Bi-Connexive Variants of Heyting-Brouwer Logic

(joint work with Norihiro Kamide)

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In (Wansing 2008), sixteen variants of **Heyting-Brouwer logic**, HB, also known as **bi-intuitionistic logic**, BiInt, are presented semantically and as display sequent systems.

For want of a better terminology and notation they were referred to as systems \((I_i, C_j)\), where \(i\) and \(j\) range over four ways of interpreting negated implications and co-implications, respectively.

In this talk, I will focus on the system \((I_2, C_2)\) with a connexive reading of negated implications and co-implications. The system \((I_2, C_2)\) is thus a **bi-intuitionistic connexive logic** (or **connexive Heyting-Brouwer logic**), and we therefore refer to it here as BCL.

The logic BCL may also be seen as an extension of the connexive logic C from (Wansing 2005) by the co-implication of BiInt, presuming a connexive understanding of negated co-implications.
Systems of connexive logic and the bi-intuitionistic logic Bilnt have been carefully studied since the 1960s and 1970s with various philosophical and mathematical motivations. The characteristic principles of connexive logic are usually traced back to Aristotle and Boethius, and the co-implication of Bilnt can be traced back to Thoralf Skolem and Grigore Moisil.

A distinctive feature of connexive logics is that they validate the so-called

**Aristotle’s theses:** \( \sim(\alpha \rightarrow \sim \alpha) \) and \( \sim(\sim \alpha \rightarrow \alpha) \), and

**Boethius’ theses:** \( (\alpha \rightarrow \beta) \rightarrow \sim(\alpha \rightarrow \sim \beta) \) and \( (\alpha \rightarrow \sim \beta) \rightarrow \sim(\alpha \rightarrow \beta) \).
A **constructive connexive modal logic**, CK, which is a constructive connexive analogue of the smallest normal modal logic K, was introduced in (Wansing 2005) by extending the mentioned constructive connexive logic, C, which is a connexive variant of **Nelson’s paraconsistent logic**.

A **classical connexive modal logic** called CS4, which is based on the positive normal modal logic S4, was introduced in (Kamide and Wansing 2011) as a Gentzen-type sequent calculus. The Kripke-completeness and cut-elimination theorems for CS4 were shown, and CS4 was shown to be embeddable into positive S4 and to be decidable. Moreover, it was shown that the basic constructive connexive logic C can be faithfully embedded into CS4 and into a subsystem of CS4 lacking syntactic duality between necessity and possibility.
Heyting-Brouwer logic, which is an extension of both dual-intuitionistic logic, DualInt, and intuitionistic logic, Int, was introduced by Rauszer (1974, 1977, 1980), who proved algebraic and Kripke completeness theorems for Bilnt.

As was shown by Uustalu in 2003, the original Gentzen-type sequent calculus by Rauszer does not enjoy cut-elimination, and various kinds of sequent systems for Bilnt have been presented in the literature, including cut-free display sequent calculi.

Moreover, Bilnt is known to be a logic that has a faithful embedding into the future-past tense logic KtT4 (Łukowski 1996), and a modal logic based on Bilnt was studied in (Łukowski 2002).
Dual-intuitionistic logics are logics which have a Gentzen-type sequent calculus in which sequents have the restriction that the antecedent contains at most one formula. This restriction of being singular in the antecedent is syntactically dual to that in Gentzen’s sequent calculus LJ for intuitionistic logic, which is singular in the consequent.

Historically speaking, the logics in the set of logics containing Czermak’s (1977) *dual-intuitionistic calculus*, Goodman’s (1981) *logic of contradiction* or *anti-intuitionistic logic*, and Urbas’s (1996) extensions of Czermak’s and Goodman’s logics were collectively referred to by Urbas as dual-intuitionistic logics.
The dual-intuitionistic logic referred to as DualInt in (Goré 2000 and Wansing 2013) is the implication-free fragment of BiInt (in a language with constants $\bot$ and $\top$, but without intuitionistic negation as primitive).

An interpretation of DualInt as the *logic of scientific research* was presented by Shramko (2005).

In this talk I will combine the two approaches and introduce the **bi-intuitionistic connexive logic** (or *connexive Heyting-Brouwer logic*), BCL, as a Gentzen-type sequent calculus.
We will proceed as follows.

First, the logic BCL is introduced as a Gentzen-type sequent calculus, and a dual-valuation-style Kripke semantics for BCL is defined.

Gentzen-type sequent calculi ICL, DCL, BL, IL and DL for intuitionistic connexive logic, dual-intuitionistic connexive logic, bi-intuitionistic logic, intuitionistic logic and dual-intuitionistic logic, respectively, are straightforwardly defined as subsystems of BCL.
Next, some theorems for syntactically and semantically embedding BCL into BL are proved, and using these theorems, the completeness theorem with respect to the Kripke semantics for BCL is shown as a central result of this talk.

The cut-elimination theorems for ICL and DCL are shown using some restricted versions of the syntactical embedding theorem of BCL into BL.

The cut-elimination theorem does not hold for the sequent systems BCL and BL.
Then some theorems for syntactically embedding ICL into DCL and vice versa are shown. These theorems reveal that ICL and DCL are syntactically dual to each other in a certain sense. Thus, it is shown in these theorems that BCL is constructed based on a duality principle of the characteristic subsystems.

Finally, we present a sound and complete tableau calculus for BCL and its subsystems ICL, DCL, BL, IL, and DL using triply-signed formulas.
Formulas are constructed from countably many propositional variables \( p, q, \ldots \), the binary connectives \( \land \) (conjunction), \( \lor \) (disjunction), \( \rightarrow \) (implication), \( \leftarrow \) (co-implication), the constants \( \top \) and \( \bot \), and the unary \( \sim \) (paraconsistent, strong negation). Greek small letters \( \alpha, \beta, \ldots \) are used to denote formulas, and Greek capital letters \( \Gamma, \Delta, \ldots \) are used to represent finite (possibly empty) sets of formulas.

The symbol \( \equiv \) is used to denote the equality of symbols.
A sequent is an expression of the form $\Gamma \Rightarrow \Delta$. An expression $L \vdash \Gamma \Rightarrow \Delta$ means that $\Gamma \Rightarrow \Delta$ is provable in a sequent calculus $L$. A rule $R$ of inference is said to be admissible in a sequent calculus $L$ if the following condition is satisfied: For any instance 

$$
\frac{S_1 \cdots S_n}{S}
$$

of $R$, if $L \vdash S_i$ for all $i$, then $L \vdash S$.

The bi-intuitionistic connexive logic BCL is introduced below as a Gentzen-type sequent calculus.
Definition (BCL)

The initial sequents of BCL are of the following form, for any propositional variable $p$: $p \Rightarrow p$, $\sim p \Rightarrow \sim p$, $\Gamma \Rightarrow \Delta, \top \setminus, \Gamma \Rightarrow \Delta$, $\sim \top, \Gamma \Rightarrow \Delta$, $\Gamma \Rightarrow \Delta, \sim \bot$.

The structural inference rules of BCL are of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \quad \text{(cut)}$$

$$\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \quad \text{(we-left)} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha} \quad \text{(we-right)}.$$

The positive logical inference rules of BCL are of the form:

$$\frac{\Gamma \Rightarrow \Delta}{\top, \Gamma \Rightarrow \Delta} \quad \text{(\top-left)} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \bot} \quad \text{(\bot-right)}$$

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \land \beta, \Gamma \Rightarrow \Delta} \quad \text{(\land-left)} \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \alpha \land \beta} \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \alpha \land \beta} \quad \text{(\land-right)}$$
Definition (BCL continued)

\[
\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \lor \beta, \Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \lor \beta} \quad (\lor \text{left})
\]

\[
\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\alpha \rightarrow \beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi} \quad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta} \quad (\rightarrow \text{left})
\]

\[
\frac{\alpha \Rightarrow \Delta, \beta}{\alpha \leftarrow \beta \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \alpha \leftarrow \beta} \quad (\leftarrow \text{right})
\]

The negative logical inference rules of BCL are of the form:

\[
\frac{\Gamma \Rightarrow \Delta}{\neg \bot, \Gamma \Rightarrow \Delta} \quad (\neg \bot \text{-left}) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg \top} \quad (\neg \top \text{-right})
\]

\[
\frac{\alpha, \Gamma \Rightarrow \Delta}{\neg \neg \alpha, \Gamma \Rightarrow \Delta} \quad (\neg \neg \text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \neg \neg \alpha} \quad (\neg \neg \text{right})
\]

\[
\frac{\neg \alpha, \Gamma \Rightarrow \Delta \quad \neg \beta, \Gamma \Rightarrow \Delta}{\neg (\alpha \land \beta), \Gamma \Rightarrow \Delta} \quad (\neg \land \text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \neg \alpha, \neg \beta}{\Gamma \Rightarrow \Delta, \neg (\alpha \land \beta)} \quad (\neg \land \text{right})
\]
Sequent calculi ICL, DCL, BL, IL and DL for intuitionistic connexive logic, dual-intuitionistic connexive logic, bi-intuitionistic logic, intuitionistic logic and dual-intuitionistic logic, respectively, are defined as subsystems of BCL.
Definition (Subsystems of BCL)

- **ICL** is the \(\rightarrow\)-free part of BCL.
- **DCL** is the \(\Rightarrow\)-free part of BCL.
- **BL** is the \(\sim\)-free part of BCL.
- **IL** is the \(\rightarrow\)-free part of BL.
- **DL** is the \(\Rightarrow\)-free part of BL.
We may note the following:

- Let $L \in \{\text{BCL, ICL, DCL, BL, IL, DL}\}$. The sequents of the form $\alpha \Rightarrow \alpha$ for any formula $\alpha$ are provable in $L$. This fact can be shown by induction on $\alpha$.

- ($\rightarrow \text{right}$) and ($\leftarrow \text{left}$) in BCL satisfy the single-succedent restriction and the single-antecedent restriction, respectively. These rules are the usual inference rules for the standard Gentzen-type sequent calculi LJ and DJ for intuitionistic logic and dual-intuitionistic logic, respectively. The same restrictions are also imposed to ($\sim \rightarrow \text{right}$) and ($\sim \leftarrow \text{left}$) in BCL.

- ($\sim \rightarrow \text{left}$), ($\sim \rightarrow \text{right}$), ($\sim \leftarrow \text{left}$) and ($\sim \leftarrow \text{right}$) correspond to the following characteristic axiom schemes for connexive logic:
  - $\sim(\alpha \rightarrow \beta) \leftrightarrow \alpha \rightarrow \sim \beta$,
  - $\sim(\alpha \leftarrow \beta) \leftrightarrow \sim \alpha \leftarrow \beta$. 


A Gentzen-type sequent calculus LBiI for Bilnt was presented in (Pinto and Uustalu 2010) based on Dragalin’s sequent calculus for Int. LBiI has the logical inference rules of the form:

\[
\frac{\alpha \rightarrow \beta, \Gamma \Rightarrow \Delta, \alpha}{\alpha \rightarrow \beta, \Gamma \Rightarrow \Delta}
\frac{\beta, \Gamma \Rightarrow \Delta}{\alpha \rightarrow \beta, \Gamma \Rightarrow \Delta}
\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \Delta, \alpha \rightarrow \beta}
\frac{\alpha \Rightarrow \Delta, \beta}{\alpha \leftarrow \beta, \Gamma \Rightarrow \Delta}
\frac{\Gamma \Rightarrow \Delta, \alpha}{\beta, \Gamma \Rightarrow \Delta, \alpha \leftarrow \beta}
\frac{\Gamma \Rightarrow \Delta, \alpha \leftarrow \beta}{\Gamma \Rightarrow \Delta, \alpha \rightarrow \beta}.
\]

The cut-elimination theorem does not hold for LBiI.

BL is theorem-equivalent to LBiI, and the cut-elimination theorem does not hold for BL and BCL. On the other hand, the cut-elimination theorem holds for ICL, DCL and DL.
A counterexample of the failure of the cut-elimination in BL and BCL is presented as follows. This example is the same as that in (Pinto and Uustalu 2010). The sequent $p \Rightarrow q, r \rightarrow ((p \leftarrow q) \land r)$ where $p$, $q$ and $r$ are distinct propositional variables is not provable in BL without (cut), but provable in BL with (cut) by:

\[
\begin{align*}
&\vdots \\
p \leftarrow q \Rightarrow p \leftarrow q & \quad r \Rightarrow r \\
&\vdots \\
r, p \leftarrow q, p \Rightarrow p \leftarrow q & \quad r, p \leftarrow q, p \Rightarrow r \\
p \Rightarrow p & \quad q \Rightarrow q \\
\hline
p \Rightarrow q, p \leftarrow q & \quad r, p \leftarrow q, p \Rightarrow (p \leftarrow q) \land r \\
\hline
r, p \leftarrow q, p \Rightarrow r \rightarrow ((p \leftarrow q) \land r) & \quad (\rightarrow \text{right}) \\
\hline
p, p \Rightarrow q, r \rightarrow ((p \leftarrow q) \land r) & \quad (\text{cut}) \\
\hline
p \Rightarrow q, r \rightarrow ((p \leftarrow q) \land r) 
\end{align*}
\]
The cut-elimination theorem holds for IL, which is logically equivalent to a slightly modified version of Maehara’s LJ’ for intuitionistic logic.

Intuitionistic negation is definable by $\neg_i \alpha := \alpha \rightarrow \bot$, and co-negation is definable by $\neg_d \alpha := \top \leftarrow \alpha$. Moreover, in the presence of implication, $\top$ can be defined as $p \rightarrow p$, and in the presence of co-implication, $\bot$ can be defined as $p \leftarrow p$, for some fixed propositional variable $p$.

We introduce a Kripke semantics for BCL, where a distinction is drawn between positive valuations $\models^+$ and negative ones $\models^-$.

**Definition**

A Kripke frame is a structure $\langle M, \leq \rangle$ satisfying the following conditions:

- $M$ is a nonempty set (of states),
- $\leq$ is a preorder (i.e., a reflexive and transitive binary relation) on $M$. 
Definition

Connexive valuations $\models^+$ and $\models^-$ on a Kripke frame $\langle M, \leq \rangle$ are mappings from the set $\Phi$ of propositional variables to the power set $2^M$ of $M$ such that for any $\star \in \{+, -, \}$, any $p \in \Phi$ and any $x, y \in M$, if $x \models^\star (p)$ and $x \leq y$, then $y \models^\star (p)$. We will write $x \models^\star p$ for $x \models ^\star (p)$. The connexive valuations $\models^+$ and $\models^-$ are extended to mappings from the set of all formulas to $2^M$ by:

- $x \models^+ \top$ always,
- $x \models^+ \bot$ never,
- $x \models^+ \alpha \wedge \beta$ iff $x \models^+ \alpha$ and $x \models^+ \beta$,
- $x \models^+ \alpha \vee \beta$ iff $x \models^+ \alpha$ or $x \models^+ \beta$,
- $x \models^+ \alpha \rightarrow \beta$ iff $\forall y \in M \ [x \leq y \ and \ y \models^+ \alpha \ imply \ y \models^+ \beta]$,
- $x \models^+ \alpha \leftarrow \beta$ iff $\exists y \in M \ [x \geq y, \ y \models^+ \alpha \ and \ not-(y \models^+ \beta)]$,
- $x \models^+ \sim \alpha$ iff $x \models^- \alpha$,
Definition (continued)

- $x \models^- \top$ never,
- $x \models^- \bot$ always,
- $x \models^- \alpha \land \beta$ iff $x \models^- \alpha$ or $x \models^- \beta$,
- $x \models^- \alpha \lor \beta$ iff $x \models^- \alpha$ and $x \models^- \beta$,
- $x \models^- \alpha \rightarrow \beta$ iff $\forall y \in M \ [x \leq y$ and $y \models^+ \alpha$ imply $y \models^- \beta]$,
- $x \models^- \alpha \leftarrow \beta$ iff $\exists y \in M \ [x \geq y$, $y \models^- \alpha$ and not-$(y \models^+ \beta)]$,
- $x \models^- \neg \alpha$ iff $x \models^+ \alpha$.

The following hereditary condition holds for $\models^+$ and $\models^-$: For any $\star \in \{+, -, \}$, any formula $\alpha$ and any $x, y \in M$, if $x \models^\star \alpha$ and $x \leq y$, then $y \models^\star \alpha$. 
**Definition**

A connexive Kripke model is a structure $\langle M, \leq, \models^+ , \models^- \rangle$ such that

- $\langle M, \leq \rangle$ is a Kripke frame,
- $\models^+$ and $\models^-$ are connexive valuations on $\langle M, \leq \rangle$.

A formula $\alpha$ is true in a connexive Kripke model $\langle M, \leq, \models^+, \models^- \rangle$ if $x \models^+ \alpha$ for any $x \in M$. A formula $\alpha$ is BCL-valid in a Kripke frame $\langle M, \leq \rangle$ if it is true for all connexive valuations $\models^+$ and $\models^-$ on the Kripke frame. A formula $\alpha$ is dually BCL-valid in a Kripke frame $\langle M, \leq \rangle$ if for all connexive valuations $\models^+$ and $\models^-$ on the Kripke frame, $x \models^- \alpha$ for any $x \in M$. A set of formulas $\Gamma$ entails a formula $\alpha$ in BCL ($\Gamma \models_{\text{BCL}} \alpha$) if whenever all formulas in $\Gamma$ are BCL-valid in a Kripke frame, then so is $\alpha$; $\Gamma$ dually entails $\alpha$ in BCL ($\Gamma \models_{\text{d BCL}}^d \alpha$) if whenever all formulas in $\Gamma$ are dually BCL-valid in a frame, then so is $\alpha$. 
Obviously, in the presence of $\sim$, the notion of dual BCL-validity in a Kripke frame is definable in terms of BCL-validity. A formula $\alpha$ is dually BCL-valid in a Kripke frame iff $\sim\alpha$ is BCL-valid in that frame.

When we turn to tableaux, we also consider a notion of dual provability.

Next, we present a Kripke semantics for BL. It has been emphasized in (Wansing 2008) that in the relational semantics of intuitionistic logic and Heyting-Brouwer logic, only verification conditions and no falsification conditions of formulas are specified. The reason why negative valuations $\models^-$ have not been considered in the literature on BiInt presumably is that its language lacks strong negation, $\sim$. Accordingly, the semantics of BL is presented in terms of ordinary Kripke models.
Definition

A valuation $\models$ on a Kripke frame $\langle M, \leq \rangle$ is a mapping from the set $\Phi$ of propositional variables to the power set $2^M$ of $M$ such that for any $p \in \Phi$ and any $x, y \in M$, if $x \in \models (p)$ and $x \leq y$, then $y \in \models (p)$. We will write $x \models p$ for $x \in \models (p)$. This valuation $\models$ is extended to a mapping from the set of all formulas to $2^M$ by:

- $x \models \top$ always,
- $x \models \bot$ never,
- $x \models \alpha \land \beta$ iff $x \models \alpha$ and $x \models \beta$,
- $x \models \alpha \lor \beta$ iff $x \models \alpha$ or $x \models \beta$,
- $x \models \alpha \rightarrow \beta$ iff $\forall y \in M \ [x \leq y \ and \ y \models \alpha \ imply \ y \models \beta]$,
- $x \models \alpha \leftarrow \beta$ iff $\exists y \in M \ [x \geq y \ and \ y \models \alpha \ and \ not-(y \models \beta)]$.

The following hereditary condition holds for $\models$: For any formula $\alpha$ and any $x, y \in M$, if $x \models \alpha$ and $x \leq y$, then $y \models \alpha$. 
A Kripke model is a structure $\langle M, \leq, \models \rangle$ such that

1. $\langle M, \leq \rangle$ is a Kripke frame,
2. $\models$ is a valuation on $\langle M, \leq \rangle$.

A formula $\alpha$ is true in a Kripke model $\langle M, \leq, \models \rangle$ if $x \models \alpha$ for any $x \in M$. A formula $\alpha$ is BL-valid in a Kripke frame $\langle M, \leq \rangle$ if it is true for every valuation $\models$ on the Kripke frame. A set of formulas $\Gamma$ entails a formula $\alpha$ in BL ($\Gamma \models_{BL} \alpha$) if whenever all formulas in $\Gamma$ are BL-valid in a frame, then so is $\alpha$.

In addition one may define a formula $\alpha$ to be dually BL-valid in a Kripke frame $\langle M, \leq \rangle$ if for all connexive valuations $\models^+$ and $\models^-$ on the Kripke frame, $x \models^- \alpha$ for any $x \in M$ and say that $\Gamma$ dually entails $\alpha$ in BL ($\Gamma \models_{dBL} \alpha$) if whenever all formulas in $\Gamma$ are dually BL-valid in a Kripke frame, then so is $\alpha$. 
Proposition (Completeness for BL)

For any finite set of formulas $\Gamma \cup \{\alpha\}$, $\text{BL} \vdash \Gamma \Rightarrow \alpha$ iff $\Gamma \models_{\text{BL}} \alpha$.

In the following, we introduce a translation of BCL into BL, and by using this translation, we show two theorems for syntactically and semantically embedding BCL into BL. A similar translation has been used by Gurevich (1977) Rautenberg (1979), and Vorob’ev (1952) to embed Nelson’s constructive logic N3 into positive intuitionistic logic.
We fix a set $\Phi$ of propositional variables and define the set $\Phi' := \{ p' \mid p \in \Phi \}$ of propositional variables. The language $\mathcal{L}_{\text{BCL}}$ of BCL, introduced above, is based on $\Phi$, $\top$, $\bot$, $\land$, $\lor$, $\to$, $\multimap$ and $\sim$. The language $\mathcal{L}_{\text{BL}}$ of BL is that of BCL without $\sim$. A mapping $f$ from $\mathcal{L}_{\text{BCL}}$ to $\mathcal{L}_{\text{BL}}$ is defined inductively as follows.

- for any $p \in \Phi$, $f(p) := p$ and $f(\sim p) := p' \in \Phi'$,
- $f(\#) := \#$ with $\# \in \{\top, \bot\}$,
- $f(\alpha \# \beta) := f(\alpha) \# f(\beta)$ with $\# \in \{\land, \lor, \to, \multimap\}$,
- $f(\sim \sim \alpha) := f(\alpha)$,
- $f(\sim \top) := \bot$, $f(\sim \bot) := \top$,
- $f(\sim (\alpha \land \beta)) := f(\sim \alpha) \lor f(\sim \beta)$,
- $f(\sim (\alpha \lor \beta)) := f(\sim \alpha) \land f(\sim \beta)$,
- $f(\sim (\alpha \to \beta)) := f(\alpha) \to f(\sim \beta)$,
- $f(\sim (\alpha \multimap \beta)) := f(\sim \alpha) \multimap f(\beta)$. 
An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula $\alpha$ in $\Gamma$ by an occurrence of $f(\alpha)$. The same notation is used for the other mappings discussed later. Note that the function $f$ is surjective but not injective.

**Theorem (Syntactical embedding from BCL into BL)**

Let $\Gamma, \Delta$ be finite sets of formulas in $\mathcal{L}_{\text{BCL}}$, and $f$ be the mapping defined above.

- $\text{BCL} \vDash \Gamma \Rightarrow \Delta$ iff $\text{BL} \vDash f(\Gamma) \Rightarrow f(\Delta)$,
- $\text{BCL} - (\text{cut}) \vDash \Gamma \Rightarrow \Delta$ iff $\text{BL} - (\text{cut}) \vDash f(\Gamma) \Rightarrow f(\Delta)$.

**Proof.** We show only the first claim.

$(\Rightarrow)$ : By induction on the proofs $P$ of $\Gamma \Rightarrow \Delta$ in BCL. We distinguish the cases according to the last inference of $P$, and show some cases.
• Case ($\neg p \Rightarrow \neg p$): The last inference of $P$ is of the form: $\neg p \Rightarrow \neg p$ for any $p \in \Phi$. In this case, we obtain $\text{BL} \vdash f(\neg p) \Rightarrow f(\neg p)$, i.e., $\text{BL} \vdash p' \Rightarrow p'$ ($p' \in \Phi'$), by the definition of $f$.

• Case ($\neg \rightarrow \text{right}$): The last inference of $P$ is of the form:

\[
\alpha, \Gamma \Rightarrow \neg \beta \\
\Gamma \Rightarrow \neg (\alpha \rightarrow \beta) \quad (\neg \rightarrow \text{right}).
\]

By induction hypothesis, we have $\text{BL} \vdash f(\alpha), f(\Gamma) \Rightarrow f(\neg \beta)$. Then, we obtain the required fact:

\[
\vdots \\
f(\alpha), f(\Gamma) \Rightarrow f(\neg \beta) \\
f(\Gamma) \Rightarrow f(\alpha \rightarrow f(\neg \beta)) \quad (\rightarrow \text{right})
\]

where $f(\alpha) \rightarrow f(\neg \beta)$ coincides with $f(\neg(\alpha \rightarrow \beta))$ by the definition of $f$. 
Case \(\sim \rightarrow \text{left}\): The last inference of \(P\) is of the form:
\[
\Gamma \Rightarrow \Delta, \alpha \sim \beta, \Sigma \Rightarrow \Pi \quad (\sim \rightarrow \text{left}).
\]

By induction hypothesis, we have \(\text{BL} \vdash f(\Gamma) \Rightarrow f(\Delta), f(\alpha)\) and \(\text{BL} \vdash f(\sim \beta), f(\Sigma) \Rightarrow f(\Pi)\). Then, we obtain the required fact:
\[
\vdots \quad \vdots 
\]
\[
f(\Gamma) \Rightarrow f(\Delta), f(\alpha) \quad f(\sim \beta), f(\Sigma) \Rightarrow f(\Pi) \quad (\rightarrow \text{left})
\]
where \(f(\alpha) \rightarrow f(\sim \beta)\) coincides with \(f(\sim(\alpha \rightarrow \beta))\) by the definition of \(f\).

Case \(\sim \sim \text{left}\): The last inference of \(P\) is of the form:
\[
\alpha, \Gamma \Rightarrow \Delta \quad (\sim \sim \text{left}).
\]

By induction hypothesis, we have the required fact \(\text{BL} \vdash f(\alpha), f(\Gamma) \Rightarrow f(\Delta)\) where \(f(\alpha)\) coincides with \(f(\sim \sim \alpha)\) by the definition of \(f\).
• Case (cut): The last inference of $P$ is of the form:

$$
\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \quad \text{(cut)}.
$$

By induction hypothesis, we have $\text{BL} \vdash f(\Gamma) \Rightarrow f(\Delta), f(\alpha)$ and $\text{BL} \vdash f(\alpha), f(\Sigma) \Rightarrow f(\Pi)$. Then, we obtain the required fact:

$$
\vdots
$$

$$
\frac{f(\Gamma) \Rightarrow f(\Delta), f(\alpha) \quad f(\alpha), f(\Sigma) \Rightarrow f(\Pi)}{f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)} \quad \text{(cut)}.
$$
(⇐): By induction on the proofs \( Q \) of \( f(\Gamma) \Rightarrow f(\Delta) \) in BL. We distinguish the cases according to the last inference of \( Q \), and show some cases.

- Case (\( \sim\sim \text{left} \)): The last inference of \( Q \) is of the form:

\[
\frac{f(\alpha), f(\Gamma) \Rightarrow f(\Delta)}{f(\sim\sim\alpha), f(\Gamma) \Rightarrow f(\Delta)} \quad (\sim\sim \text{left})
\]

where \( f(\sim\sim\alpha) \) coincides with \( f(\alpha) \) by the definition of \( f \). By induction hypothesis, we have the required fact \( \text{BCL} \vdash \alpha, \Gamma \Rightarrow \Delta \).
• **Case (cut):** The last inference of $Q$ is of the form:

\[
\quad \frac{f(\Gamma) \Rightarrow f(\Delta), \beta, f(\Sigma) \Rightarrow f(\Pi)}{f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)} \quad \text{(cut)}.
\]

In this case, $\beta$ is a formula of BL. We then have the fact $\gamma = f(\gamma)$ for any formula $\gamma$ in BL. This can be shown by induction on $\gamma$. Thus, $Q$ is of the form:

\[
\quad \frac{f(\Gamma) \Rightarrow f(\Delta), f(\beta), f(\Sigma) \Rightarrow f(\Pi)}{f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)} \quad \text{(cut)}.
\]

By induction hypothesis, we have $\text{BCL} \vdash \Gamma \Rightarrow \Delta, \beta$ and $\text{BCL} \vdash \beta, \Sigma \Rightarrow \Pi$. Then, we obtain the required fact:

\[
\quad \frac{\Gamma \Rightarrow \Delta, \beta, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \quad \text{(cut)}.
\]
Theorem (Syntactical embedding from ICL into IL)

Suppose that $\mathcal{L}_{\text{ICL}}$ and $\mathcal{L}_{\text{IL}}$ are obtained from $\mathcal{L}_{\text{BCL}}$ and $\mathcal{L}_{\text{BL}}$, respectively, by deleting $\neg$. Let $\Gamma, \Delta$ be finite sets of formulas in $\mathcal{L}_{\text{ICL}}$, and $f$ be the mapping from $\mathcal{L}_{\text{ICL}}$ into $\mathcal{L}_{\text{IL}}$, which is obtained from the previous definition by deleting the conditions of $\neg$.

- $\text{ICL} \vdash \Gamma \Rightarrow \Delta$ iff $\text{IL} \vdash f(\Gamma) \Rightarrow f(\Delta)$,
- $\text{ICL} - \text{(cut)} \vdash \Gamma \Rightarrow \Delta$ iff $\text{IL} - \text{(cut)} \vdash f(\Gamma) \Rightarrow f(\Delta)$.

Theorem (Syntactical embedding from DCL into DL)

Suppose that $\mathcal{L}_{\text{DCL}}$ and $\mathcal{L}_{\text{DL}}$ are obtained from $\mathcal{L}_{\text{BCL}}$ and $\mathcal{L}_{\text{BL}}$, respectively, by deleting $\to$. Let $\Gamma, \Delta$ be finite sets of formulas in $\mathcal{L}_{\text{DCL}}$, and $f$ be the mapping from $\mathcal{L}_{\text{DCL}}$ into $\mathcal{L}_{\text{DL}}$, which is obtained from the previous definition by deleting the conditions of $\to$.

- $\text{DCL} \vdash \Gamma \Rightarrow \Delta$ iff $\text{DL} \vdash f(\Gamma) \Rightarrow f(\Delta)$,
- $\text{DCL} - \text{(cut)} \vdash \Gamma \Rightarrow \Delta$ iff $\text{DL} - \text{(cut)} \vdash f(\Gamma) \Rightarrow f(\Delta)$.
Theorem (Cut-elimination for ICL and DCL)

Let $L$ be ICL or DCL. The rule \((\text{cut})\) is admissible in cut-free $L$.

Proof. We show only the case for ICL. Suppose that $\text{ICL} \vdash \Gamma \Rightarrow \Delta$. Then, we have $\text{IL} \vdash f(\Gamma) \Rightarrow f(\Delta)$, and hence $\text{IL} - (\text{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$ by the cut-elimination theorem for IL (this is known). But then we also obtain $\text{ICL} - (\text{cut}) \vdash \Gamma \Rightarrow \Delta$. □

Next, we show a theorem for semantically embedding BCL into BL.

Lemma

Let $f$ be the mapping defined above. For any connexive Kripke model $\langle M, \leq, \models^+, \models^- \rangle$, we can construct a Kripke model $\langle M, \leq, \models \rangle$ such that for any formula $\alpha$ and any $x \in M$,

- $x \models^+ \alpha$ iff $x \models f(\alpha)$,
- $x \models^- \alpha$ iff $x \models f(\sim \alpha)$.
\textbf{Proof.} Let \( \Phi \) be a set of propositional variables and \( \Phi' \) be the set \( \{ p' \mid p \in \Phi \} \) of propositional variables. Suppose that \( \langle M, \leq, \models^+, \models^- \rangle \) is a connexive Kripke model where \( \models^+ \) and \( \models^- \) are mappings from \( \Phi \) to the power set \( 2^M \) of \( M \), and that the hereditary condition with respect to \( p \in \Phi \) holds for \( \models^+ \) and \( \models^- \). Suppose that \( \langle M, \leq, \models \rangle \) is a Kripke model where \( \models \) is a mapping from \( \Phi \cup \Phi' \) to \( 2^M \), and that the hereditary condition with respect to \( p \in \Phi \cup \Phi' \) holds for \( \models \). Suppose moreover that these models satisfy the following conditions: For any \( x \in M \) and any \( p \in \Phi \),

\begin{itemize}
  \item \( x \models^+ p \iff x \models p \),
  \item \( x \models^- p \iff x \models p' \quad (p' = f(\sim p)) \).
\end{itemize}

Then, the lemma is proved by (simultaneous) induction on the complexity of \( \alpha \).
Lemma

Let $f$ be the mapping defined above. For any Kripke model $\langle M, \leq, \models \rangle$, we can construct a connexive Kripke model $\langle M, \leq, \models^+, \models^- \rangle$ such that for any formula $\alpha$ and any $x \in M$,

- $x \models f(\alpha)$ iff $x \models^+ \alpha$,
- $x \models f(\sim \alpha)$ iff $x \models^- \alpha$.

Theorem (Semantical embedding from BCL into BL)

Let $f$ be the mapping defined above. For any any finite set of formula $\Gamma \cup \{\alpha\}$, $f(\Gamma) \models_{\text{BL}} f(\alpha)$ iff $\Gamma \models_{\text{BCL}} \alpha$.

Proof. By the previous two lemmas. \qed
Theorem (Completeness for BCL)

For any finite set of formula \( \Gamma \cup \{ \alpha \} \), \( \text{BCL} \vdash \Gamma \Rightarrow \alpha \) iff \( \Gamma \models_{\text{BCL}} \alpha \) (and thus also \( \text{BCL} \vdash \neg \Gamma \Rightarrow \neg \alpha \) iff \( \Gamma \models_{d} \text{BCL} \alpha \)).

Proof. \( \text{BCL} \vdash \Gamma \Rightarrow \alpha \) iff \( \text{BL} \vdash f(\Gamma) \Rightarrow f(\alpha) \) iff \( f(\Gamma) \models_{\text{BL}} f(\alpha) \) iff \( \Gamma \models_{\text{BCL}} \alpha \). \( \square \)

One can show that ICL and DCL can be syntactically embedded into each other. These results show that BCL is constructed based on a duality between ICL and DCL. Firstly, we introduce a translation from ICL into DCL. The idea of this translation comes from (Czermak 1977, Urbas 1996).
Definition

We fix a common set $\Phi$ of propositional variables. The language $\mathcal{L}_{\text{ICL}}$ of ICL is defined using $\Phi$, $\bot$, $\land$, $\lor$, $\rightarrow$ and $\neg$. The language $\mathcal{L}_{\text{DCL}}$ of DCL is defined using $\Phi$, $\top$, $\land$, $\lor$, $\leftarrow$ and $\neg$.

A mapping $f$ from $\mathcal{L}_{\text{ICL}}$ to $\mathcal{L}_{\text{DCL}}$ is defined inductively as follows.

1. $f(p) := p$ for any $p \in \Phi$,
2. $f(\bot) := \top$,
3. $f(\alpha \land \beta) := f(\alpha) \lor f(\beta)$,
4. $f(\alpha \lor \beta) := f(\alpha) \land f(\beta)$,
5. $f(\alpha \rightarrow \beta) := f(\beta) \leftarrow f(\alpha)$,
6. $f(\neg \alpha) := \neg f(\alpha)$. 
We then obtain a theorem for syntactically embedding ICL into DCL.

**Theorem (Syntactical embedding from ICL into DCL)**

Let $\Gamma$ and $\Delta$ be finite sets of formulas in $\mathcal{L}_{ICL}$, and $f$ be the mapping just defined.

- $\text{ICL} \vdash \Gamma \Rightarrow \Delta$ iff $\text{DCL} \vdash f(\Delta) \Rightarrow f(\Gamma)$,
- $\text{ICL} - (\text{cut}) \vdash \Gamma \Rightarrow \Delta$ iff $\text{DCL} - (\text{cut}) \vdash f(\Delta) \Rightarrow f(\Gamma)$.

**Proof.** By induction on proofs of $\Gamma \Rightarrow \Delta$ in ICL and proofs of $f(\Delta) \Rightarrow f(\Gamma)$ in DCL. \qed
Similarly, we can introduce a translation from DCL into ICL.

**Definition**

Φ, \( L_{DCL} \) and \( L_{ICL} \) are defined as above. A mapping \( g \) from \( L_{DCL} \) to \( L_{ICL} \) is defined inductively as follows.

1. \( g(p) := p \) for any \( p \in \Phi \),
2. \( g(\alpha \land \beta) := g(\alpha) \lor g(\beta) \),
3. \( g(\alpha \lor \beta) := g(\alpha) \land g(\beta) \),
4. \( g(\alpha \rightarrow \beta) := g(\beta) \rightarrow g(\alpha) \),
5. \( g(\neg \alpha) := \neg g(\alpha) \).

We can obtain a theorem for syntactically embedding DCL into ICL.
Theorem (Syntactical embedding from DCL into ICL)

Let $\Gamma$ and $\Delta$ be finite sets of formulas in $\mathcal{L}_{DCL}$ and $g$ be the mapping just defined.

- $DCL \vdash \Gamma \Rightarrow \Delta$ iff $ICL \vdash g(\Delta) \Rightarrow g(\Delta)$,
- $DCL - (cut) \vdash \Gamma \Rightarrow \Delta$ iff $ICL - (cut) \vdash g(\Delta) \Rightarrow g(\Delta)$.

Note that the following holds for ICL and DCL:

- $ICL \vdash gf(\Gamma) \Rightarrow gf(\Delta)$ iff $ICL \vdash \Gamma \Rightarrow \Delta$,
- $DCL \vdash fg(\Gamma) \Rightarrow fg(\Delta)$ iff $DCL \vdash \Gamma \Rightarrow \Delta$.

Similarly, we can introduce translations from IL into DL and vice versa, and can show the syntactical embedding theorems based on these translations.

Using these theorems, we can obtain alternative proofs of the cut-elimination theorems for ICL, DCL, IL and DL.
A tableau calculus for connexive Heyting-Brouwer logic

A sound and complete tableau calculus for connexive Heyting-Brouwer logic can be obtained by modifying the tableau calculus for the modal logic BS4 from (Odintsov and Wansing 2010), which was obtained by modifying Priest’s (2008a, 2008b) tableau calculus for the modal logic $S4_{FDE}$ (or $K_{FDE}^{\rho \tau}$ as Priest calls the latter system), i.e., $S4$ based on first-degree entailment logic.

We assume some familiarity with the tableau method as applied by Priest. We define tableau calculi for BCL and its subsystems ICL, DCL, BL, IL, and DL.
Since the languages of BL, IL, and DL lack strong negation, support of falsity for a formula $\alpha$ cannot be captured as the support of truth of $\sim \alpha$, and the tableau nodes have to provide information concerning:

- support of truth, indicated by $+T$,
- failure to support truth, indicated by $+F$,
- support of falsity, indicated by $-T$,
- failure to support falsity, indicated by $-F$.

Moreover, the nodes have to provide information about “accessibility” between states.
Accordingly, in tableaux for BCL the tableau entries are of the form $\alpha, + Ti$, or $\alpha, + Fi$, or $\alpha, - Ti$, or $\alpha, - Fi$, or $irj$, where $\alpha$ is a formula from $\mathcal{L}_{BCL}$, $i$ and $j$ are natural numbers representing states, and $irj$ is to be understood as $i \leq j$.

We distinguish between single conclusion derivability statement $\Delta \vdash \beta$ and single conclusion dual derivability statement $\Delta \vdash^d \beta$.

Tableaux for a single conclusion derivability statement $\Delta \vdash \beta$ start with nodes of the form $\alpha, + T0$ for every premise $\alpha$ from the finite premise set $\Delta$ and a node of the form $\beta, + F0$.

Tableaux for a single conclusion dual derivability statement $\Delta \vdash^d \beta$ start with nodes of the form $\alpha, - T0$ for every premise $\alpha$ from the finite premise set $\Delta$ and a node of the form $\beta, - F0$. 
Tableau rules are applied to tableau nodes, thereby leading to more complex, expanded tableaux.

A branch of a tableau closes iff it contains a pair of nodes $\alpha, +Ti$ and $\alpha, +Fi$ or a pair of nodes $\alpha, -Ti$ and $\alpha, -Fi$.

The tableau closes iff all of its branches close. If a tableau (tableau branch) is not closed, it is called open.

A tableau branch is said to be completed iff no more rules can be applied to expand it. A tableau is said to be completed iff each of its branches is completed.
**Definition**

The triply-signed tableau calculus for BCL consists of the following rules:

**Decomposition rules**

<table>
<thead>
<tr>
<th>Rule</th>
<th>Logic Formulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha \land \beta, + Ti)</td>
<td>(\alpha, + Ti) (\beta, + Ti)</td>
</tr>
<tr>
<td>(\alpha \land \beta, + Fi)</td>
<td>(\alpha, + Fi) (\beta, + Fi)</td>
</tr>
<tr>
<td>(\alpha \land \beta, - Ti)</td>
<td>(\alpha, - Ti) (\beta, - Ti)</td>
</tr>
<tr>
<td>(\alpha \land \beta, - Fi)</td>
<td>(\alpha, - Fi) (\beta, - Fi)</td>
</tr>
<tr>
<td>(\alpha \lor \beta, + Ti)</td>
<td>(\alpha, + Ti) (\beta, + Ti)</td>
</tr>
<tr>
<td>(\alpha \lor \beta, + Fi)</td>
<td>(\alpha, + Fi) (\beta, + Fi)</td>
</tr>
<tr>
<td>(\alpha \lor \beta, - Ti)</td>
<td>(\alpha, - Ti) (\beta, - Ti)</td>
</tr>
<tr>
<td>(\alpha \lor \beta, - Fi)</td>
<td>(\alpha, - Fi) (\beta, - Fi)</td>
</tr>
</tbody>
</table>
### Definition (continued)

<table>
<thead>
<tr>
<th>Rule</th>
<th>Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha \rightarrow \beta, +Ti$</td>
<td>$\alpha \rightarrow \beta, +Fi$</td>
</tr>
<tr>
<td>$\triangleright \downarrow_{irj} \triangleright$</td>
<td>$\downarrow_{irj}$</td>
</tr>
<tr>
<td>$\alpha, +Fj \quad \beta, +Tj$</td>
<td>$\alpha, +Tj \quad \beta, +Fj$</td>
</tr>
<tr>
<td>$\alpha, +Fj$</td>
<td>$\alpha, +Tj$</td>
</tr>
<tr>
<td>$\beta, +Tj$</td>
<td>$\beta, -Fj$</td>
</tr>
<tr>
<td>$\beta \leftarrow \alpha, +Ti$</td>
<td>$\beta \leftarrow \alpha, +Fi$</td>
</tr>
<tr>
<td>$\downarrow_{jri}$</td>
<td>$\downarrow_{jri}$</td>
</tr>
<tr>
<td>$\alpha, +Fj$</td>
<td>$\alpha, +Tj$</td>
</tr>
<tr>
<td>$\beta, +Tj$</td>
<td>$\beta, -Tj$</td>
</tr>
<tr>
<td>$\sim \alpha, +Ti$</td>
<td>$\sim \alpha, +Fi$</td>
</tr>
<tr>
<td>$\downarrow$</td>
<td>$\downarrow$</td>
</tr>
<tr>
<td>$\alpha, -Ti$</td>
<td>$\alpha, -Fi$</td>
</tr>
<tr>
<td>$\alpha, +Fi$</td>
<td>$\alpha, +Ti$</td>
</tr>
<tr>
<td>$\bot, +Fi$</td>
<td>$\sim \bot, +Ti$</td>
</tr>
<tr>
<td>$\top, +Ti$</td>
<td>$\sim \top, +Fi$</td>
</tr>
</tbody>
</table>
Definition (continued)

*Structural rules (for capturing the reflexivity and transitivity of the relation ≤) and rules for capturing the hereditary condition*

\[
\begin{array}{cccc}
\cdot & irj & p, + Ti & p, - Ti \\
\downarrow & jrk & \downarrow irj & \downarrow irj \\
iri & irk & p, + Tj & p, - Tj \\
\end{array}
\]
The decomposition rules and the hereditary rules in which a statement $ir_j$ appears above an arrow are applied whenever a node $ir_j$ occurs on the branch.

The decomposition rules in which a statement $ir_j$ appears above an arrow require the introduction of a new natural number $j$ not already occurring in the tableau. In applications the smallest natural number not already occurring in the tableau is chosen.

If the application of a rule would result in creating a node that is already on the branch, the rule is not applied. The structural rule for reflexivity may be used to introduce a node $iri$ when $i$ occurs on the branch. Note that due to the transitivity rule, tableaux may be infinite.
We define notions of provability and dual provability.

**Definition**

Let $\Delta \cup \{\alpha\}$ be a finite set of $\mathcal{L}_{BCL}$-formulas. We say that $\alpha$ is provable from $\Delta$ ($\Delta \vdash \alpha$) iff there exists a closed and completed tableau for a list of nodes consisting of $\alpha$, $+F0$ and $\beta$, $+T0$, for every $\beta \in \Delta$. We say that $\alpha$ is dually provable from $\Delta$ ($\Delta \vdash^d \alpha$) iff there exists a closed and completed tableau for a list of nodes consisting of $\alpha$, $-F0$ and $\beta$, $-T0$, for every $\beta \in \Delta$.

As an example of a tableau proof we present a proof of Uustalu’s counterexample to cut-elimination in the Gentzen-type sequent calculus by Rauszer, $p \vdash q \lor (r \rightarrow ((p \rightarrow q) \land r))$:
\[ p, +T0 \]
\[ q \lor (r \rightarrow ((p \rightarrow q) \land r)), +F0 \]
\[ \downarrow \]
\[ q, +F0 \]
\[ r \rightarrow ((p \rightarrow q) \land r), +F0 \]
\[ \downarrow \]
\[ 0r1 \]
\[ r, +T1 \]
\[ (p \rightarrow q) \land r, +F1 \]
\[ \downarrow \]
\[ \downarrow \]
\[ p \leftarrow q, +F1 \]
\[ \downarrow \]
\[ \downarrow \]
\[ p, +F0 \]
\[ \downarrow \]
\[ q, +T0 \]
Definition

Let $\mathcal{M} = \langle M, \leq, \models^+, \models^- \rangle$ be any connexive Kripke model and let $br$ be a tableau branch. The model $\mathcal{M}$ is said to be faithful to $br$ iff there exists a function $f$ from the set of all natural numbers into $M$ such that:

1. for every node $\alpha$, $+Ti$ on $br$, $f(i) \models^+ \alpha$;
2. for every node $\alpha$, $+Fi$ on $br$, $f(i) \models^+ \alpha$;
3. for every node $\alpha$, $-Ti$ on $br$, $f(i) \models^- \alpha$;
4. for every node $\alpha$, $-Fi$ on $br$, $f(i) \not\models^- \alpha$;
5. for every node $jrk$ on $br$, $f(j) \leq f(k)$.

The function $f$ is said to show that $\mathcal{M}$ is faithful to branch $br$. 
Lemma (Soundness)

Let $\mathcal{M}$ be any connexive Kripke model and $br$ be any tableau branch. If $\mathcal{M}$ is faithful to $br$ and a tableau rule is applied to $br$, then the application produces at least one extension $br'$ of $br$, such that $\mathcal{M}$ is faithful to $br'$.

Proof. By induction on the construction of tableaux. \hfill $\square$

Definition

Let $br$ be a completed and open tableau branch. Then the structure $\mathcal{M}_{br} = \langle M_{br}, \leq_{br}, \models^+_{br}, \models^-_{br} \rangle$ induced by $br$ is defined as follows:

1. $M_{br} := \{ x_j \mid j \text{ occurs on } br \}$,
2. $x_j \leq_{br} x_k$ iff $jrk$ occurs on $br$,
3. $x_j \in \models^+_{br} (p)$ iff $p, +Tj$ occurs on $br$,
4. $x_j \in \models^-_{br} (p)$ iff $p, -Tj$ occurs on $br$. 
Since $br$ is a completed branch, $\leq_{br}$ is reflexive and transitive, and thus $\langle M_{br}, \leq_{br} \rangle$ is a Kripke frame and $M_{br}$ is a connexive Kripke model.

The hereditary condition is satisfied because for any $\star \in \{+, -\}$, if $x_j \models^\star p$ and $x_j \leq_{br} x_k$, then $p, \star Tj$ and $jrk$ occur on $br$. Since $br$ is completed, the hereditary rule has been applied. Thus, $p, \star Tk$ occurs on $br$ and hence $x_k \models^\star p$.

Moreover, since $br$ is an open branch, $x_j \not\models^+ (p)$ if the node $p, +Fj$ occurs on $br$, and $x_j \not\models^- (p)$ if the node $p, -Fj$ occurs on $br$. 
Lemma (Completeness)

Suppose that $br$ is a completed and open tableau branch, and let $\mathcal{M}_{br} = \langle M_{br}, \leq_{br}, \models^+_br, \models^-_{br} \rangle$ be the model induced by $br$. Then

- If $\alpha, +Ti$ occurs on $br$, then $\mathcal{M}_{br}, x_i \models^+ \alpha$
- If $\alpha, +F$ occurs on $br$, then $\mathcal{M}_{br}, x_i \not\models^+ \alpha$
- If $\alpha, -Ti$ occurs on $br$, then $\mathcal{M}_{br}, x_i \models^- \alpha$
- If $\alpha, -Fi$ occurs on $br$, then $\mathcal{M}_{br}, x_i \not\models^- \alpha$.

Proof. By simultaneous induction on the construction of $\alpha$. □
From the previous two lemmas, it follows that the above tableau calculus is sound and complete for BCL with respect to both validity and dual validity.

**Theorem**

Let \( \Delta \cup \{ \alpha \} \) be a finite set of \( \mathcal{L}_{BCL} \)-formulas. Then 1. \( \Delta \models_{BCL} \alpha \) iff \( \Delta \vdash \alpha \), and 2. \( \Delta \models_{d}^{BCL} \alpha \) iff \( \Delta \vdash_{d}^{BCL} \alpha \).

**Proof.** We prove the second claim; the proof for the first claim is analogous.

Soundness: Suppose, by contraposition, that it is not the case that \( \Delta \models_{d}^{BCL} \alpha \) and let \( \Delta = \{ \beta_1, \ldots, \beta_n \} \). Then there is a connexive Kripke model \( \mathcal{M} \) with a state \( x \in \mathcal{M} \) such that \( x \models^= \beta_1 \ldots x \models^= \beta_n \) but \( x \not\models^= \alpha \). The model \( \mathcal{M} \) is faithful to the tableau branch consisting of \( \beta_1, -T0, \ldots, \beta_n, -T0, \alpha, -F0 \). By the soundness lemma, a completed tableau obtained from that list contains at least one branch to which \( \mathcal{M} \) is faithful. Clearly, this branch and hence the tableau must be open.
Completeness: Suppose, by contraposition, that it is not the case that $\Delta \vdash^d \alpha$. Then there is a completed open tableau starting with $\beta_1, -T0, \ldots, \beta_n, -T0, \alpha, -F0$, where $\Delta = \{ \beta_1, \ldots, \beta_n \}$. Let $br$ be an open branch of that tableau. By the completeness lemma, in the model induced by $br$, the state $x_0$ reveals that $\Delta \not\vdash^d_{BCL} \alpha$. \hfill $\Box$

Sound and complete tableau calculi for the subsystems ICL, DCL, BL, IL, and DL of BCL can be obtained from the tableau calculus for BCL by deleting the decomposition rules for the connective that is left out in the respective subsystem.
Open questions for future research with both a formal and a more philosophical concern:

- Are there any specific applications of connexive Heyting-Brouwer logic in addition to already known applications of systems of connexive logic?

- Applications to modelling syllogistic reasoning call for an extension to first order, and so does the discussion about co-implication in Heyting-Brouwer logic as a constructive connective.

- Another topic of interest is functional completeness for connexive Heyting-Brouwer logic, either along proof-theoretic or model-theoretic lines.


Sequent calculus and Kripke semantics
Embedding and completeness theorems
Duality between subsystems
Tableau calculus


Y. Shramko, “Dual intuitionistic logic and a variety of negations: The logic of scientific research”, *Studia Logica* 80 (2005), 347-367.


