(g(x)) = g(x) + c_1g(x)^2 + c_2g(x)^3 + c_3g(x)^4 + c_4g(x)^5 + \ldots \quad (\mod x^6) \\ &\equiv x + b_1x^2 + b_2x^3 + b_3x^4 + b_4x^5 \\ &\quad +c_1(x^2 + 2b_1x^3 + (2b_2 + b_1^2)x^4 + (2b_3 + 2b_1b_2)x^5) \\ &\quad +c_2(x^3 + 3b_1x^4 + (3b_2 + 3b_1^2)x^5 + c_3(x^4 + 4b_1x^5) + c_4x^5 \end{aligned}$$

Comparing coefficients gives the system of equations

$$\begin{aligned} 0 &= c_1 + b_1 \\ 0 &= c_2 + 2b_1c_1 + b_2 \\ 0 &= c_3 + 3b_1c_2 + c_1(2b_2 + b_1^2) + b_3 \\ 0 &= c_4 + 4b_1c_3 + c_2(3b_2 + 3b_1^2) + c_1(2b_3 + 2b_1b_2) + b_4 \end{aligned}$$

resulting in

$$\begin{aligned} c_1 &= -b_1 \\ c_2 &= 2b_1^2 - b_2 \\ c_3 &= -5b_1^3 + 5b_1b_2 - b_3 \\ c_4 &= 14b_1^4 - 21b_1^2b_2 + 6b_1b_3 + 3b_2^2 - b_4. \end{aligned}$$

Applying the residue theorem of complex analysis proves the following (as done in [Ada74, p. 65 Prop. (7.5)]):

Proposition 3 *Denoting the degree $2n$-part of an inhomogeneous polynomial with a lower index n we have*

$$c_n = \frac{1}{n+1}(\sum_{i\geq 0} b_i)_n^{-(n+1)} \text{ and } \quad b_n = \frac{1}{n+1}(\sum_{i\geq 0} c_i)_n^{-(n+1)}.$$

Next we explicitly calculate ${}^gF_K(x,y) = g(g^{-1}x + g^{-1}y + vg^{-1}xg^{-1}y)$:

$$\begin{aligned}
{}^gF_K(x,y) \quad = \quad & x + y + (v+2b_1)xy + (b_1v - 2b_1^2 + 3b_2)(x^2y + xy^2) \\
& +(2vb_2 - 2vb_1^2 + 4b_3 - 8b_1b_2 + 4b_1^3)(x^3y + xy^3) \\
& +(v^2b_1 - 3vb_1^2 + 2b_1^3 - 6b_1b_2 + 6vb_2 + 6b_3)x^2y^2 \\
& +(5vb_1^3 - 8vb_1b_2 + 25b_1^2b_2 + 3vb_3 - 10b_1^4 - 14b_1b_3 - 6b_2^2 + 5b_4) \\
& \quad \times (x^4y + xy^4) \\
& +(4vb_1^3 - 18vb_1b_2 - 4b_1^4 + 8b_1^2b_2 - 2v^2b_1^2 + 3v^2b_2 - 3b_2^2 \\
& \quad -16b_1b_3 + 12vb_3 + 10b_4) \times (x^3y^2 + x^2y^3) \\
& + \text{higher order terms.}
\end{aligned}$$

This implies:

$$\begin{aligned}
a_{11} & \mapsto v + 2b_1 \\
a_{21} & \mapsto vb_1 - 2b_1^2 + 3b_2 \\
a_{31} & \mapsto 2vb_2 - 2vb_1^2 + 4b_3 - 8b_1b_2 + 4b_1^3 \\
a_{22} & \mapsto v^2b_1 - 3vb_1^2 + 2b_1^3 - 6b_1b_2 + 6vb_2 + 6b_3 \\
a_{41} & \mapsto 5vb_1^3 - 8vb_1b_2 + 25b_1^2b_2 + 3vb_3 - 10b_1^4 - 14b_1b_3 - 6b_2^2 + 5b_4 \\
a_{32} & \mapsto 4vb_1^3 - 18vb_1b_2 - 4b_1^4 + 8b_1^2b_2 - 2v^2b_1^2 + 3v^2b_2 - 3b_2^2 - 16b_1b_3 + 12vb_3 + 10b_4
\end{aligned}$$

Recall that the complex manifold $\mathbb{CP}^n$ defines an element $[\mathbb{CP}^n] \in \pi_{2n}MU$. The Hurewicz homomorphism

$$\pi_*MU \to H_*MU$$

tells us that the image of $[\mathbb{CP}^n]$ in $H_{2n}MU$ is $(n+1)c_n$ since the formula $(\sum_{i\geq 0} b_i)_n^{-(n+1)}$ gives the normal Chern numbers of $\mathbb{CP}^n$. The most important formula for us will be

$$[\mathbb{CP}^n] = (n+1)c_n = (\sum_{i\geq 0} a_{1i})_n^{-1}$$

leading to

$$\boxed{\begin{aligned}
[\mathbb{CP}^1] &= -a_{11} \\
[\mathbb{CP}^2] &= -a_{12} + a_{11}^2 \\
[\mathbb{CP}^3] &= -a_{13} - a_{11}^3 + 2a_{11}a_{12} \\
[\mathbb{CP}^4] &= -a_{14} + a_{11}^4 + a_{12}^2 + 2a_{11}a_{13}
\end{aligned}}$$

Substituting these formulas we get

$$\begin{aligned}
[N] \quad = \quad & -112vb_1^3 + 340vb_1b_2 + 256b_1^2b_2 - 60vb_3 - 184b_1^4 + 40b_1b_3 \\
& +12b_2^2 - 40b_4 + 48v^2b_2 + 58v^2b_1^2 + 22v^3b_1
\end{aligned}$$

and

$$\frac{1}{4}[K3^2] = v^4 + 24v^3b_1 + 120v^2b_1^2 + 48v^2b_2 - 288vb_1^3 + 448vb_1b_2 \\ +144b_1^4 - 576b_1^2 + 576b_2^2.$$

Defining

$$M := \frac{1}{4}K3^2 + 12N$$

we get

$$[M] = v^4 + 16 \cdot (18v^3b_1 + 51v^2b_1^2 + 39v^2b_2 - 102vb_1^3 + 283vb_1b_2 \\ -45vb_3 - 129b_1^4 + 30b_1b_3 + 156b_1^2b_2 + 45b_2^2 - 30b_4).$$

3 Construction of an SU-manifold with $\hat{A} = 1$

To split off the spectrum T_ζ from MSU one essentially uses the existence of an Artin-Schreier class $b \in KO_0MSU$ satisfying $\psi^3 b = b + 1$. Via Miscenkos formula we have seen that such a class can be constructed with the logarithm construction if there is a Bott manifold whose associated K-theory class is congruent to v^4 modulo 16. Essentially we have to find a Bott manifold in SU bordism, i.e. an SU-manifold M with $\hat{A}([M]) = 1$ giving a periodicity element in MSU_*.

Main idea

The Hopf bundle $\sigma : S^7 \to S^4$ with fiber $S^3 \cong SU(2)$ on the one hand admits an SU structure and on the other hand generates $Im(J)_7 \cong \Omega_7^{fr} \cong \pi_7^{st} \cong \mathbb{Z}/240$. Since $Td(D(\sigma)) = 1/240$ and since $240[\sigma] = 0$ in Ω_7^{fr} implies the existence of a framed manifold R^8 with $\partial R^8 = -240\sigma$, we define

$$B := 240D(\sigma) \cup_{240\sigma} R^8$$

which serves as the desired Bott manifold, i.e. $Td(B) = \hat{A}(B) = 1$.

$Sp(1)$-principal bundles over S^4

With the identifications $Sp(1) \cong SU(2) \cong S^3$ and $Sp(2)/Sp(1) \cong S^7$ and

$$\frac{Sp(2)}{Sp(1) \times Sp(1)} \cong \mathbb{HP}^1 \cong S^4$$

we take the canonical $Sp(1)$-principal bundle over S^4

$$\begin{array}{ccc} Sp(1) \cong S^3 & \longrightarrow & S^7 \\ & & \downarrow \\ & & S^4 \end{array}$$

i.e. the bundle whose associated line bundle

$$E := S^7 \times_{Sp(1)} \mathbb{H}^1 \to S^4$$

satisfies $\langle c_2(E), [S^4] \rangle = 1$. We know that every G-principal bundle is given as the pullback of the universal G-principal bundle via the classifying map

$$\begin{array}{ccc} f^*EG & \longrightarrow & EG \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & BG. \end{array}$$

In other words the functor $G\text{-}Pb(-)$ is representable by BG and

$$[B, BG] \cong G\text{-}Pb(B) \text{ via } f \mapsto f^*EG.$$

In the case of $Sp(1)$-principal bundles over S^4 we get

$$[S^4, BSp(1)] = [\Sigma S^3, BSp(1)] \cong [S^3, \Omega BSp(1)] = [S^3, Sp(1)] = [S^3, S^3] \cong \mathbb{Z}.$$

The canonical $Sp(1)$-principal bundle over S^4 is associated to $1 \in \mathbb{Z}$. We see that the disk bundle $Q := D(E)$ with $\pi : Q \to S^4$ has as boundary $\partial Q = \partial D(E) = S(E)$ the original principal bundle.

Splitting of the tangent bundle TQ

In general for a smooth vector bundle $\xi : E \to M$ the total space E is again a smooth manifold. Now we are interested in the structure of the tangential bundle TE. There are two induced bundles, namely the induced tangential bundle and that of the total space:

$$\begin{array}{ccc} \xi^*TM & \longrightarrow & E \\ \downarrow & & \downarrow \xi \\ TM & \longrightarrow & M \end{array} \quad \text{and} \quad \begin{array}{ccc} \xi^*E & \longrightarrow & E \\ \downarrow & & \downarrow \xi \\ E & \xrightarrow{\xi} & M \end{array}$$

These already give an isomorphism

$$TE \cong \xi^*TM \oplus \xi^*E.$$

Such a splitting of a tangent bundle is geometrically called a connection. With the notation of above restricting the tangent bundle of the vector bundle to the disk bundle we get the splitting

$$TQ \cong \pi^*E \oplus \pi^*TS^4;$$

note that the second summand is stably trivial.

The Hopf bundle is an SU manifold

The Hopf bundle $\sigma : S^7 \to S^4$ with fiber $S^3 \cong SU(2)$ is not only an $SU(2)$-bundle but also an SU manifold. A manifold M has an SU structure if its stable tangent bundle TM is a complex vector bundle with a trivialization of its determinant bundle $det(TM) \cong 1_{\mathbb{C}}$.

$$\begin{array}{ccccc} D(\sigma) & \longrightarrow & \lambda_{taut} & \longrightarrow & \lambda_{taut} \\ \downarrow & & \downarrow & & \downarrow \\ S^4 & \longrightarrow & BSU & \longrightarrow & BU. \end{array}$$

From the splitting above we see the SU structure, since the 8-dimensional bundle splits into two 4-dimensional bundles and TS^4 is stably trivial and E is chosen to have vanishing c_1.

Evaluation of the Todd genus

We recall $Td = e^{c_1/2}\hat{A}$ and see that for SU manifolds the Todd-genus and the $\hat{A}$-genus coincide. From [Hi56] the degree 8-term of the Todd genus is given in Chern classes by:

$$T_4 = \frac{1}{720}(-c_4 + c_3c_1 + 3c_2^2 + 4c_2c_1^2 - c_1^4).$$

For the evaluation the Chern classes c_1 and c_4 do not contribute, due to the SU structure and since Q is a homotopy 4-sphere, respectively. Next we emphasize that while for closed stably almost complex manifolds the Todd genus maps to the integers; the situation for (U, fr) manifolds is different. A (U, fr) manifold M^n is a differentiable manifold M with a given complex structure on its stable tangent bundle TM and a given compatible framing of TM restricted to the boundary ∂M. Their Chern numbers depend only on the bordism classes in $\Omega_n^{U,fr}$ and hence we have a Todd genus

$$Td : \Omega_{2n}^{U,fr} \to \mathbb{Q}.$$

Moreover there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_{2n}^{U} & \longrightarrow & \Omega_{2n}^{U,fr} & \longrightarrow & \Omega_{2n-1}^{fr} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Td} & & \downarrow{\scriptstyle Td} & & \downarrow{\scriptstyle e_{\mathbb{C}}} & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \end{array}$$

where $e_{\mathbb{C}}$ is the Adams e-invariant. This is worked out in [CF66a]. As done on page 95 of [CF66a] we can now evaluate the Todd genus

$$\begin{aligned} \langle Td(TQ), [Q, \partial Q] \rangle &= \langle \frac{1}{720} 3c_2^2(E), [Q, \partial Q] \rangle = \frac{1}{240} \langle c_2^2(E), [Q, \partial Q] \rangle \\ &= \frac{1}{240} \langle c_2(E), [S^4] \rangle = \frac{1}{240}. \end{aligned}$$

Remark on the relation to K-theory

In modern formulation the Todd genus is associated to the multiplicative formal group law and therefore to K-theory. Let $P(x)$ be a power series with 1 as constant coefficient. Its logarithm g is given by

$$g^{-1}(x) = \frac{x}{P(x)}.$$

Complex oriented cohomology theories always come with a formal group law $F(x,y)$ which can be expressed as

$$F(x,y) = g^{-1}(g(x)+g(y)).$$

For the Todd genus we have $P(x) = \frac{x}{1-e^{-x}}$ implying $y = g^{-1}(x) = 1 - e^{-x}$. This gives us $g(x) = -\ln(1-x)$ and thus

$$\begin{aligned} F(x,y) &= 1 - \exp[-(-\ln(1-x) - \ln(1-y))] \\ &= 1 - \exp(\ln(1-x) + \ln(1-y)) \\ &= 1 - (1-x)(1-y) = x + y - xy, \end{aligned}$$

which is the multiplicative formal group law coming from complex K-theory.

Definition of the Bott manifold

Since $\partial Q = S^7$ is framed and $[\partial Q] \in \Omega_7^{fr} \cong \pi_7^s \cong \mathbb{Z}/240$ we have $240[\partial Q] = 0$, i.e. there exists a framed manifold R^8 with $\partial R^8 = -240\partial Q$. We define a Bott-manifold by

$$B := 240Q \cup_{240\partial Q} R^8$$

and see that indeed $\hat{A}(B) = Td(B) = 240Td(Q) + 0 = 1$.

4 Construction of an Artin-Schreier class

Having a Bott manifold with associated K-theory class congruent to v^4 modulo 16 we can use the power series of the logarithm

$$\log(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

to define

$$b = -\frac{\log([M])}{\log(3^4)}.$$

Proposition 4 *The class b is an Artin-Schreier class.*

Proof:

$$\psi^3 b = -\frac{\log([M]/3^4)}{\log(3^4)} = -\frac{\log([M])}{\log(3^4)} + \frac{\log(3^4)}{\log(3^4)} = b + 1.$$

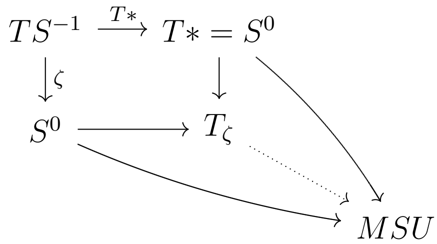
Here the stable Adams operation $\psi^k : K \to K$ is defined levelwise by $\frac{\Psi^k}{k^n} : K_{2n} \to K_{2n}$ with Ψ^k being the unstable Adams operation. Inverting powers of $k \in \mathbb{Z}_2^\times$ is not a problem since everything is 2-completed. □

5 Construction of an E_∞ map $T_\zeta \to MSU$

The fiber sequence $X \to KO \wedge X \xrightarrow{\psi^3-1} KO \wedge X$ induces the unit map $\pi_0 KO \to \pi_{-1} S^0$ mapping $1 \mapsto \zeta$. Now we define T_ζ to be the homotopy pushout in the category of $K(1)$-local E_∞ ring spectra:

$$\begin{array}{ccc} TS^{-1} & \xrightarrow{T*} & T* = S^0 \\ \downarrow{\scriptstyle \zeta} & & \downarrow \\ S^0 & \longrightarrow & T_\zeta \end{array}$$

with TX the free E_∞ spectrum generated by the pointed space X. As the Hurewicz image of $\zeta \in \pi_{-1} MSU$ is zero we get a map $T_\zeta \to MSU$:

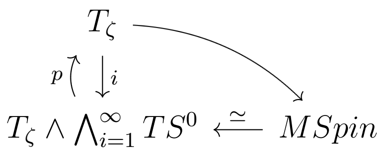


6 Split map - direct summand argument

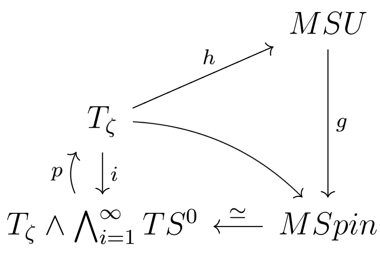
To get T_ζ as a direct summand, one has to construct a split p such that the composition

$$T_\zeta \xrightarrow{i} MSU \xrightarrow{p} T_\zeta$$

is the identity. This can be done using the *Spin* splitting of Laures [Lau03]



and showing that the extended triangle commutes



6.1 Comparison of the Artin-Schreier classes

The *SU* Artin-Schreier class constructed above is naturally also a *Spin* Artin-Schreier class. Refering to [Lau02] we have

Lemma 5 *Let b and b' be two Artin-Schreier elements of $\pi_0 KO \wedge MSpin$. Then there is an E_∞ self homotopy equivalence κ of $MSpin$ which carries b to b'.*

Proof: The short exact sequence

$$0 \to \pi_0 MSpin \to \pi_0 KO \wedge MSpin \overset{\psi^3-1}{\to} \pi_0 KO \wedge MSpin \to 0$$

with $(\psi^3 - 1)b = (\psi^3 - 1)b' = 1$ tells us that b and b' can only differ by a class $a \in \pi_0 MSpin$. Let κ be the E_∞ map of

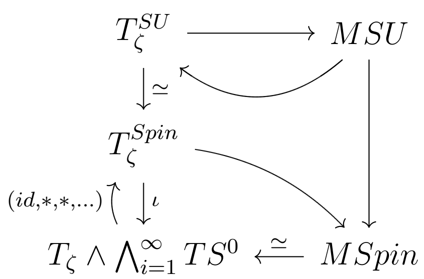
$$MSpin \cong T_\zeta \wedge \bigwedge TS^0$$

which is the identity on each TS^0 and restricts to

$$\iota + a\delta : C_\zeta \to MSpin$$

on T_ζ. Then its inverse is defined in the same way with a replaced by $-a$.□

With the notations T_ζ^{SU} and T_ζ^{Spin} for the E_∞ spectra we get from the different Artin-Schreier classes, we have the following diagram with E_∞ maps:



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