Generalized Hopf differentials

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(joint work with Harold Rosenberg, Paris 7)

The basic global results in the theory of constant mean curvature (cmc) surfaces in space forms are the theorems of A. D. Alexandrov and H. Hopf from the 1950ies [4, 8]. Alexandrov's theorem states that a closed, embedded cmc surface in $\mathbb{S}^3_+$, $\mathbb{R}^3$, or $\mathbb{H}^3$ space form is necessarily a standard distance sphere. Its proof is based on a moving planes argument that is amazingly flexible and has been applied in many other contexts since. It has even turned out to be fruitful for the theory of nonlinear elliptic equations [7].

Hopf's theorem on the other hand states that an immersed cmc sphere in a space form M^3_κ is necessarily a standard distance sphere. The basic idea in Hopf's argument is to observe that the $(2,0)$–part of the second fundamental form $h_\Sigma = \langle\,.\,, A\,.\,\rangle$ of such a cmc surface Σ^2 is a *holomorphic* quadratic differential, a fact that is also one of the foundations of the theory of cmc tori in space forms [1, 5, 6].

1. New Results for Cmc Surfaces in Product Spaces

It is straightforward to extend Alexandrov's result and prove that a closed, embedded cmc surface Σ^2 in either one of the product spaces $\mathbb{S}^2_+ \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ is an embedded, rotationally-invariant sphere S^2_H and is therefore uniquely determined up to congruence by the value of its mean curvature H. In the first case the restriction to hemispheres is again crucial and, in fact, much more serious than in the case of the 3–sphere.

Hopf's result on the other hand is not that easy to generalize; the $(2,0)$–part $\pi_{2,0}(h_\Sigma)$ of the second fundamental form is in general *not holomorphic*, since the ambient space curvature term in the Codazzi equations does not vanish anymore. Our basic new result is that for cmc surfaces in the product spaces $M^2_\kappa \times \mathbb{R}$ holomorphicity can be restored with the help of an explicit, geometrically defined correction term [3]:

Theorem 1. *Let $(\kappa, H) \neq 0$, and let L be the symmetric bilinear form corresponding to the field of projectors onto the vertical lines in the product space $M^2_\kappa \times \mathbb{R}$. Then the expression*

$$Q := 2H \cdot \pi_{2,0}(h_\Sigma) - \kappa \cdot \pi_{2,0}(\iota^\star L) \ .$$

defines a natural holomorphic quadratic differential on any immersed cmc surface $\iota\colon \Sigma^2 \looparrowright M^2_\kappa \times \mathbb{R}$ with mean curvature H.

The remaining two theorems in this section are also established in [3, see]. First of all, applying ODE techniques to the fundamental equations of surface theory, we can classify the cmc surfaces with $Q \equiv 0$.

Theorem 2. *Let $(\kappa, H) \neq 0$, and let $\iota\colon \Sigma^2 \looparrowright M^2_\kappa \times \mathbb{R}$ be a complete surface with constant mean curvature H and vanishing holomorphic quadratic differential Q.*

Furthermore, let $\theta := \arcsin(d\xi\cdot\nu)$ *denote the angle between the unit normal field* ν *and the vertical lines. Then the following holds*:

- *if* $\kappa+4H^2>0$*, then* Σ^2 *is congruent to one of the embedded, rotationally-invariant cmc spheres* S^2_H*.*
- *if* $\kappa+4H^2\le 0$*, then* Σ^2 *is a complete* open *surface. Depending on the sign of the function* $4H^2+\kappa\cos^2(\theta)$*, it is either congruent to a disk-like surface* D^2_H *or a particular parabolic surface* P^2_H *or a surface* C^2_H *of catenoidal type.*

Since the space of holomorphic quadratic differentials on the sphere $\mathbb{S}^2=\mathbb{CP}^1$ is trivial, Theorems 1 and 2 yield the following analogue of Hopf's result:

Theorem 3. *Any immersed cmc sphere* S^2 *in a product space* $M^2_\kappa\times\mathbb{R}$ *is congruent to one of the embedded, rotationally-invariant cmc spheres* S^2_H*.*

2. Further Generalizations

In this section we investigate the *scope* where our generalized Hopf differentials can be defined. In particular, we ask for which (orientable) Riemannian 3–manifold (M^3,g) does there exist a correction field L that induces a holomorphic quadratic differential on any immersed cmc surface $\iota\colon\Sigma^2\looparrowright(M^3,g)$. In this generality, it is of course no longer possible to write down an explicit expression for the correction L. However, the following holds:

Theorem 4. *Fix some constant* $H\in\mathbb{R}$*. Let* (M^3,g) *be an oriented Riemannian manifold, and let* L_0 *be a* $\mathbb{C}$*–valued, traceless, symmetric bilinear form on* M^3*. Then the expression*

$$Q := \pi_{2,0}(h_\Sigma + \iota^\star L_0)$$

defines a holomorphic *quadratic differential on any surface* $\iota\colon\Sigma^2\looparrowright(M^3,g)$ *with constant mean curvature* H*, if and only if* L_0 *solves the differential equation*

$$(*)\qquad \mathrm{D}_X L_0 \;=\; \tfrac{1}{2}\,i\cdot\left[\star X, G-2H\,L_0\right]\,.$$

Here the square brackets denote the commutator, and $\star X$ stands for the skew-symmetric endomorphism $Y\mapsto X\times Y$ induced by the cross-product. Restricting to traceless fields L_0 is in fact a mere normalization. The ODE-system $(*)$ is strongly *overdetermined*, and thus one should expect that the corresponding integrability conditions impose serious *restrictions on the geometry* of the underlying Riemannian 3–manifold:

Theorem 5. *Let* $(\tilde M^3,g)$ *be a simply-connected, oriented Riemannian manifold, and let* $H\in\mathbb{R}$ *be some real constant. Then equation* $(*)$ *is solvable if and only if* $(\tilde M^3,g)$ *is a* homogeneous space *with an at least* 4*-dimensional isometry group.*

Recall that homogeneous Riemannian 3–manifolds $(\tilde M^3,g)$ come with 6–, 4–, or 3–dimensional isometry groups. Those with 6–dimensional isometry groups are the space forms, whereas those with 4–*dimensional isometry groups* admit natural equivariant Riemannian submersions with 1–dimensional, totally-geodesic fibers [10, 11]. Up to isometry they are classified by the curvature κ of the quotient surface and the bundle curvature τ. In this class of homogeneous 3–manifolds, one distinguishes six different *homogeneous structures*:

	$\kappa > 0$	$\kappa = 0$	$\kappa < 0$
$\tau = 0$	$\mathbb{S}^2 \times \mathbb{R}$	$\mathbb{R}^3$	$\mathbb{H}^2 \times \mathbb{R}$
$\tau \neq 0$	$\mathbb{S}^3_{\text{Berger}}$	$\mathsf{Nil}(3)$	$\widetilde{\mathsf{Sl}}(2, \mathbb{R})$

At this point we have constructed holomorphic quadratic differentials on cmc surfaces in homogeneous 3–manifolds corresponding to 7 of the eight maximal homogeneous structures that appear in Thurston theory; only the geometries corresponding to $\mathsf{Solv}(3)$ are missing.

Holomorphic quadratic differentials on cmc surfaces in the target spaces listed in the second row of the table were not known beforehand. Inspecting the proof of Theorem 5, one finds that equation $(*)$ always admits a homogeneous solution L_0. Following the argument leading to Theorem 2, it is possible to classify the cmc surfaces where the corresponding holomorphic quadratic differential Q vanishes identically, and thus we can generalize Hopf's result even further:

Theorem 6. *Any immersed cmc sphere S^2 in a simply-connected homogeneous space $(\tilde{M}^3, g)$ with an at least 4–dimensional isometry group is in fact an embedded, rotationally-invariant cmc sphere.*

It seems natural to think of the holomorphic quadratic differential Q constructed in Theorems 4 and 5 as a *family of first integrals for the cmc equation* that is due to the 1–dimensional isotropy groups of the bundle geometries and the 3–dimensional isotropy groups of the space forms, respectively. For the proofs of all 3 theorems presented in this section we refer the reader to the forthcoming paper [2].

3. Conclusions

It is a common feature of the bundle geometries that the isotropy group of any point p contains the 180°–rotations around all horizontal geodesics through p. This property makes it feasible to construct global minimal surfaces from Plateau solutions with suitable boundary polygons, using the Schwarz reflection principle. Together with the principal results of the preceding section, this observation provides a lot of evidence for the *thesis* that homogeneous 3–manifolds with at least 4–dimensional isometry groups are the proper setting for studying global properties of minimal surfaces and cmc surfaces.

The talk ended discussing this thesis in the context of the Heisenberg group. After describing the equivariant minimal surfaces in $\mathsf{Nil}(3)$ as classified by Mercuri and Pedrosa [9], we presented some local and global analogues of the doubly-periodic Scherk surface. With this background, we discussed the possibility of half-space theorems and Bernstein theorems for minimal surfaces in $\mathsf{Nil}(3)$. These results will be the subject of a forthcoming joint paper.

References

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