Calculating Adams operations on $K_*\mathbb{CP}^\infty$ via 2-adic analysis

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1 Mahler series in p-adic analysis

The binomial polynomials define continuous functions $\binom{\cdot}{k} : \mathbb{Z}_p \to \mathbb{Z}_p$. Since $\mathbb{N}$ is dense in $\mathbb{Z}_p$, we have $\left\|\binom{\cdot}{k}\right\| = \sup_{\mathbb{N}} \left|\binom{n}{k}\right| \leq 1$. Because of $\binom{k}{k} = 1$, equality holds in fact. In p-adic analysis we know that for a given sequence $(a_i)_{i\geq 0}$ in $\mathbb{C}_p$ with $|a_i| \to 0$, the series $\sum_{k\geq 0} a_k \binom{\cdot}{k}$ is a continuous function $f : \mathbb{Z}_p \to \mathbb{C}_p$. It is quite remarkable that conversely, every continuous function $\mathbb{Z}_p \to \mathbb{C}_p$ can be represented this way. This result has been obtained by Mahler. With the notation of the norm $\|f\| = \sup_{\mathbb{Z}_p} |f(x)|$ the finite-difference operator ∇

$$(\nabla f)(x) = f(x+1) - f(x)$$

and its k-fold iterated version ∇^k are defined and the following theorem holds.

Theorem 1 (Mahler) *Let $f : \mathbb{Z}_p \to \mathbb{C}_p$ be a continuous function and put $a_k = \nabla^k f(0)$. Then $|a_k| \to 0$, and the series $\sum_{k\geq 0} a_k \binom{\cdot}{k}$ converges uniformly to f. Moreover $\|f\| = \sup_{k\geq 0} |a_k|$.*

1.1 The ring of numerical polynomials

The ring of numerical functions

$$A := \{f \in \mathbb{Q}[\omega] \text{ such that } f(\mathbb{Z}) \subset \mathbb{Z}\},$$

can be completed with respect to the p-adic norm

$$\hat{A}_p = \{f \in \mathbb{Q}_p[\![\omega]\!] : f(\mathbb{Z}_p) \subset \mathbb{Z}_p\}$$

being the ring of continuous functions $f : \mathbb{Z}_p \to \mathbb{Z}_p$. Integrally we can identify $K_0\mathbb{CP}^\infty \cong A$, i.e. the K-homology of $\mathbb{CP}^\infty$ equals the ring of numerical polynomials. The duality

$$K^0\mathbb{CP}^\infty \cong \mathrm{Hom}(K_0\mathbb{CP}^\infty, \mathbb{Z})$$

is given as follows: The series $\sum a_i t^i \in \mathbb{Z}[\![t]\!] \cong K^0\mathbb{CP}^\infty$ maps to the homomorphism given by $\binom{w}{i} \mapsto a_i$ on basis elements.

2 Alternative description of the Adams operations

We have seen that $K_*\mathbb{CP}^\infty$ is the ring of continuous functions on $\mathbb{Z}_p$ which are given as Mahler series. Its module generators $\beta_i \in K_{2i}\mathbb{CP}^\infty$ represent the function $\beta_i(T) = \binom{T}{i}$. Application of a base change leads to an interesting observation: At the prime 2 we have:

$$\begin{aligned} \binom{3T}{1} &= 3\binom{T}{1} \\ \binom{3T}{2} &= 9\binom{T}{2} + 3\binom{T}{1} \\ \binom{3T}{3} &= 27\binom{T}{3} + 18\binom{T}{2} + \binom{T}{1} \end{aligned}$$

and this is exactly the Adams operation $\psi^{3^{-1}}$ on β_k with respect to the generator $x = L - 1 \in K^*\mathbb{CP}^\infty$. The generator used before results in the same operation up to an alternating sign.

Lemma 2 *The Adams operation $\psi^{3^{-1}}$ on $K_*\mathbb{CP}^\infty$ is given by*

$$\psi^{3^{-1}}\beta_i(T) = \beta_i(3T),$$

or, equivalently as the Mahler series

$$\psi^{3^{-1}}\binom{T}{i} = \binom{3T}{i} = \sum_{j\geq 1} a_j \binom{T}{j},$$

where

$$a_j = \sum_{s+t=i-j} \binom{j}{s}\binom{s}{t} 3^{j-t}.$$

Proof: Due to Mahler's theorem, the j^{th} coefficient satisfies

$$a_j = \nabla^j \binom{3x}{i}\Big|_{x=0},$$

i.e. it can be expressed using the j-fold iterated finite difference operator. Starting the calculations we get

$$\nabla \binom{3T}{i} = 3\binom{3T}{i-1} + 3\binom{3T}{i-2} + \binom{3T}{i-3}.$$

Comparing this to the calculation of the Adams operation in $K^*\mathbb{CP}^\infty$ with respect to the generator $x = L - 1$ we get

$$\psi^3 x = (1+x)^3 - 1 = 3x + 3x^2 + x^3$$

and see that taking the j^{th} power of $3x + 3x^2 + x^3$ is exactly the same as taking the j^{th} iterated finite difference operator ∇^j. Hence the calculations coincide and the claim follows.

Corollary 3 *For a 2-adic unit $k \in \mathbb{Z}_2^\times$ we have*

$$\left(\psi^{k^{-1}}\beta_i\right)(T) = \beta_i(kT) = \binom{kT}{i}.$$

Proof: Since 3 is a topological generator of $\mathbb{Z}_2^\times$, the sequence $a_n = 3^n$ contains a subsequence $(a_{i_n})_n$ converging to k, and we have

$$\begin{aligned}
\psi^{k^{-1}}\beta_i(T) &= \lim_n \psi^{a_{i_n}^{-1}}\beta_i(T) = \lim_n \psi^{3^{-a_{i_n}}}\beta_i(T) \\
&= \lim_n \psi^{3^{-1}} \cdots \psi^{3^{-1}}\beta_i(T) = \lim_n \beta_i(3^{a_{i_n}}T) \\
&= \beta_i(kT).
\end{aligned}$$

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