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$K(1)$-local topological modular forms

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Abstract. We construct the Witten orientation of the topological modular forms spectrum *tmf* in the $K(1)$-local setting by attaching E_∞ cells to the bordism theory $MO\langle 8\rangle$. We also identify the KO-homology of *tmf* with the ring of divided congruences.

1. Introduction and statement of results

In his analysis of the Dirac operator on loop spaces [53] Witten introduced a bordism invariant for spin manifolds with trivialized Pontryagin class $p_1/2$. This so called Witten genus associates to each such $O\langle 8\rangle$-manifold an integral modular form. In [49][20][29] et al. the hope was formulated that the genus allows a generalization to families of $O\langle 8\rangle$-manifolds. From the topological point of view, this means that there is a multiplicative transformation of generalized cohomology theories from the bordism theory $MO\langle 8\rangle$ to some new cohomology theory E. For spin bordism, the $\hat{A}$-genus of a family takes its values in real K-theory. Hence one may regard E as some higher version of K-theory.

There are several candidates for such a theory E. The first was introduced by Landweber, Ravenel and Stong in [40]. Its coefficient ring is given by modular functions over $\mathbb{Z}[1/2]$ for the congruence subgroup $\Gamma_0(2)$ which are holomorphic on the upper half plane. For this and other elliptic theories in which 2 is invertible the Witten orientation was constructed in [41]. The difficulty appears at the prime 2 since here $MO\langle 8\rangle$ is not well understood yet.

There are elliptic cohomology theories for which 2 is not a unit (cf [25] et al.) All theories share the property that they are complex orientable and that the associated formal group comes from the formal completion of an

Mathematics Subject Classification (2000): 55P43, 55N22, 55P43, 55P48, 55P60, 55S12, 55S25, 11G07 elliptic curve. It is the goal of [31] to setup and rigidify a diagram of such elliptic spectra as a topological model for the stack of generalized elliptic curves. Its homotopy inverse limit is the spectrum *tmf* of topological modular forms (cf [29].) The new theory *tmf* is not complex oriented but it maps to any such E in a canonical way. Hence, an $O\langle 8\rangle$ orientation of *tmf* furnishes the desired orientation of E.

The work in hand relies on a different construction of *tmf* in a local setting. For the Witten orientation it is enough to look at the $K(1)$- and $K(2)$-localizations of *tmf* (cf [5].) The theory $K(n)$ denotes the nth Morava K-theory at the prime 2. There is an explicit decomposition of the $K(1)$-local *tmf* into E_∞ cells [30] as follows. Let T_ζ be the E_∞ cone over a generator ζ of the -1st homotopy group of the $K(1)$-local sphere. Then *tmf* is obtained from T_ζ by killing one homotopy class. In particular, there is an E_∞ map from T_ζ to *tmf*.

In the present work we combine these results with the splitting of $K(1)$-localized E_∞ spectra

$$MO\langle 8\rangle \cong T_\zeta \wedge \bigwedge_{i=1}^{\infty} TS^0. \tag{1}$$

Here, TS^0 is the free E_∞ spectrum generated by the sphere spectrum. The formula was shown in [42] for spin bordism but the natural map from $MO\langle 8\rangle$ to $MSpin$ is a $K(1)$-equivalence. The reason is that the reduced $K(1)$-homology of $K(\mathbb{Z}, 3)$ vanishes (cf [32][48]) and hence the bar spectral sequence of the fibration

$$K(\mathbb{Z}, 3) \longrightarrow BO\langle 8\rangle \longrightarrow BSpin \tag{2}$$

collapses.

The splitting formula enables us to explicitly list the E_∞ cells which are needed to obtain *tmf* from $MO\langle 8\rangle$: loosely speaking, the cells kill the free part of $MO\langle 8\rangle$ and turn the operation ψ into the Frobenius operator for trivialized elliptic curves.

Theorem 1. *In the category of $K(1)$-local spectra there are classes $x_0, x_1, x_2, \ldots \in \pi_0 MO\langle 8\rangle$ and an E_∞ map*

$$W : MO\langle 8\rangle \longrightarrow tmf$$

with the property that the diagram

$$\begin{tikzcd}
\bigwedge_{i=1}^{\infty} TS^0 \arrow[r, "*"] \arrow[d, "{(x_0,x_1,x_2,\ldots)}"'] & T* = S^0 \arrow[d] \\
MO\langle 8\rangle \arrow[r, "W"] & tmf
\end{tikzcd}$$

is a homotopy pushout of E_∞ ring spectra.

The map W is related to the original Witten orientation in the following way: let $z(\xi)$ be the KO-Thom class associated to a spin bundle ξ as it was constructed by Atiyah-Bott-Shapiro (ABS) in [10]. Let $S_t(\xi)$ be the formal power series in t with the symmetric powers of ξ as coefficients. Then the system of Thom classes

$$\xi \mapsto z(\xi) \otimes \bigotimes_n S_{q^n}(\dim \xi - \xi). \tag{3}$$

induces a map of ring spectra

$$W_q : MO\langle 8\rangle \longrightarrow MSpin \longrightarrow KO \wedge S\mathbb{Z}[\![q]\!] = KO[\![q]\!] \quad . \tag{4}$$

Here, SR is the Moore spectrum associated to the ring R.

Theorem 2. *The diagram of $K(1)$-localized ring spectra*

$$\begin{tikzcd} MO\langle 8\rangle \arrow[r, "W"] \arrow[d, "W_q"'] & tmf \arrow[d] \\ KO[\![q]\!] \arrow[r] & K[\![q]\!] \end{tikzcd}$$

commutes up to homotopy. Here, the right vertical map is induced by the Tate curve and the lower horizontal one by the complexification map.

It is not hard to see that the ring map W is in fact characterized by the commutativity of the above diagram. The proofs of the Theorems 1, 2 and the following isomorphism of Conner-Floyd type are postponed to the last section.

Corollary 1. *In the category of $K(1)$-local spectra the natural map*

$$\pi_* MO\langle 8\rangle \wedge X \otimes_{\pi_* MO\langle 8\rangle} \pi_* tmf \longrightarrow \pi_* tmf \wedge X$$

induced by the Witten genus W_* is an isomorphism for all spectra X.

There is a well known relationship between the ring of divided congruences D [36] and the K-homology rings of certain elliptic cohomology theories (cf [21][11][41].) The next result generalizes this connection to *tmf* at the prime 2 in the $SZ/2$-local category:

Theorem 3. *There is a homotopy equivalence of ring spectra*

$$L_{S\mathbb{Z}/2}(KO \wedge SD) \xrightarrow{\cong} L_{S\mathbb{Z}/2}(KO \wedge tmf).$$

In particular, we have $\pi_0 KO \wedge tmf \cong D$ after 2-completion.

The work is organized as follows. We first collect some basic notions and results on the theory of elliptic curves over arbitrary rings. We then recall the work of N. Katz on divided congruences. Although we work with a different notion of a generalized elliptic curve it turns out that the ring of 2-adic modular forms coincides with the ring $D_2^\wedge$ of divided congruences in the sense of Katz. As a consequence, a 2-adic modular form is determined by its q-expansion.

In the third section we recall from [29] the notion of an elliptic spectrum E and construct a canonical class $b = b(E)$ in $\pi_0 KO \wedge E$ which plays a central role in the investigation of E_∞ structures on elliptic spectra.

In the next section we recall the results of [30] about E_∞ elliptic spectra E and their associated θ-algebras. It is shown that for any such E the class $f = \psi b - b$ satisfies a universal relation $\theta(f) = h(f)$ for some fixed 2-adic convergent series h. Then we define M to be the initial E_∞ spectrum whose homotopy contains an element f which satisfies this relation. Hence M maps into any $K(1)$-local E_∞ elliptic E in a natural way. Since M has the right coefficients it is used as the $K(1)$-localization of *tmf* in the remainder of the paper.

We then turn to the proof of Theorem 3 and give an explicit isomorphism. The sixth section shows that the Witten map W_q induces a map of θ-algebras. In the last section we conduct some geometry to lift the canonical class b to $\pi_0 KO \wedge MO\langle 8\rangle$. The orientation W with its desired properties then becomes a consequence of the splitting formula (1).

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2. Modular forms and divided congruences

A *Weierstrass equation over a ring* R is an equation of the form

$$C : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

with $a_1, a_2, a_3, a_4, a_6 \in R$. Associated to such an equation there are the following standard quantities (compare [22]):

$$b_2 = a_1^2 + 4a_2 \qquad c_4 = b_2^2 - 24b_4$$

$$b_4 = 2a_4 + a_1 a_3 \qquad c_6 = -b_2^3 + 36 b_2 b_4 - 216 b_6$$

$$b_6 = a_3^2 + 4a_6 \qquad \omega_{can} = dx/(2y + a_1 x + a_3)$$

$$b_8 = a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2 \qquad j = c_4^3/\Delta$$

$$\Delta = -b_2^2 b_8 - 8b_4^3 - 27 b_6^2 + 9 b_2 b_4 b_6$$

The 1-form ω_{can} is called the canonical nowhere vanishing differential. Two Weierstrass equations define the same marked curve iff there is a change of variables of the form

$$x = u^2x' + r, \quad y = u^3y' + u^2sx' + t \tag{5}$$

with $r, s, t \in R$ and $u \in R^\times$. Under these transformations we have

$$u^4c_4' = c_4,\ u^6c_6' = c_6,\ u^{12}\Delta' = \Delta,\ u^{-1}\omega_{can}' = \omega_{can},\ j' = j. \tag{6}$$

By a *generalized elliptic curve* over a ring R we mean a marked curve over $\operatorname{spec}(R)$ which locally is isomorphic to a Weierstrass curve (cf [6]B2.) By a *modular form* over R of weight k we mean a rule (cf [33]) which assigns to every pair $(C/S, \omega)$ consisting of a generalized elliptic curve over an R-algebra S together with a nowhere vanishing invariant section of $\Omega^1_{C/S}$ an element $f(C/S, \omega) \in S$ in such a way that

(i) f only depends on the isomorphism class of the pair.
(ii) for all algebra maps $g : S \longrightarrow S'$ we have

$$g(f(C/S, \omega)) = f(C/S', \omega_{S'}),$$

that is, f commutes with arbitrary base change.
(iii) for all $a \in S^\times$ we have

$$f(C/S, a\omega) = a^{-k} f(C/S, \omega).$$

For example, by (6) the quantities c_4, c_6 and Δ define modular forms over $\mathbb{Z}$ of weights 4, 6 and 12, respectively. In fact, it turns out (cf [22]) that the ring of modular forms over $\mathbb{Z}$ is precisely given by

$$mf_* = \mathbb{Z}[c_4, c_6, \Delta]/\big(1728\Delta - c_4^3 + c_6^2\big). \tag{7}$$

Example 1. The curve [12]38f

$$C_j : y^2 + xy = x^3 - 36(j - 1728)^{-1}x - (j - 1728)^{-1}$$

is defined over $\mathbb{Z}_2[j^{-1}]^\wedge_2$ since the series $1/(j - 1728) = \sum 2^{6s}3^{3s}j^{-s-1}$ converges 2-adically. For the Weierstrass equation C_j and its canonical differential, we have

$$c_4 = -c_6 = j(j - 1728)^{-1},\ \Delta = j^2(j - 1728)^{-3}.$$

Hence, its j-invariant is j.

Example 2. The Tate curve is defined by the equation

$$Tate : y^2 + xy = x^3 + a_4x + a_6.$$

The coefficients are given by $a_4 = (1 - E_4)/48$, $a_6 = (1 - E_4)/576 + (E_6 - 1)/864$ and take values in $\mathbb{Z}[\![q]\!]$ ([52]422ff.) Here,

$$E_{2k}(q) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

are the Eisenstein series, $\sigma_k(n) = \sum_{d|n} d^k$ and B_k are the Bernoulli numbers defined by

$$\frac{te^t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

For *Tate* we have

$$c_4 = E_4, \;\; c_6 = -E_6, \;\; \Delta = \frac{E_4^3 - E_5^2}{1728}. \tag{8}$$

The Tate curve is already defined over the subring $D \subset \mathbb{Z}[\![q]\!]$ of *divided congruences*. The elements of D are those sums $\sum f_i$ of modular forms f_i of weight i over $\mathbb{C}$ with the property that the q-expansion

$$\sum f_i(q) = \sum f_i(Tate/\mathbb{C}[\![q]\!], \omega_{can}) \in \mathbb{C}[\![q]\!]$$

has coefficients in $\mathbb{Z}$.

Congruences between modular forms correspond to elements of D. For example, the congruence $E_4 \equiv 1 \mod 240$ gives the class $(E_4 - 1)/240 \in D$. The presence of congruences means that mf_* does not give a pure subring of D, that is, the cokernel of the inclusion map $mf_* \subset D$ has torsion. However, we can make it into one by introducing a new parameter v which keeps track of the grading:

$$\lambda_* : mf_* \longrightarrow D[v^{\pm}], \quad f \mapsto f(Tate/D[v^{\pm}], v^{-1}\, dx/(2y + x)).$$

That is,

$$\lambda_*(c_4) = E_4 v^4, \quad \lambda_*(c_6) = -E_6 v^6.$$

Moreover, λ_* admits a factorization after 2-completion

$$mf_* \xrightarrow{\iota} \mathbb{Z}_2[j^{-1}]_2^{\wedge}[v^{\pm 2}] \xrightarrow{\lambda_v} D_2^{\wedge}[v^{\pm}] \tag{9}$$

which is defined as follows: with $u = v\sqrt[4]{1 - 1728/j}$ set

$$\iota(c_4) = c_4\big(C_j/\mathbb{Z}_2[j^{-1}]_2^{\wedge}[v^{\pm 2}], u^{-1}\, dx/(2y + x)\big) = v^4$$
$$\iota(c_6) = c_6\big(C_j/\mathbb{Z}_2[j^{-1}]_2^{\wedge}[v^{\pm 2}], u^{-1}\, dx/(2y + x)\big) = -\sqrt{1 - 1728/j}\, v^6$$

and

$$\lambda_v(j^{-1}) = \lambda(j^{-1}), \quad \lambda_v(v^2) = \sqrt{E_4}\, v^2. \tag{10}$$

Remark 1. In Sect. 4 we will see that

$$\pi_* L_{K(1)} tmf \cong \pi_* KO[j^{-1}]^{\wedge}_2.$$

The factorization (9) appears as bottom row of the commutative diagram of graded rings

$$\begin{tikzcd}
\pi_* tmf \arrow[r] \arrow[d] & \pi_* L_{K(1)} tmf \arrow[r] \arrow[d] & \pi_*(KO \wedge tmf)^{\wedge}_2 \arrow[d] \\
mf_* \arrow[r] & \mathbb{Z}_2[j^{-1}]^{\wedge}_2[v^{\pm 2}] \arrow[r] & D^{\wedge}_2[v^{\pm}]
\end{tikzcd}$$

in which the middle vertical map sends the real Bott class to v^4 and the left one is the complexification map into

$$\pi_*(K \wedge tmf)^{\wedge}_2 \cong D^{\wedge}_2[v^{\pm}].$$

The completion of a generalized elliptic curve C at its origin O is a formal group (cf [6] p 671) and is denoted by $\hat{C}$ in the sequel. If the curve is given in Weierstrass form we can choose the formal parameter $t = -x/y$ in a neighborhood U of $O = [0, 1, 0]$. Then the addition is described by the following formal group law (cf [51] p 115)

$$\hat{C}(x, y) = x + y - a_1 xy - a_2(x^2 y + xy^2) \tag{11}$$
$$-(2a_3 x^3 y - (a_1 a_2 - 3a_3)x^2 y^2 + 2a_3 xy^3) + \cdots . \tag{12}$$

Example 3. There is a canonical isomorphism $\varphi_{can} : \hat{G}_m \longrightarrow \widehat{Tate}$ over $\mathbb{Z}[\![q]\!]$ which pulls back the canonical differential ω_{can} to du/u on the multiplicative formal group $\hat{G}_m$ (cf [52] 422ff.) In fact, φ_{can} is already defined over the pure subring D since strict isomorphisms are unique over torsion free algebras and they always exist over $\mathbb{Q}$-algebras.

Let p be a prime number and let $\hat{F}$ be a 1-dimensional formal group law over a $\mathbb{Z}_{(p)}$-algebra R. Recall from [27] that the Hazewinkel elements $v_1, v_2, \ldots \in R$ of its p-typicalization $\hat{F}^p$ can be obtained from the formal expansion of its Frobenius operation

$$f_p(T) = \sum_{i=0}^{\infty}{}^{\hat{F}^p} v_{i+1} T^{p^i}.$$

For the universal formal group law $\hat{F}_U$ over the Lazard ring $L_{(p)}$, the classes $v_1, v_2, \ldots$ freely generate the ring L^p which carries the universal p-typical formal group law $\hat{F}^p_U$. The coefficients $m_i \in \mathbb{Q}[v_1, v_2, \ldots]$ of its logarithm

$$\log_{\hat{F}^p_U}(T) = \sum_{i=0}^{\infty} m_i T^{p^i}$$

are related to the generators by the formula (cf [27][26][2])

$$pm_{n+1} = \sum_{i+j=n} m_j v_{i+1}^{p^j}. \tag{13}$$

Lemma 1. *The Hazewinkel elements of the formal group $\hat{C}^p$ (cf (11)) over $\mathbb{Z}_{(p)}[a_1, a_2, \dots, a_6]$ are given for $p = 2$ by the formulae*

$$v_1 = a_1, \quad v_2 = a_3 + a_1 a_2. \tag{14}$$

Proof. Compute

$$\begin{aligned}\log_{\hat{C}}(T) &= \int \left(\tfrac{\partial}{\partial y}_{|(T,0)} \hat{C} \right)^{-1} dT \\ &= T + \frac{a_1}{2}T^2 + \frac{a_2 + a_1^2}{3}T^3 + \frac{2a_3 + 2a_1a_2 + a_1^3}{4}T^4 + O(T^5).\end{aligned}$$

Hence the result follows from (13). □

A strict isomorphism between p-typical formal group laws is specified by the target $\hat{F}$ and a power series which formally expands in the form

$$\varphi(T) = {\sum_{i=0}^{\infty}}^{\hat{F}} t_i T^{p^i}, \ t_0 = 1.$$

For $t = (t_1, t_2, \dots)$ let $(L^p, L^p[t])$ be the Hopf algebroid which classifies p-typical formal group laws and strict isomorphisms between them. Then the right unit η_R (which corepresents the source of a morphism) satisfies the formula (cf [27] p 167)

$$\eta_R(m_k) = \sum_{i+j=k} m_i t_j^{p^i}. \tag{15}$$

Lemma 2. *Let C be a generalized elliptic curve over a $\mathbb{Z}_{(p)}$-algebra R. Suppose that $\varphi : \hat{C} \longrightarrow \hat{G}_m$ is an isomorphism of formal groups over R. Then $c_4 = c_4(C/R, \varphi^* du/u)$ and $c_6 = c_6(C/R, \varphi^* du/u)$ satisfy the congruences*

$$c_4 \equiv 1 \ mod \ 16 \tag{16}$$
$$3c_4 + 2c_6 \equiv 1 \ mod \ 64. \tag{17}$$

Proof. We may assume that C is given in Weierstrass form and that the canonical differential coincides with $\varphi^* du/u$. Then φ gives a strict isomorphism between the formal group law (11) and

$$\hat{G}_m(x, y) = x + y + xy$$

as well as between their p-typicalizations. The Hazewinkel elements of $\hat{G}_m^p$ are given by $v_1 = -1$, $v_2 = 0$ and the ones for $\hat{C}^p$ were computed in Lemma 1. Hence, in R we have with (15)

$$-1 = a_1 + 2t_1 \tag{18}$$
$$0 = a_3 + a_1a_2 - 3a_1^2t_1 - 5a_1t_1 - 4t_1^3 + 2t_2. \tag{19}$$

Set

$$b = t_1 + t_1^2 + 2a_2 - t_1^4 + a_2^2 + a_4 + t_2 + t_1t_2. \tag{20}$$

Then these relations show

$$c_4 \equiv 1 + 16b \bmod 64,$$
$$c_6 \equiv -1 + 8b \bmod 32$$

and the claim follows. □

Recall from [35] that a *trivialized elliptic curve* over a ring R is a pair $(C/R, \varphi)$, consisting of a generalized elliptic curve over R and an isomorphism $\varphi : \hat{C} \longrightarrow \hat{G}_m$. A *Katz modular form* over R is a rule f which assigns to any trivialized elliptic curve $(E/S, \varphi)$ over a p-adically complete R-algebras an element $f(E/S, \varphi) \in S$ in such a way that

(i) f only depends on the isomorphism class of the pair $(E/S, \varphi)$.
(ii) f commutes with arbitrary base change of p-adically complete R-algebras.

The ring of all Katz modular forms over R is itself a p-adically complete R-algebra and is denoted by V_R. In the sequel, we restrict our attention to the prime $p = 2$.

Proposition 1. *(cf [35]) The algebra V_R corepresents the functor*

$$M^{triv}(S) = \{\text{isomorphism classes of triv. elliptic curves over } S\}$$

in the category of 2*-adically complete R-algebras.*

Proof. Lemma 2 shows that the values

$$d_4 = \frac{1 - c_4}{48}, \ d_6 = \frac{1}{1728} - \frac{c_4}{576} - \frac{c_6}{864}$$

define Katz modular forms over $\mathbb{Z}_2$ by carefully associating to $(C/R, \varphi)$ the value $d_i(C/R, \varphi^* du/u)$ in a way that it commutes with arbitrary base change. To achieve this, observe that the algebra

$$W^{triv} = \mathbb{Z}_{(2)} \otimes_{\hat{G}_m^2} L^2[t] \otimes_{\hat{C}^2} \mathbb{Z}_{(2)}[a_1, a_2, a_3, a_4, a_6]$$

classifies Weierstrass equations C over $\mathbb{Z}_{(2)}$-algebras together with strict isomorphisms between $(\hat{C}, \omega_{can})$ and $(\hat{G}_m, du/u)$. Hence, a Katz modular form is determined by its value on the pair $(C/(W^{triv})^{\wedge}_2, \varphi_{can})$. Moreover, $(W^{triv})^{\wedge}_2$ is torsion free since tensoring with the L^2-module $\mathbb{Z}_{(2)}$ is exact in the category $\mathcal{BP}$ of $L^2[t]$-comodules which are finitely presented as modules [39]. We conclude that the curve

$$Tate : y^2 + xy = x^3 + d_4 x + d_6$$

is well defined over $V = V_{Z_2}$ and hence over V_R. The notation is justified by the fact that the curve $Tate/V$ coincides with $Tate/D^{\wedge}_2$ under the evaluation map

$$\iota : V \longrightarrow D^{\wedge}_2, \;\; f \mapsto f(Tate/D^{\wedge}_2, \varphi_{can})$$

where φ_{can} was described in Example 3. Similarly, for each trivialized elliptic curve $(C/S, \varphi)$ there is an evaluation map from V_R to S and hence a Tate curve $Tate(C/S, \varphi)$. The claim follows from the following two lemmas. □

Lemma 3. *For each trivialized elliptic curve $(C/R, \varphi)$ there is a strict isomorphism*

$$\rho : (C/R, \varphi^* du/u) \xrightarrow{\sim} (Tate(C/R, \varphi), \omega_{can})$$

which commutes with arbitrary base change.

Proof. We may assume that C is given by a Weierstrass equation and $\varphi^* du/u$ coincides with the canonical differential. Then the map ρ is defined by the transformation of variables

$$x \mapsto x - \frac{1}{3}\big(a_2 + t_1 + t_1^2\big)$$
$$y \mapsto y + (1 + t_1)x - \frac{1}{3}\big(2a_2 + 4a_2 t_1 + 5t_1 + 12t_1^2 - 10t_1^3 + 3t_2\big)$$

as one verifies by an elementary calculation with the help of the relations (18) and (19). □

Lemma 4. *There is a strict isomorphism φ_{can} between $(\widehat{Tate}, \omega_{can})$ and $(\hat{G}_m, du/u)$ over V_R. Moreover, for each Katz modular form f we have the identity*

$$f = f(Tate/V, \varphi_{can}).$$

Proof. It suffices to consider the case $R = \mathbb{Z}_2$. We have already seen that V is a subring of the torsion free ring $(W^{triv})^{\wedge}_2$. By Lemma 3, there is a strict isomorphism φ_{can} between $\widehat{Tate}$ and the multiplicative formal group law $\hat{G}_m$ over $(W^{triv})^{\wedge}_2$. Its coefficients can be expressed in terms of rational polynomials in c_4 and c_6. Hence they are invariant under any change of variables which preserve the Weierstrass form and the canonical differential. Thus they give Katz modular forms and φ_{can} is already defined over V. Finally, observe that for the inclusion map $i : V \subset (W^{triv})^\wedge_2$ we have

$$\begin{aligned} i(f) &= f\big(C/(W^{triv})^\wedge_2, \varphi_{can}\big) \\ &= f\big(Tate/(W^{triv})^\wedge_2, \varphi_{can}\big) \\ &= i(f(Tate/V, \varphi_{can})) \end{aligned}$$

which proves the last claim. □

Also the following result is due to N. Katz [34] for a slightly different notion of a generalized elliptic curve.

Proposition 2. *The map $\iota : V \longrightarrow D^\wedge_2$ is an isomorphism. In particular, a Katz modular form is determined by its q-expansion.*

Proof. We have already seen that V has no 2-torsion and that c_4 and c_6 define Katz modular forms. Hence, it suffices to show that the map

$$\iota_{\mathbb{F}_2} : V \otimes \mathbb{F}_2 \cong V_{\mathbb{F}_2} \longrightarrow \mathbb{F}_2[\![q]\!]$$

is injective. Over $\mathbb{F}_2$-algebras the Tate curve takes the form

$$y^2 + xy = x^3 + bx + b$$

where b was defined in (20). Hence by Lemma 3 and the universal property, the ring $V_{\mathbb{F}_2}$ coincides with

$$\mathbb{Z}_{(2)} \otimes_{\hat{G}_m^2} L^2[t] \otimes_{\widehat{Tate}^2} \mathbb{F}_2[b].$$

In the tensor product, the left module is flat in $\mathcal{BP}$. We conclude that the completion map

$$V_{\mathbb{F}_2} \longrightarrow V_{\mathbb{F}_2} \otimes_{\mathbb{F}_2[b]} \mathbb{F}_2[\![b]\!]$$

is an inclusion. Let w be the inverse power series of

$$b(q) = \sum_{n \geq 1} \sigma_3(n) q^n = \sum_{n \geq 1} \frac{q^n}{1 - q^n} \in \mathbb{F}_2[\![q]\!].$$

Then the composite

$$V_{\mathbb{F}_2} \xrightarrow{\iota} \mathbb{F}_2[\![q]\!] \xrightarrow{w(b)} \mathbb{F}_2[\![b]\!] \xrightarrow{1 \otimes id} V_{\mathbb{F}_2} \otimes_{\mathbb{F}_2[b]} \mathbb{F}_2[\![b]\!]$$

defines the same Tate curve as the completion map. The resulting strict isomorphisms can differ by a strict automorphism of $\hat{G}_m$. This defect can easily be fixed by an automorphism of the range with the help of the comodule structure of $V_{\mathbb{F}_2}$. Thus ι is injective. □

For $a \in \mathbb{Z}_2^\times \subset Aut(\hat{G}_m)$ the Diamond operation $[a]$ acts on V by

$$[a]f(C/R, \varphi) = f(C/R, a^{-1}\varphi).$$

Clearly, the action is trivial on Katz modular which only depend on the isomorphism class of the elliptic curve. The converse also holds.

Lemma 5. *The invariant Katz modular forms under the action of the Diamond operations admit a 2-adic convergent expansion in j^{-1}.*

Proof. The observation is due to Serre (cf [50] 202f) but we provide an alternative proof here. It is not hard to see that the curve C_j is isomorphic to the Tate curve if j^{-1} is q-expanded. Hence over the ring

$$R = \mathbb{F}_2[\![q_L]\!] \otimes_{\mathbb{F}_2[j^{-1}]} \mathbb{F}_2[\![q_R]\!]$$

the curve C_j admits trivializations φ_L and φ_R. Since $\mathbb{Z}_2^\times$ acts transitively on the automorphisms of $\hat{G}_m/R$ we obtain for each invariant $f \in V_{\mathbb{F}_2}$ with the help of the base change property

$$f(C_j/R, \varphi_L) = f(C_j/R, \varphi_R) \in \mathbb{F}_2[j^{-1}] = \mathbb{F}_2[\![q_L]\!] \cap \mathbb{F}_2[\![q_R]\!].$$

The claim follows from Proposition 2 since this polynomial in j^{-1} has the same q-expansion as f. □

For a trivialized elliptic curve $(E/R, \varphi)$ let E_{reg} bet the abelian R-group scheme of nonsingular points in E (cf [6] 670f.) Let $E_{can} \subset E_{reg}$ be the group scheme which extends the subgroup $\varphi^{-1}(\mu_2)$ of the formal completion of E. Let π be the projection onto the quotient and $\breve{\pi} : E_{reg}/E_{can} \longrightarrow E_{reg}$ be the dual isogeny. Since $\breve{\pi}$ is etale it induces an isomorphism of the associated formal groups. The Frobenius endomorphism ψ of V is defined by

$$\psi f = (\psi f)(Tate, \varphi_{can}) = f(Tate_{reg}/Tate_{can}, \varphi_{can}\breve{\pi}). \tag{21}$$

This expression apriori lies in $V[\Delta^{-1}]$ but we claim that it actually lies in V. In terms of q-expansions we have

$$(\psi f)(q) = f(q^2) \tag{22}$$

(see [36] p 246 for this computation.). Write $\psi f = \Delta^{-k} g$ for some $g \in V$. Then for all $a \in \mathbb{Z}_2^\times$ it follows from

$$\psi([a]f) = [a](\psi f) = a^{-k}\Delta^{-k}([a]g)$$

that $[a]g$ q-expands in $q^k\mathbb{Z}_2[\![q]\!]$. Suppose the series $\sum g_i$ of modular forms g_i of weight i 2-adically converges to g. Then the first k coefficients $g_{i,r}$ for $0 \leq r \leq k-1$ of their q-expansions satisfy for all $a \in \mathbb{Z}_2^\times$

$$\sum a^i g_{i,r} = 0$$

which implies that they all vanish. Consequently, g_i is divisible by Δ^k and ψf lies in V.

3. Elliptic cohomology theories and Schreier classes

Let E be a multiplicative cohomology theory. Then E is called *complex orientable* if there is a class $x \in \widetilde{E}^2\mathbb{C}P^\infty$ which restricts to $\Sigma^2 1 \in \widetilde{E}^2 S^2$ under the inclusion map of the bottom cell. Any choice of x is a complex orientation of E. An orientation of E supplies a system of Thom and Euler classes for complex vector bundles (cf [2].) The Euler class of a tensor product of line bundles defines a formal group law

$$\hat{G}_E(x, y) = e(L_1 \otimes L_2) \in E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong \pi_* E[\![x, y]\!]$$

with $x = e(L_1)$, $y = e(L_2)$. For instance, if a ring theory E is *even* ($\pi_{odd}E = 0$) and *periodic* ($\pi_2 E$ contains a unit) then the Atiyah-Hirzebruch spectral sequence shows that E is complex orientable. In this case, we write $\hat{G}_E$ for the formal group $\mathrm{spf}(E^0\mathbb{C}P^\infty)$ over $\pi_0 E$. Recall from [29]

Definition 1. *An elliptic spectrum is a triple (E, C, κ) consisting of*

(i) an even periodic ring spectrum E
(ii) a generalized elliptic curve C_E over $\pi_0 E$
(iii) an isomorphism of the formal completion $\hat{C}_E$ of this curve with the formal group $\hat{G}_E$ associated to E over $\pi_0 E$.

Example 4. Suppose we are given a Weierstrass equation over a torsion free $\mathbb{Z}_{(2)}$-algebra. Then by (14) the formal group law (11) associated to the coordinate t satisfies the Landweber exactness conditions if a_1 is invertible (see [46] and [39] for details.) In particular, we obtain the homology theory

$$W_* X = \mathbb{Z}_{(2)}[a_1, \ldots, a_6]\left[a_1^{-1}\right] \otimes_{MU_*} MU_* X.$$

The coefficients a_i of the universal Weierstrass equation have the degree $2i$ and the transformation

$$x = a_1^2 x', \quad y = a_1^3 y' \tag{23}$$

gives a curve over $\pi_0 W$. The homology theory canonically identifies a spectrum W since there are no phantom maps between elliptic spectra (cf [25].) Note that $K_0 W$ concides with the ring $W^{triv}[a_1^{-1}]$ considered in the previous section.

Similarly, there are elliptic spectra Ord and K_{tate} for which the elliptic curves $C_j/Z_2[j^{-1}]^\wedge_2$ and $Tate/D$ are part of the data:

$$Ord_* X = \cong \mathbb{Z}_2[j^{-1}]^\wedge_2[v^\pm] \otimes_{MU_*} MU_* X \tag{24}$$
$$K_{tate*} X = K_*(X; D) \cong D[v^\pm] \otimes_{MU_*} MU_* X. \tag{25}$$

Let E be an elliptic theory and assume that its elliptic curve is given in Weierstrass form. Then we can choose a unit $u \in \pi_2 E$ and transform the Weiserstrass equation for $C_E/\pi_0 E$ into an equation for $C_E/\pi_* E$ with $a_i \in \pi_{2i} E$. We want to be more careful in the choice of the unit in order to apply the formula (11). Moreover, we claim that an elliptic spectrum comes with a preferred nowhere vanishing differential ω_{can} for its elliptic curve over $\pi_* E$ which in a way is due to the suspension isomorphism. Pick a Weierstrass equation for C_E and set $t = -x/y \in E^0 \mathbb{C}P^\infty$ as in equation (11). The formal parameter t gives a unit $u \in \pi_2 E$ when it is restricted to the bottom cell. Now use this unit to transform the curve as in (23). Define the differential ω_{can} such that

$$\omega_{can|U} = (u^{-1} t^0 + \text{ terms of higher order in } t)\, dt.$$

It coincides with the canonical differential for the new equation of $C_E/\pi_* E$ by (6).

Lemma 6. *The differential ω_{can} does not depend on the Weierstrass equation for $C_E/\pi_0 E$.*

Proof. Suppose we are given a change of variables of the form

$$x = v^2 x' + p, \;\; y = v^3 y' + v^2 q x' + r$$

with $v \in \pi_0 E^\times$. Then we have $t = -x/y = v^{-1} t' + O(t'^2)$. Hence t' restricts to $vu \in \pi_2 E$ and we obtain

$$\omega'_{can|U} = ((vu)^{-1} + O(t'))dt' = (u^{-1} + O(t))dt = \omega_{can|U}.$$

□

For the complex orientation $t = -x/y \in E^2 \mathbb{C}P^\infty$ of a normalized Weierstrass equation of $C_E/\pi_* E$ we computed the Hazewinkel generators in (14) and hence get with (15)

$$a_1 \equiv v \bmod 2 \quad \in \pi_2 K \wedge E.$$

Thus the map $E \longrightarrow E[a_1^{-1}]$ is a $K(1)$-equivalence and we conclude

Lemma 7. *For all $K(1)$-localized elliptic spectra E and for all normalized Weierstrass equations of $C_E/\pi_* E$ the class $a_1 \in \pi_2 E$ is a unit. Hence, the canonical class $c_4 = c_4(C_E, \omega_{can}) \in \pi_8 E$ is invertible and $j^{-1} \in \pi_0 E$ is well defined.* □

We now turn to the construction of the canonical class b mentioned in the introduction. To keep the notation simple we make the

Convention. *All statements in the sequel take place in the category $\mathcal{C}$ of $K(1)$-local spectra or in the category $\mathcal{A}$ of Ext-2-complete (cf [17] 172ff) abelian groups unless stated otherwise.*

In $\mathcal{C}$ we have the fibre sequence (cf [13][45])

$$E \longrightarrow KO \wedge E \xrightarrow{(\psi^3 - 1)\wedge 1} KO \wedge E \tag{26}$$

where ψ^3 is the third stable Adams operation.

Definition 2. *A class $b \in \pi_0 KO \wedge E$ is an Artin-Schreier (AS) class if it satisfies the equation*

$$\psi^3 b = b + 1.$$

Lemma 8. *For all $K(1)$-localized elliptic spectra E the sequences*

$$0 \longrightarrow \pi_* KO \wedge E \longrightarrow \pi_* K \wedge E \longrightarrow \pi_{*-2} KO \wedge E \longrightarrow 0$$

$$\pi_0 E \longrightarrow \pi_0 KO \wedge E \xrightarrow{(\psi^3-1)\wedge 1} \pi_0 KO \wedge E$$

are exact. The second sequence is short exact if $\pi_* E$ is torsionfree. In particular, any two AS classes coincide up to translation by a class in the image of $\pi_0 E$.

Proof. The first exact sequence is induced by the fibration (cf [3]) $\Sigma KO \xrightarrow{\eta} KO \longrightarrow K$. It is short exact since $\eta \in \pi_1 S^0$ vanishes in the even theory E. The second exact sequence follows from the first one and the fibration (26). It remains to show that $\pi_1 K \wedge E$ vanishes if $\pi_* E$ is torsionfree. For the unlocalized $K \wedge E$ this is a consequence of the Landweber exactness of K since $E_* MU \otimes_{MU_*} \pi_* K$ is concentrated in even degrees. Hence, after $S\mathbb{Z}/2$-localizing we obtain (cf [13] p 262)

$$\pi_1 L_{S\mathbb{Z}/2}(K \wedge E) = \mathrm{Hom}(\mathbb{Z}/2^\infty, K_0 E) = 0$$

since $K_0 E$ is torsionfree. □

The following result was sketched in [30]:

Proposition 3. *For all K(1)-localized elliptic spectra E there is a canonical AS class b which is natural with respect to maps of elliptic spectra.*

Proof. Let

$$\log(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

be the 2-adic logarithm and set

$$b = -\frac{\log(c_4)}{\log(3^4)} \in V. \tag{27}$$

The logarithm converges 2-adically and is divisible by 16 in the torsion free group V by Lemma 2. In fact, the Katz modular form b coincides with the class b given in (20) over $\mathbb{F}_2$. For each elliptic spectrum E there is a canonical strict isomorphism φ_{can} between $(\hat{C}_E/\pi_0 K \wedge E, \omega_{can})$ and $(\hat{G}_m, du/u)$. (We think of $\pi_0 K \wedge E$ as the quotient $\pi_* K \wedge E/v - 1$.) This gives a natural class

$$b(C_E/\pi_0 K \wedge E, \varphi_{can}) \in \pi_0 K \wedge E.$$

Moreover, the class b is a complex AS class:

$$\psi^3 b = -\tfrac{\log(c_4/3^4)}{\log(3^4)} = -\tfrac{\log(c_4)}{\log(3^4)} + \tfrac{\log(3^4)}{\log(3^4)} = b + 1. \tag{28}$$

Next we show that b gives rise to a unique real class for the torsion free elliptic theory W. Since v^4 concides with the real Bott class under the complexifaction map the class $\log(3^4)b \in \pi_0 K \wedge E$ admits a real lift. Hence, the claim follows from Lemma 8 since the cokernel of the complexifaction map is torsion free.

Next let E be a $K(1)$-localized elliptic spectrum and suppose that its elliptic curve $C_E/\pi_* E$ is given in a normalized Weierstrass form. Then define b to be the image of the AS class of W under the obvious map. Finally for arbitrary E, the scheme $\mathrm{Spec}(\pi_0 E)$ can be covered by open subschemes of the form $\mathrm{Spec}(\pi_0 E[f^{-1}])$ for certain $f \in \pi_0 E$ with the property that $C_E/\pi_0 E[f^{-1}]$ is isomorphic to a Weierstrass curve. The AS class (27) of each ellitptic spectrum $E[f^{-1}]$ only depends on the isomorphism class of the elliptic spectrum by Lemma 6. Hence, they can be patched together to give the desired AS class of E. □

4. E_∞ elliptic spectra and the Frobenius operator

H_∞ structures on complex oriented theories E provide unstable operations: let

$$f : \pi_0 E \longrightarrow R$$

be a ring map and let $H \subset f^* \hat{G}_E$ be a closed finite subgroup. Then there is a new map

$$\psi_H : \pi_0 E \longrightarrow R \tag{29}$$

and an isogeny $f^* \hat{G}_E \longrightarrow \psi_H^* \hat{G}_E$ with kernel H (cf [8][30].)

If the formal group is isomorphic to the multiplicative and H is the canonical subgroup of order 2 then we may take f to be identity map. In this case ψ_H coincides with the operation ψ considered in [7][30][42]. The ring map ψ comes with another operation

$$\theta : \pi_0 E \longrightarrow \pi_0 E$$

of Dyer-Lashof type with the property

$$\psi(x) = x^2 + 2\theta(x) \tag{30}$$

for all $x \in \pi_0 E$. These data impart the structure of a θ-algebra on $\pi_0 E$ which has been extensively studied in [14][15][16].

Example 5. For all spaces X the function spectrum K^{X_+} admits an H_∞ structure since K does (cf [18] VIII.) The resulting θ-algebra structure of $\pi_0 K^{X_+} = K(X)$ is provided by Atiyah's operation θ^2 [9] and ψ is the second unstable Adams operation.

Example 6. Let C be the ring of continuous functions from $\mathbb{Z}_2^\times/\pm$ to $\mathbb{Z}_2$. Recall from [30][47][42] the isomorphism

$$\pi_0 KO \wedge KO \xrightarrow{\cong} C \tag{31}$$

which associates to a class $f \in \pi_0 KO \wedge KO$ the continuous 2-adic function

$$f(\lambda) : S^0 \xrightarrow{f} KO \wedge KO \xrightarrow{1\wedge\psi^\lambda} KO \wedge KO \xrightarrow{\mu} KO$$

for $\lambda \in \mathbb{Z}_2^\times/\pm$. We remind the reader of the convention that we work in the category $\mathcal{C}$ of $K(1)$-local spectra. Hence, KO is 2-adically completed and the stable Adams operations ψ^λ define self maps. The θ-algebra structure of $\pi_0 KO \wedge KO$ is given by $\psi(f) = f$ (see [42] et al.) Since the ring is torsion free the action of θ is forced by (30). Similar things hold for complex K-theory.

Example 7. The free θ-algebra on a generator x is defined by

$$T\{x\} = \mathbb{Z}_2[x, x_1, x_2, \ldots]$$

with $\theta(x) = x_1, \theta(x_i) = x_{i+1}$. Once more, the reader is reminded of the convention that we work in the category $\mathcal{A}$ of Ext-2-complete abelian groups. Hence, the polynomial ring is 2-adically completed. Let TS^0 be the free E_∞ spectrum generated by the sphere. Then McClure showed in [18]IX that the natural map

$$\pi_* E \otimes T\{x\} \longrightarrow \pi_* E \wedge TS^0 \tag{32}$$

is an isomorphism for all $K(1)$-local E_∞ ring spectra E where

$$x : S^0 \xrightarrow{1_T} TS^0 \xrightarrow{1\wedge id} E \wedge TS^0$$

(see [42] for an alternative proof.)

Example 8. In (21) we considered the Frobenius operator ψ of Katz modular forms. Since ψ is a ring map and by (22)

$$\psi f(q) \equiv f(q)^2 \bmod 2$$

there is a unique way to define the operation θ such that (30) is satisfied. Moreover, the Diamond operations commute with ψ and θ. Hence the ring $\mathbb{Z}_2[j^{-1}]$ of invariant Katz modular forms is a sub θ-algebra of V by Lemma 5. Thus the inclusions

$$\mathbb{Z}_2[j^{-1}] \subset V \subset \mathbb{Z}_2[\![q]\!]$$

are compatible with the θ-algebra structures given by $\psi f(q) = f(q^2)$.

The following definition can be found in [30].

Definition 3. *An E_∞ elliptic spectrum is an E_∞ spectrum E with the following data and properties:*

1. *E is an elliptic spectrum*
2. *each isogeny described above extends to an isogeny of the elliptic curve associated to E.*

In [30] a universal relation is constructed which holds for all E_∞ elliptic theories E as follows: consider the Katz modular form

$$f = \psi b - b \in V \tag{33}$$

where b was defined in (27). The Diamand operations act trivially since

$$\begin{aligned}[3]f(E,\varphi) &= f(E, 3^{-1}\varphi) = b(E/H, 3^{-1}\varphi) - b(E, 3^{-1}\varphi)\\ &= \psi(b(E,\varphi)+1) - (b(E,\varphi)+1) = f(E,\varphi).\end{aligned}$$

Thus f is a 2-adic convergent series in j^{-1} by Lemma 5. In fact, it is not hard to check with (22) that f coincides with j^{-1} mod 2. Hence we have

Lemma 9. *(cf [30]]) The map*

$$\mathbb{Z}_2[f] \longrightarrow \mathbb{Z}_2[j^{-1}]$$

is an isomorphism.

Let h be the 2-adically convergent power series with

$$h(f) = \theta(f) \in \mathbb{Z}_2[j^{-1}].$$

Then our construction gives:

Proposition 4. *For all $K(1)$-local E_∞ elliptic spectra E we have the relation*

$$(\theta(f) - h(f))(C_E) = 0 \ \in \pi_0 E. \tag{34}$$

Proof. The evaluation map

$$\mathbb{Z}_2[j^{-1}] \longrightarrow \pi_0 E,\ j^{-1} \mapsto j^{-1}(C_E)$$

is a map of θ-algebras. □

We now turn to the construction of an E_∞ model for the spectrum *tmf* in the $K(1)$-local category. The generator ζ of $\pi_{-1}S^0$ is defined by the image of the unit under the map

$$\pi_0 KO \longrightarrow \pi_{-1}S^0,\ 1 \mapsto \zeta$$

which is induced by the fibre sequence (26). Its Hurewicz image in $\pi_{-1}E$ vanishes for all elliptic spectra E. Let T_ζ be the cone defined by the homotopy pushout diagram of K(1)-local E_∞ spectra

$$\begin{tikzcd} TS^{-1} \arrow[r, "T*"] \arrow[d, "\zeta"] & T* = S^0 \arrow[d] \\ S^0 \arrow[r] & T_\zeta \end{tikzcd} \cdot$$

(TX is the free E_∞ spectrum generated by the pointed space X.) Then there is a quite canonical E_∞ map from the cone T_ζ into each E_∞ elliptic spectrum E. However, it turns out that T_ζ is still far off to be the desired universal object *tmf*. To see this, recall from [30] or [42] that

$$\pi_* KO \wedge T_\zeta \cong \pi_* KO \otimes T\{b\} \tag{35}$$

where b is an AS-class. In fact, this is an immediate consequence of the vanishing of $KO \wedge \zeta$ and McClure's result (32). The homotopy groups of T_ζ are more difficult to determine. The following result of [30] is sketched in the appendix.

Proposition 5. *The sequence*

$$\pi_* T_\zeta \longrightarrow \pi_* KO \wedge T_\zeta \xrightarrow{\psi^3 - 1} \pi_* KO \wedge T_\zeta$$

is short exact. Moreover, there is an isomorphism of θ-algebras

$$\pi_* T_\zeta \cong \pi_* KO \wedge T\{f\}$$

with $f = \psi b - b$.

Set $y = \theta(f) - h(f) \in \pi_0 T_\zeta$. Then the universal relation (34) tells us that y vanishes for all E_∞ elliptic spectra E under the canonical map from T_ζ. Hence, define M to be the E_∞ homotopy pushout of the diagram

$$\begin{tikzcd} TS^0 \arrow[r, "*"] \arrow[d, "y"] & T* = S^0 \arrow[d] \\ T_\zeta \arrow[r] & M \end{tikzcd} \cdot$$

Then we have shown

Corollary 2. *For all E_∞ elliptic spectra E there is a quite canonical E_∞ ring map from M to E.*

Also the following result of [30] is sketched in the appendix.

Proposition 6. *There are isomorphisms*

$$\pi_* KO \wedge M \cong \pi_* KO \otimes T\{b\} \otimes_{T\{y\}} \mathbb{Z}_2$$
$$\pi_* M \cong \pi_* KO[j^{-1}].$$

In the introduction we discussed the construction of the spectrum *tmf* by some homotopy inverse limit of (preferably E_∞) elliptic spectra E. Corollary 2 provides us with a map from M to *tmf*. It can be shown to be a homotopy equivalence since it induces an isomorphism in homotopy by the previous proposition. This justifies the

Convention. *In the rest of the paper, references to tmf are actually references to the two cells complex M.*

Corollary 3. *There is a homotopy equivalence of $K(1)$-local spectra*

$$tmf \cong KO[j^{-1}].$$

Proof. By Lemma 18 of the appendix there is a projection ρ of the free $T\{f\}$-module $T\{b\}$ onto the summand $T\{f\}$. Hence Lemma 9 gives a natural transformation of homology theories

$$\begin{aligned}
\pi_* tmf \wedge X &\longrightarrow \pi_* tmf \wedge KO \wedge X \\
&\cong \pi_* KO \wedge tmf \otimes_{\pi_* KO} \pi_* KO \wedge X \\
&\cong \pi_* KO \wedge X \otimes T\{b\} \otimes_{T\{f\}} T\{f\} \otimes_{T\{y\}} \mathbb{Z}_2 \\
&\xrightarrow{1\otimes\rho} \pi_* KO \wedge X[j^{-1}]
\end{aligned}$$

which is an isomorphism on the coefficients. □

5. The proof of Theorem 3

At this point we do not know wether the spectrum K_{tate} (see (25)) carries the structure of an E_∞ elliptic spectrum. Hence we are not in a position simply to smash a canonical map from *tmf* to K_{tate} with KO. However, Proposition 3 provides the AS element

$$b\big(\widehat{Tate}/\pi_0 KO \wedge K_{tate}, \varphi_{can}\big) = \frac{\log\big(c_4\big(Tate, v_R^{-1} dx/2y + x\big)/v_L^4\big)}{\log(3^4)}$$

with v_L, v_R the left and right complex Bott classes. Moreover, the isomorphism

$$\pi_0 KO \wedge K_{tate} \cong C \otimes D$$

gives the left hand side a θ-algebra structure. We thus may define a map of θ-algebras $g : \pi_* KO \wedge tmf \longrightarrow \pi_* KO \wedge K_{tate}$ by requiring that the AS class b of Proposition 6 is sent to the above AS class b_{tate}. This makes sense since the universal relation (34) holds by construction for K_{tate}. Theorem 3 follows from

Theorem 4. *The composite*

$$\pi_0 KO \wedge tmf \xrightarrow{g} \pi_0 KO \wedge K_{tate} \xrightarrow{\mu} \pi_0 K_{tate} \cong D$$

is an isomorphism.

The rest of the section assembles results to prove Theorem 4. Consider the ring map $\lambda_v : \pi_* KO[j^{-1}] \longrightarrow \pi_* K \otimes D$ given by

$$\lambda_v(j^{-1}) = 1 \otimes j^{-1},\ \lambda_v(2v^2) = 2v^2 \otimes \sqrt{E_4},\ \lambda_v(v^4) = v^4 \otimes E_4.$$

Let ν be the composite

$$\pi_* KO \otimes \pi_* KO[j^{-1}] \overset{\mu(1\otimes i)}{\longrightarrow} \pi_* KO \wedge T_\zeta[j^{-1}] \longrightarrow \pi_* KO \wedge tmf$$

where i is defined in (40). Then we have

Lemma 10. *The diagram*

$$\begin{tikzcd}
\pi_* KO \otimes \pi_* KO[j^{-1}] \arrow[r, "1\otimes\lambda_v"] \arrow[d, "\nu"] & \pi_* KO \otimes \pi_* K \otimes D \arrow[r, "\mu\otimes 1"] & \pi_* K \otimes D \arrow[d, "\cong"] \\
\pi_* KO \wedge tmf \arrow[r, "g"] & \pi_* KO \wedge K_{tate} \arrow[r, "\mu"] & \pi_* K_{tate}
\end{tikzcd}$$

is commutative.

Proof. Since g is a map of θ-algebras which sends b to b_{tate} the class f is send to f_{tate} and hence j^{-1} really deserves its name. Moreover, all arrows of the diagrams are maps of left $\pi_* KO$ modules. Hence, the claim follows from

$$g\nu(v^{-4} \otimes v^4) = g(3^{-4b}) = 3^{-4b_{tate}} = (v_R/v_L)^4 E_4. \tag{36}$$

□

Lemma 11. *The map*

$$\lambda g : \pi_0 KO \wedge tmf \longrightarrow \pi_0 KO \wedge K_{tate} \cong C \otimes D \subset C[\![q]\!]$$

is the inclusion of a pure subgroup.

Proof. Let $s' : C[\![q]\!] \longrightarrow C[\![q]\!]$ be the $\mathbb{Z}_2[\![q]\!]$-linear extension of

$$C \overset{s}{\longrightarrow} T\{b\} \cong \pi_0 KO \wedge T_\zeta \longrightarrow \pi_0 KO \wedge tmf \overset{\lambda g}{\longrightarrow} C[\![q]\!]$$

where s is given in Lemma 18. Since s is a section and C is free it is not hard to see that s' is an isomorphism. Moreover, in the commutative diagram

$$\begin{tikzcd}
C[j^{-1}] \arrow[r, "\mu(s\otimes i)"] \arrow[d, "\lambda"] & \pi_0 KO \wedge tmf \arrow[d, "\lambda g"] \\
C[\![q]\!] \arrow[r, "s'"] & C[\![q]\!]
\end{tikzcd}$$

the upper horizontal arrow is an isomorphism by Proposition 6 and (18). The left vertical q-expansion map λ is pure and hence λg is so as well. □

Proof of Theorem 4. Consider the composite

$$\pi_0 KO \wedge tmf \xrightarrow{\mu g} D \longrightarrow C[\![q]\!]$$

in which the last map sends a sum $f = \sum f_i$ in D to the function

$$a \mapsto [a]f = \sum a^i f_i(q).$$

One readily verifies with Lemma 10 that it coincides with λg and hence is injective mod 2 by Lemma 11. We conclude that μg is an injection mod 2 as well. To construct its inverse consider the composite

$$mf_* \xrightarrow{\iota} \mathbb{Z}_2[j^{-1}, v^{\pm 2}] \longrightarrow \pi_0 K \wedge tmf$$

where ι was defined in (9) and the last map is given by

$$v^2 \mapsto \nu(v^{-2} \otimes v^2) \in \pi_* K \wedge tmf.$$

One easily checks with Lemma 10 that it is a section of μg. Hence for all finite congruences there is a multiple which can be lifted to $\pi_0 KO \wedge tmf$. Since the cokernel of μg is torsion free by Lemma 11 we can actually lift the congruence. □

Remark 2. It follows that the equivalence

$$tmf \cong KO[j^{-1}]$$

is only additive: the idempotents of the ring D are trivial whereas the ring $\pi_0 KO \wedge KO[j^{-1}]$ has many idempotents. Hence, the situation is similar to the one of the elliptic cohomology which was defined by Kreck and Stolz in [38].

Remark 3. Theorem 3 shows that the elliptic spectrum K_{tate} is in fact an E_∞ elliptic spectrum: there is a homotopy equivalence of ring spectra between K_{tate} and the spectrum $K \wedge tmf$. Since K-theory admits an E_∞ ring structure [24] so does $K \wedge tmf$. Furthermore, we obtain a canonical E_∞ map

$$tmf \longrightarrow K_{tate}$$

which is the one considered in Theorem 2 when q-expanded.

6. Witten's map of θ-algebras

In order to show that the Witten orientation W_q of (4) induces a map of θ-algebras in KO-homology we could proceed as follows: it is known from [19]VIII that the ABS-orientation $z : MSpin \longrightarrow KO$ is an H_∞ map.

Presumably the exponential formula

$$S_{q^k}(V \otimes W) = S_{q^k}V \otimes S_{q^k}W$$

is sufficiently equivariant to imply that $w(V) = \bigotimes_k S_{q^k}(\dim V - V)$ defines an H_∞ map

$$w : \Sigma^\infty BSpin_+ \longrightarrow KO[\![q]\!].$$

If so, then it follows that

$$W_q : MSpin \longrightarrow MSpin \wedge BSpin_+ \xrightarrow{z \wedge w} KO \wedge KO[\![q]\!] \longrightarrow KO[\![q]\!]$$

is an H_∞ map and hence induces a map of θ-algebras.

In this section we offer a different way to proceed which relies on Theorem C of [42]:

Theorem 5. *The θ-algebra structure of $\pi_0 K \wedge MU$ is determined by the equation*

$$\sum_{i \geq 0} \psi(b_i) x^i = g(2)^{-1} g(1 + \sqrt{1-x}) g(1 - \sqrt{1-x}).$$

Here, the coefficients of the power series $g(x) = \sum_i b_i x^i$ are the usual free generators of $\pi_0 K \wedge MU$ (cf [2].)

Let

$$\theta(q, u) = (1-u) \prod_{n \geq 1} \frac{(1-q^n u)(1-q^n u^{-1})}{(1-q^n)^2}$$

be the normalized θ-function and let $x_K : MU \longrightarrow K$ be the standard orientation. Then for $u = 1 - x_K$ the power series

$$\theta(q, u) \in \pi_* K[\![q, x_K]\!] \cong K^* \mathbb{C}P^\infty[\![q]\!]$$

defines a new orientation t of $K[\![q]\!]$. Its behavior in K-homology is described by

Lemma 12. *Identify $\pi_0 K \wedge K$ with the ring of continuous functions C' from $\mathbb{Z}_2^\times$ to $\mathbb{Z}_2$ as in Example 6. Then for all $k \in \mathbb{Z}_2^\times$ we have the formula*

$$\sum_{i \geq 0} t_* b_i(k)\, x^i = \frac{\theta(q, (1-x)^k)}{x}.$$

Proof. For line bundles L the ordinary Thom class $z_K = 1 - L^*$ is related to the new one z_t by the formula

$$z_t = \theta(q, 1 - x_K)x_K^{-1} z_K.$$

Hence for all $k \in \mathbb{Z}_2^\times$ we have

$$\psi^k(z_t) = \frac{\psi^k(z_K)\theta\big(q, (1-x_K)^k\big)}{1-(1-x_K)^k} = \frac{\theta\big(q, (1-x_K)^k\big)}{x_K} z_K.$$

which gives the desired formula (compare [42] 3.) $\square$

Proposition 7. *The t-orientation induces a map of θ-algebras*

$$\pi_* K \wedge MU \longrightarrow \pi_* K \wedge K[\![q]\!].$$

Proof. By Theorem 5 it suffices to check the equality

$$\textstyle\sum_i \psi(t_* b_i)x^i = t_*(g(2)^{-1} g(1+\sqrt{1-x})g(1-\sqrt{1-x})).$$

With Lemma 12 this equation reads

$$\frac{\theta(q^2, (1-x)^k)}{x} = \frac{\theta(q, -\sqrt{1-x}^k)\theta(q, \sqrt{1-x}^k)2}{(1+\sqrt{1-x})(1-\sqrt{1-x})\theta(q, -1)}$$

which is easily verified by expanding the θ-series. $\square$

The t-orientation is related to W_q of (4) by the

Lemma 13. *The diagram*

$$\begin{tikzcd} MSU \arrow[r] \arrow[d] & MSpin \arrow[r, "W_q"] & KO[\![q]\!] \arrow[d] \\ MU \arrow[rr, "t"] & & K[\![q]\!] \end{tikzcd}$$

commutes.

Proof. This is readily verified with the Chern character. $\square$

Lemma 14. *The map $W_{q_*} : \pi_0 KO \wedge MO\langle 8\rangle \longrightarrow \pi_0 KO \wedge K[\![q]\!]$ is a map of θ-algebras.*

Proof. The induced map from MSU to $MSpin$ is surjective in K-homology. Hence the assertion follows from Proposition 7 and Lemma 13. $\square$

We next lift the W_q-orientation of $K[\![q]\!]$ to K_{tate}. In (2) we have seen that the natural map from $MO\langle 8\rangle$ to $MSpin$ is a $K(1)$-equivalence. The Anderson-Brown-Peterson (ABP) splitting [4] implies that $MSpin$ is a sum of KO-theories. Hence the universal coefficient spectral sequence [1] for the K-module spectrum K_{tate} gives an isomorphism

$$K^0_{tate}(MO\langle 8\rangle) \cong \mathrm{Hom}_{cts}(\pi_0 K \wedge MO\langle 8\rangle, \pi_0 K_{tate}).$$

Thus it suffices to lift the map

$$\pi_0 K \wedge MO\langle 8\rangle \longrightarrow \mathbb{Z}[\![q]\!]$$

induced by W_q to the ring of divided congruences D.

The Weiserstrass σ-function is related to θ by the formula [52]

$$\sigma(u,q) = e^{-G_2 x^2} u^{-1/2}\theta(u,q)$$

for $u = e^{2\pi i x}$ and admits an expansion of the form

$$\sigma(u,q) = x\ \exp\Big(-\sum_{k=2}^{\infty}\frac{2}{(2k)!}G_{2k}x^{2k}\Big)$$

where $G_{2k} = (-B_{2k}/4k)E_{2k}$ are the divided Eisenstein series. We conclude that σ expands in $D \otimes \mathbb{Q}[\![x]\!]$ and G_2 is the only term which keeps us from lifting W_q. This term disappears for bundles with vanishing first Pontryaging class in the computation of the Chern character of its Witten Thom class (cf [28] 84f). We conclude that W_q lifts to a unique ring map from $MO\langle 8\rangle$ to K_{tate}.

Proposition 8. *The composite*

$$W'_* : \pi_0 KO \wedge MO\langle 8\rangle \longrightarrow \pi_0 KO \wedge K_{tate} \longrightarrow \pi_0 K_{tate} \cong \pi_0 KO \wedge tmf$$

is a map of θ-algebras. Moreover, the $K(1)$-local Witten genus uniquely lifts to $\pi_0 tmf$.

Proof. The first claim follows from Lemma 14. For the second, observe that the map W'_* is compatible with the Adams operations. □

7. The proofs of the Theorems 1 and 2

In this section we show that the Witten orientation comes from a map of E_∞ ring spectra into *tmf*. For that, we construct an explicit lift of the AS-class b of *tmf* to $MO\langle 8\rangle$ with geometrical methods.

We let $\pi^j \in KO(BSpin)$ be the jth KO-characteristic class of the universal stable Spin bundle. Without changing the notation we consider the same class as an object of $\tilde{KO}(MSpin)$ via the Thom isomorphism. For any non ordered sequence of positive numbers (partition) $J = (j_1, \dots, j_n)$ we set

$$\pi^J = \pi^{j_1} \cdots \pi^{j_n} : MSpin \longrightarrow KO.$$

Then the ABP-splitting map [4] induces an isomorphism

$$(\pi^{\emptyset}, \pi^{(2)}) : \pi_8 MSpin \cong \mathbb{Z} \oplus \mathbb{Z}.$$

and $\pi^{\emptyset}$ is the $\hat{A}$-genus. Hence there are 8-dimensional Spin manifolds with $\hat{A}$-genus 1. We want to make a convenient choice of such a Bott manifold. The quaternian projective space $\mathbb{H}P^2$ admits a metric of positive scalar curvature and so its $\hat{A}$-genus vanishes [44][43]8.9. Hence any Bott manifold can be altered by a multiple of $\mathbb{H}P^2$. Since the signature $\mathrm{sig}(\mathbb{H}P^2)$ is 1 we can find a Bott manifold $\mathfrak{B}$ with vanishing signature.

To make $\mathfrak{B}$ explicit, consider the Kummer surface

$$\mathfrak{K} = \left\{ z \in \mathbb{C}P^3 \middle| \, z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0 \right\}.$$

The homogeneous polynomial defines a section of the fourth power of the Hopf bundle over $\mathbb{C}P^3$ which transversally intersects the 0-section in $\mathfrak{K}$. Hence this line bundle gives the normal bundle of $\mathfrak{K}$ in $\mathbb{C}P^3$. The surface $\mathfrak{K}$ intersects $[\mathbb{C}P^1]$ in 4 points. Thus the image of its fundamental class $[\mathfrak{K}]$ under the map induced by the inclusion is $4[\mathbb{C}P^2] \in H_4\mathbb{C}P^3$. With these data it is elementary to check that

$$\hat{A}(\mathfrak{K}) = 2, \ \mathrm{sig}(\mathfrak{K}) = -16$$

which gives the relation

$$\mathfrak{B} = \mathfrak{K}^2/4 - 64\,\mathbb{H}P^2. \tag{37}$$

Lemma 15. *The sequence*

$$\pi_8 MO\langle 8\rangle \longrightarrow \pi_8 MSpin \xrightarrow{p_1^2} \mathbb{Z}$$

is exact. Moreover, any $O\langle 8\rangle$-manifold is spin bordant to a multiple of

$$\mathfrak{M} = \mathfrak{B} - 224\,\mathbb{H}P^2. \tag{38}$$

Proof. It is elementary to check that

$$p_1^2(\mathfrak{M}) = 0, \ \mathrm{sig}(\mathfrak{M}) = -224.$$

Kervaire and Milnor [37] showed the existence of an almost parallelizable manifold $\mathfrak{M}'$ with signature 224. Hence, $\mathfrak{M}$ is spin bordant to an $O\langle 8\rangle$-manifold. Since its $\hat{A}$-genus is 1 any other class in the kernel must be a multiple of $\mathfrak{M}$ and the claim follows. □

Proposition 9. *The class*

$$b^W = -\frac{\log(\mathfrak{M}/v^4)}{\log(3^4)} \in \pi_0 KO \wedge MO\,\langle 8\rangle$$

is an AS class. Moreover, the map W'_* sends b^W to the canonical AS class $b \in \pi_0 KO \wedge tmf$.

Proof. By a calculation similar to (28) it suffices to show that the quotient is well defined. This will be done in Lemma 16. For the second claim observe that the Witten genus of $\mathfrak{M}$ is a modular form of weight 4 and hence a multiple of E_4. Since the constant term of its q-expansion is the $\hat{A}$-genus we have $W_{q_*}(\mathfrak{M}) = E_4$. Hence the AS classes coincide by (27) and the explicit isomorphism given in Theorem 4. □

Lemma 16. *The class $v^4 - \mathfrak{M}$ is divisible by 16 in $\pi_8 KO \wedge MO\,\langle 8\rangle$.*

Proof. Compute for all $\lambda \in \mathbb{Z}_2^\times$

$$\big(\mu\big(\psi^\lambda \wedge \pi_*^\emptyset\big)\big)(v^4 - \mathfrak{M}) = (\lambda^4 - 1)\, v^4 \equiv 0 \bmod 16.$$

and

$$\big(\mu\big(\psi^\lambda \wedge \pi_*^{(2)}\big)\big)(v^4 - \mathfrak{M}) = \pi^{(2)}(\mathfrak{M})v^4.$$

To evaluate the latter we compute with the topological Riemann-Roch formula

$$\pi^{(2)}(\mathfrak{K}^2) = \big\langle \hat{A}(T\mathfrak{K}^2)ch(\pi^{(2)}(\nu) \otimes \mathbb{C}), [\mathfrak{K}^2]\big\rangle = 2304.$$

which shows

$$\pi^{(2)}(\mathfrak{M}) \equiv 0 \bmod 16.$$

Hence, the assertion follows from the ABP-splitting and the interpretation of $\pi_* KO \wedge KO$ given in Example 6. □

Remark 4. Since the Kummer surface admits an SU-structure on its stable tangent bundle one might hope to obtain an Artin-Schreier class in $\pi_0 KO \wedge MSU$ this way. For that, consider $\mathfrak{K}$ as an object of

$$\pi_* K \wedge MSU \subset \pi_* K \wedge MU \cong \pi_* K[b_1, b_2, \dots].$$

To determine the polynomial we can first work out its Chern numbers to obtain

$$\mathfrak{K} = 18(\mathbb{C}P^1)^2 - 16\mathbb{C}P^2.$$

Then an easy calculation with Miscenko's formula [2] gives

$$\mathfrak{K} = 48b_2 - 24b_1^2 - 24vb_1 + 2v^2 \in \pi_4 K \wedge MSU.$$

Hence $\mathfrak{M} - v^4$ is not divisible by 16 and we can not set up the logarithm to lift the AS class to MSU.

Lemma 17. *Let b and b' be two AS classes of $MSpin$. Then there is an E_∞ self homotopy equivalence $MSpin$ which sends b to b'.*

Proof. The short exact sequence (26)

$$0 \longrightarrow \pi_0 MSpin \longrightarrow \pi_0 KO \wedge MSpin \xrightarrow{\psi^3-1} \pi_0 KO \wedge MSpin \longrightarrow 0$$

tells us that b and b' can only differ by a class $a \in \pi_0 MSpin$. Let the E_∞ self map of (1)

$$MSpin \cong T_\zeta \wedge \bigwedge TS^0$$

be the identity on each TS^0. On T_ζ let it alter the given null homotopy of ζ by the class a. Then its inverse is defined in the same way with a replaced by $-a$. □

The lemma gives a splitting

$$(\tilde{\varphi}, (\tilde{x}_k)) : T_\zeta \wedge \bigwedge TS^0 \xrightarrow{\cong} MO\langle 8\rangle \tag{39}$$

for which the AS class $b \in \pi_0 KO \wedge T_\zeta$ maps to b^W. Let $e_k \in \pi_0 tmf$ be the images of the free generators $\tilde{x}_k$ under the Witten genus of Proposition 8. Define the E_∞ map

$$W : MO\langle 8\rangle \cong T_\zeta \wedge \bigwedge TS^0 \longrightarrow tmf$$

by the canonical map on T_ζ and by e_k on the free components. Its induced map in KO-homology coincides with the map W'_* on the θ-algebra generators $b^W, \tilde{x}_1, \tilde{x}_2, \ldots$ by Proposition 9 and hence on all classes by Proposition 8. Thus we conclude with (37) that W makes the diagram of Theorem 2 commutative.

For the cellular decomposition of W in Theorem 1 observe that we have a surjection by Lemma 9

$$\pi_0 T_\zeta \xrightarrow{\tilde{\varphi}} \pi_0 MO\langle 8\rangle \xrightarrow{W_*} \pi_0 tmf.$$

Hence, there are classes $t_k \in \pi_0 T_\zeta$ which lift $W_*(\tilde{x}_k)$. For $k \geq 1$ define

$$x_k = \tilde{x}_k - t_k \in \pi_0 MO\langle 8\rangle$$

and set $x_0 = \tilde{\varphi}(y)$ where y was defined in Proposition 6. Then all x_k are annihilated by the Witten genus. Thus the pushout square of Theorem 1 will be a consequence of the splitting (39) and Proposition 6 once the self map

$$(id, (x_k)) : T_\zeta \wedge \bigwedge TS^0 \longrightarrow T_\zeta \wedge \bigwedge TS^0$$

is an isomorphism. This is clear since its inverse is given by the map $(id, (\tilde{x}_k + t_k))$.

Finally, it remains to show the Conner-Floyd isomorphism of Corollary 1: by Theorem 1 and (39) we have the isomorphism

$$\pi_* MO\langle 8\rangle \wedge X \otimes_{\pi_* MO\langle 8\rangle} \pi_* tmf \cong \pi_* T_\zeta \wedge X \otimes_{T\{y\}} \mathbb{Z}_2.$$

Moreover, this tensor product gives a homology theory since it is isomorphic to $\pi_* KO \wedge X[f]$ as in Corollary 3. □

Appendix. The homotopy groups of T_ζ and M

The appendix is devoted to the computation of the homotopy groups of the cone T_ζ and the two cells complex M. The calculations can be found in [30] and are briefly reproduced here.

Define the ring map $i : \pi_* KO \longrightarrow \pi_* KO \wedge T_\zeta$ by $i(\eta) = \eta$ and

$$i(2v^2) = 2v^2 3^{-2b} = 2v^2 \sum_{n=0}^{\infty} \binom{-b}{n} 2^{3n} \in T\{b\} \tag{40}$$

$$i(v^4) = v^4 3^{-4b} = v^4 \sum_{n=0}^{\infty} \binom{-2b}{n} 2^{3n} \in T\{b\}. \tag{41}$$

We are going to construct a section s to the the canonical map of θ-algebras

$$p : \pi_* KO \otimes T\{b\} \cong \pi_* KO \wedge T_\zeta \longrightarrow \pi_* KO \wedge KO \cong C$$

which is compatible with the Adams operations and the underlying coalgebra structures. Suppose we have already found the section s. Then Proposition 5 follows from the exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow C \xrightarrow{\psi^3 - 1} C \longrightarrow 0$$

and the

Lemma 18. *The additive map*

$$\mu(s \otimes i) : C \otimes \pi_* KO \otimes T\{f\} \longrightarrow \pi_* KO \wedge T_\zeta$$

is an isomorphism.

Proof. Let $\mathbb{W} = \mathbb{Z}_2[a_0, a_1, a_2, \dots]$ be the 2-complete Witt algebra. Then $\mathbb{W}$ has a unique (2-complete) Hopf algebra structure for which the Witt vectors

$$w_n = a_0^{2^n} + 2a_1^{2^{n-1}} + \cdots + 2^n a_n$$

are primitive. There is a unique θ-algebra structure on $\mathbb{W}$ with $\psi(w_i) = w_{i+1}$. Moreover, by Dwork's Lemma [23] there is a unique isomorphism of θ-algebras from $\mathbb{W}$ to $T\{b\}$ which sends w_0 to b. The composite

$$\iota : \mathbb{W} \cong T\{b\} \longrightarrow C$$

sends a_0 to the map $3^n \mapsto n$ and

$$\iota(a_i) \equiv (3^n \mapsto n_i) \bmod 2$$

for $n = \sum n_i 2^i$. By [42] 3.10 this shows that the sequence of Hopf algebras

$$T\{f\} \xrightarrow{\psi(b)-b} T\{b\} \xrightarrow{p} C \tag{42}$$

is short exact. Since the section is compatible with the coalgebra structure the result follows from the standard Milnor-Moore argument. $\square$

In the following we identify C with the ring of continuous functions on Z_2 via our choice of the topological generator of $\mathbb{Z}_2/\pm$. Then the binomial functions

$$n \mapsto \binom{n}{k}$$

form a basis. The coalgebra structure of C is given by the Cartan formula. In order to define the desired section s it suffices to lift each binomial function to the Witt algebra in a way that all structures are preserved. For $a = (a_1, a_2, \dots)$ let $A = \mathbb{Z}_2[a]$ and formally write $(1-x)^a$ for the power series $\prod(1 - a_n x^n)$. Then the big Witt vectors w_n^B are defined by the expansion

$$x\, d\log(1-x)^a = -\sum w_n^B x^n.$$

The algebra A has a unique Hopf algebra structure for which the big Witt vectors are primitive. The resulting 'group law' takes the form

$$(1-x)^{a+b} = (1-x)^a(1-x)^b.$$

Another application of Dwork's lemma provides us with a unique map of Hopf algebras $f : A \longrightarrow W$ with the property that

$$f\big(w^B_{2^k m}\big) = w_k \text{ for all } 2 \nmid m.$$

The composite pf sends each w_n^B to the identity map. Furthermore, the series $(1-x)^a$ is sent to the map

$$n \mapsto (1-x)^n.$$

Hence, when we define c_k by the equation

$$(1-x)^a = \sum c_k(-x)^k$$

then their image under f in $\mathbb{W}$ gives the desired lift.

It remains to prove Proposition 6. The map y is the composite of

$$TS^0 \xrightarrow{\theta(x)-h(x)} TS^0 \xrightarrow{f} T_\zeta.$$

The second θ-algebra map is etale in K-homology by Lemma 18. In terms of generators the first one is

$$\mathbb{Z}_2[y_0, y_1, \dots] \longrightarrow \mathbb{Z}_2[y_0, y_1, \dots][x_0] \longrightarrow \mathbb{Z}_2[x_0, x_1, \dots].$$

Here, y_0 is sent to $x_1 - h(x_0)$ and hence y_i is mapped to x_{i+1} modulo monomials in $(x_0, x_1, \dots, x_i)$. We conclude that the last map is an isomorphism and thus y is flat in K-homology. The assertion now follows from the Eilenberg-Moore spectral sequence. □

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