Local projection stabilisation for higher order discretisations of convection-diffusion problems on Shishkin meshes

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We consider a singularly perturbed convection-diffusion equation on the unit square where the solution of the problem exhibits exponential boundary layers. In order to stabilise the discretisation, two techniques are combined: Shishkin meshes are used and the local projection method is applied. For arbitrary $r \geq 2$, the standard Q_r-element is enriched by just 6 additional functions leading to an element which contains the P_{r+1}. In the local projection norm, the difference between the solution of the stabilised discrete problem and an interpolant of the exact solution is of order $\mathcal{O}\big((N^{-1}\ln N)^{r+1}\big)$, uniformly in ε. Furthermore, it is shown that the method converges uniformly in ε of order $\mathcal{O}\big((N^{-1}\ln N)^{r+1}\big)$ in the global energy norm.

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1 Introduction

Let $\Omega = (0,1)^2$ be the unit square. We consider the singularly perturbed boundary value problem

$$\begin{aligned} -\varepsilon\triangle u + b\cdot\nabla u + cu &= f \qquad \text{in } \Omega,\\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where ε is a small positive parameter while $b : \Omega \to \mathbb{R}^2$, $c : \Omega \to \mathbb{R}$, and $f : \Omega \to \mathbb{R}$ are sufficiently smooth functions satisfying

$$b_1(x,y) \geq \beta_1 > 0, \quad b_2(x,y) \geq \beta_2 > 0, \quad c(x,y) \geq 0 \qquad \forall (x,y) \in \overline{\Omega} \tag{2}$$

and

$$c(x,y) - \frac{1}{2} \operatorname{div} b(x,y) \geq c_0 > 0 \qquad \forall (x,y) \in \overline{\Omega}. \tag{3}$$

These assumptions on the data ensure that (1) has a unique solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$. Provided the assumptions in (2) are satisfied, condition (3) can be always fulfilled for sufficiently small ε by a change of variables $v(x,y) = e^{\sigma x} u(x,y)$ with a suitable constant σ.

Due to the positivity of β_1 and β_2, the solution u of problem (1) exhibits exponential boundary layers near the sides $x = 1$ and $y = 1$. For simplicity of our analysis, we assume that neither interior layers nor parabolic layers are present. The smallness of the parameter ε causes unphysical oscillations if standard schemes on general meshes are applied to discretise problem (1). Hence, stabilised methods and/or a-priori adapted meshes are widely used in order to get discrete solution with satisfactory accuracy. An overview on these methods together with consideration of the analytic behaviour of solutions of convection-diffusion problems can be found in the survey [19].

Provided some information on the structure of the layers are available, piecewise uniform Shishkin meshes can be chosen a priori, see [15, 18, 7]. Shishkin meshes were originally introduced for finite difference schemes. The first paper which considered Shishkin meshes for finite element methods seems to be [20] where the standard Galerkin method with piecewise bilinear elements was used.

The standard Galerkin discretisation lacks stability even on a-priori adapted meshes, see the numerical results in [13]. Furthermore, the linear systems of equations which arise from the standard Galerkin method on Shishkin meshes are hardly to solve by iterative methods [11, 13].

A powerful method for stabilising convection-diffusion problems is the streamline-diffusion finite element method (SDFEM) which was proposed by Hughes and Brooks [8]. This method is known to provide good stability properties and high accuracy outside interior and boundary layers. The SDFEM was investigated by many authors, see for example [16, 6, 9, 10]. The disadvantage of the SDFEM, in particular for higher order discretisations, is that additional terms have to be included into the weak formulation in order to ensure the strong consistency of the resulting method. However, the SDFEM on Shishkin meshes is much less sensitive to the choice of the transition points as standard Galerkin discretisations, see [17].

The SDFEM with higher order finite elements applied to convection-diffusion equations on Shishkin meshes was studied by Stynes and Tobiska in [22]. They have shown for Q_r-elements, $r \geq 2$, that in the streamline-diffusion norm the difference between the SDFEM solution and a special finite element interpolant of the solution of (1) is of order $\mathcal{O}(N^{-(r+1/2)})$. The proofs in [22] are based on Lin identities and anisotropic error estimates for the special finite element interpolation. Postprocessing operators are suggested in [21, 22] which allows to achieve estimates for the error between the weak solution and the discrete solution.

A different approach for stabilising the standard Galerkin discretisation is the local projection stabilisation technique. The stabilising term is based on a projection π into a discontinuous finite element space. Stabilisation of the standard Galerkin method is achieved by adding terms which give a weighted L^2-control on the fluctuations $(id - \pi)$ of the derivatives of the quantity of interest. The local projection stabilisation method has been introduced for the Stokes problem in [3] and was extended to the transport problem in [4]. An analysis of the local projection stabilisation for the Oseen problem can be found in [5, 14]. Although the local projection stabilisation leads to discrete systems which are consistent only in a weak sense, the appearing consistency error can be bounded such that the optimal convergence order is maintained.

Originally, the local projection stabilisation technique was introduced as a two level method where the projection maps into a discontinuous finite element space which lives on patches of elements [3, 4, 5]. It is possible to use standard finite element spaces for both the approximation space and the projection space, see [5, 14]. A fundamental drawback of this approach is the increased discretisation stencil. Moreover, the necessary data structures might not be available in an existing computer code. As pointed out in [14], the key for the analysis of the local projection method is the existence of a special interpolation operator which provides the standard interpolation error estimates and an additional orthogonality property. Applying the abstract setting from [14], the enrichment approach of the local projection method can be constructed where the approximation space and the projection space live on the same mesh. The approximation space is enriched compared to standard finite element spaces. In [14], it was shown that it suffices to enrich the standard quadrilateral Q_r-element, $r \geq 2$, by just two additional functions, independent of r. Hence, the discretisation stencil remains small.

Unfortunately, the enriched finite element spaces proposed in [14] are not suited for Shishkin meshes since they fail to satisfy optimal anisotropic error estimates, see Remark 8 in Sect. 3. For any $r \geq 2$, we will enrich the standard Q_r-element by just 6 additional functions. This leads to an finite element which contains the P_{r+1}. The newly constructed finite elements provide optimal anisotropic interpolation error estimates. The general line of our convergence proof is based on ideas given in [22]. However, different arguments have to be used since Lin identities seem to be not available for the used finite element spaces.

There are two main results in this paper. First, the error between the solution of the stabilised discrete problem and an interpolation of the solution of the continuous problem is shown to be bounded in the local projection norm by $\mathcal{O}\big((N^{-1}\ln N)^{r+1}\big)$, uniformly in ε. Second, we prove that the error between the solution of the stabilised discrete problem and the solution of the continuous problem itself can be estimated in the global energy norm by $\mathcal{O}\big((N^{-1}\ln N)^{r+1}\big)$, also uniformly in ε.

This paper is organised as follows. Section 2 describes the used Shishkin meshes and introduces the local projection stabilisation. For our enrichment approach of the local projection method, we propose and analyse in Section 3 new quadrilateral finite elements which are derived from the standard Q_r element but contain the P_{r+1}. The convergence of the enrichment approach is shown in Section 4. We will prove that in the local projection norm the difference between the local projection solution and an interpolant of the exact solution is of order $\mathcal{O}\big((N^{-1}\ln N)^{r+1}\big)$, uniformly in ε. Furthermore, it is shown that the enrichment approach of the local projection method converges uniformly in ε of order $\mathcal{O}\big((N^{-1}\ln N)^{r+1}\big)$ in the global energy norm. Finally, conclusions are given in Section 5.

Notation. Throughout this paper, C denotes a generic constant which is independent of the diffusion parameter ε and mesh parameter N. Although we consider finite element of arbitrary order, the dependence of any constant on the order r will not be elaborated.

Let G be an arbitrary measurable two-dimensional subset $G \subset \Omega$. The measure of G is denoted by $|G|$. On G, the usual Sobolev spaces $W^{m,p}(G)$ with norm $\|\cdot\|_{m,p,G}$ and semi-norm $|\cdot|_{m,p,G}$ are used. In the case $p=2$, we write $H^m(G)$ instead of $W^{m,2}(G)$ and skip the index p in the norm and the semi-norm. The L^2-inner product on G is denoted by $(\cdot,\cdot)_G$. Note that the index G in norms, semi-norms, and inner products is omitted in the case $G=\Omega$. All notation are also used for the vector-valued case.

Let $P_s(K)$ denote the space of all polynomials of total degree less than or equal to s while $Q_s(K)$ is the space of all polynomials of degree less than or equal to s in each variable separately.

2 Local projection stabilisation on Shishkin meshes

2.1 Shishkin meshes

Shishkin meshes are piecewise uniform meshes which are constructed a priori such that they are refined inside the layers, see [15, 17, 18]. Let N be an even positive integer. We denote by λ_x and λ_y the transition parameters which indicate where the mesh changes from coarse to fine. These parameters are defined by

$$\lambda_x := \min\left(\frac{1}{2}, (r+2)\frac{\varepsilon}{\beta_1}\ln N\right), \qquad \lambda_y := \min\left(\frac{1}{2}, (r+2)\frac{\varepsilon}{\beta_2}\ln N\right).$$

We assume in our analysis that λ_x and λ_y take the second value inside the corresponding minimum, otherwise N^{-1} is much smaller than ε and the analysis can be simplified dramatically.

Note that the multiplier in front of $(\varepsilon \ln N)/\beta_1$ and $(\varepsilon \ln N)/\beta_2$ (which is here $r+2$) has to be chosen large enough to ensure the optimal order of convergence. We refer to [17] for details on the lowest order case.

The domain Ω is divided into four parts as shown in Figure 1, left. Let $\overline{\Omega} = \overline{\Omega}_{11} \cup \overline{\Omega}_{12} \cup \overline{\Omega}_{21} \cup \overline{\Omega}_{22}$

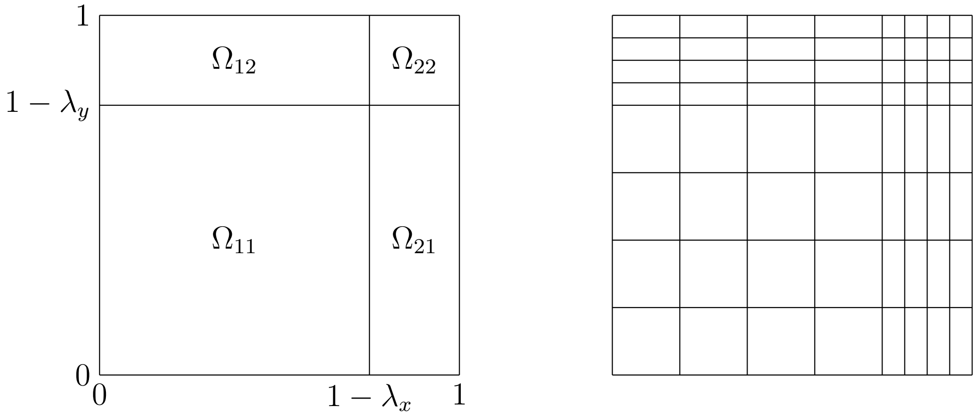

Figure 1: Division of Ω (left) and a corresponding Shishkin mesh (right).

where the subdomains are given by

$$\Omega_{11} := (0, 1-\lambda_x) \times (0, 1-\lambda_y), \qquad \Omega_{12} := (0, 1-\lambda_x) \times (1-\lambda_y, 1),$$

$$\Omega_{21} := (1-\lambda_x, 1) \times (0, 1-\lambda_y), \qquad \Omega_{22} := (1-\lambda_x, 1) \times (1-\lambda_y, 1).$$

Let $\mathcal{T}_x^N := \{(x_{i-1}, x_i) : i = 1, \ldots, N\}$ and $\mathcal{T}_y^N := \{(y_{j-1}, y_j) : j = 1, \ldots, N\}$ be two partitions of the interval $(0,1)$ where

$$x_i := \begin{cases} 2i(1-\lambda_x)/N, & i = 0, \ldots, N/2, \\ 1 - 2(N-i)\lambda_x/N, & i = N/2+1, \ldots, N, \end{cases}$$

and

$$y_j := \begin{cases} 2j(1-\lambda_y)/N, & j = 0, \ldots, N/2, \\ 1 - 2(N-j)\lambda_y/N, & j = N/2+1, \ldots, N. \end{cases}$$

Let $\mathcal{T}^N$ denote the tensor-product of $\mathcal{T}_x^N$ and $\mathcal{T}_y^N$. All cells in $\mathcal{T}^N$ are rectangles which are aligned with the coordinate axes, see Figure 1, right. Each subdomain contains $N^2/4$ cells. The cells in each subdomain are congruent to each other. Each rectangle in Ω_{11} is of size $\mathcal{O}(N^{-1}) \times \mathcal{O}(N^{-1})$. The rectangles in Ω_{22} are of size $\mathcal{O}(\varepsilon N^{-1} \ln N) \times \mathcal{O}(\varepsilon N^{-1} \ln N)$. In Ω_{12} and Ω_{21}, the size the longer edge of each rectangle is of order $\mathcal{O}(N^{-1})$ while the shorter edge size is of order $\mathcal{O}(\varepsilon N^{-1} \ln N)$. Hence, the mesh is coarse in Ω_{11}, fine in Ω_{22}, and highly anisotropic in Ω_{12} and Ω_{21}. The midpoint of $K \in \mathcal{T}^N$ is denoted by (x_K, y_K) while $h_{K,x}$ and $h_{K,y}$ are the edge sizes of K in x-direction and y-direction, respectively. Furthermore, let $h_K := \operatorname{diam} K$.

2.2 Decomposition of solution

Our subsequent analysis will rely on the precise knowledge of the behaviour of the solution u of the convection-diffusion problem (1). The typical behaviour of u is given in the following assumption.

Assumption 1. *The solution u can be decomposed as*

$$u = S + E_{12} + E_{21} + E_{22} \tag{4}$$

with $S, E_{12}, E_{21}, E_{22} \in C^{r+2}(\Omega)$. The smooth part S of the solution u fulfils

$$\left| \frac{\partial^{i+j} S}{\partial x^i \partial y^j}(x,y) \right| \le C, \qquad 0 \le i+j \le r+2, \tag{5}$$

while the layer functions satisfy

$$\left| \frac{\partial^{i+j} E_{12}}{\partial x^i \partial y^j}(x,y) \right| \le C\, \varepsilon^{-j}\, e^{-\beta_2(1-y)/\varepsilon}, \qquad 0 \le i+j \le r+2, \tag{6}$$

$$\left| \frac{\partial^{i+j} E_{21}}{\partial x^i \partial y^j}(x,y) \right| \le C\, \varepsilon^{-i}\, e^{-\beta_1(1-x)/\varepsilon}, \qquad 0 \le i+j \le r+2, \tag{7}$$

$$\left| \frac{\partial^{i+j} E_{22}}{\partial x^i \partial y^j}(x,y) \right| \le C\, \varepsilon^{-(i+j)}\, e^{-[\beta_1(1-x)/\varepsilon + \beta_2(1-y)/\varepsilon]}, \qquad 0 \le i+j \le r+2, \tag{8}$$

for all $(x,y) \in \Omega$. Here, E_{21} and E_{12} are exponential boundary layers along $x = 1$ and $y = 1$, respectively, while E_{22} is an exponential corner layer at the point $(1,1)$.

Note that the bound

$$\|S\|_{r+2} \le C \tag{9}$$

follows immediately from (5).

In [12], conditions on the right-hand side f of problem (1) were given which guarantee a decomposition of the solution into a smooth part and boundary layer parts such that lower order derivatives can be estimates by exponential bounds. The extension of these results to the case of higher order derivatives as needed in our case seems to be possible but tedious. The number of these sufficient conditions will increase rapidly with increasing differentiation order. We refer to [19, Sect. 7] for more details on these compatibility conditions.

Now we provide some estimates for integrals which involve exponential functions.

Lemma 2. *Let α, β be positive constants and $\lambda := (r+2)(\varepsilon \ln N)/\beta$. Then, the estimates*

$$\int_0^{1-\lambda} \exp\big(-\alpha\beta(1-z)/\varepsilon\big)\, dz \le C\varepsilon N^{-\alpha(r+2)} \quad \textit{and} \quad \int_{1-\lambda}^{1} \exp\big(-\alpha\beta(1-z)/\varepsilon\big)\, dz \le C\varepsilon$$

hold true.

Proof. For proving the first estimate, we get

$$\begin{aligned}\int_0^{1-\lambda} \exp\big(-\alpha\beta(1-z)/\varepsilon\big)\, dz &= \frac{\varepsilon}{\alpha\beta} \exp\big(-\alpha\beta(1-z)/\varepsilon\big)\Big|_{z=0}^{1-\lambda} \\ &\le \frac{\varepsilon}{\alpha\beta} \exp\big(-\alpha\beta\big(1-(1-\lambda)\big)/\varepsilon\big) \le C\varepsilon N^{-\alpha(r+2)}\end{aligned}$$

due to the positivity of the exponential function and the definition of λ. For the second estimate, we have

$$\begin{aligned}\int_{1-\lambda}^{1} \exp\big(-\alpha\beta(1-z)/\varepsilon\big)\, dz &= \frac{\varepsilon}{\alpha\beta} \exp\big(-\alpha\beta(1-z)/\varepsilon\big)\Big|_{z=1-\lambda}^{1} \\ &\le \frac{\varepsilon}{\alpha\beta} \exp\big(-\alpha\beta(1-1)/\varepsilon\big) \le C\varepsilon\end{aligned}$$

where again the positivity of the exponential function was exploited. □

2.3 Galerkin discretisation

Let $V := H_0^1(\Omega)$. We define the bilinear form

$$a(v, w) := \varepsilon(\nabla v, \nabla w) + (b \cdot \nabla v + cv, w).$$

A weak formulation of the convection-diffusion problem (1) reads

Find $u \in V$ such that

$$a(u, v) = (f, v) \qquad \forall v \in V. \tag{10}$$

Note that the variational formulation (10) has a unique solution due to (3).

Let $\widehat{V}$ be a finite dimensional function space on the reference square $\widehat{K} := (-1,1)^2$. The space $\widehat{V}$ will be introduced in Sect. 3.1. Furthermore, let $F_K : \widehat{K} \to K$ with

$$F_K(\hat{x},\hat{y}) = \left(x_K + \frac{h_{K,x}}{2}\hat{x}, y_K + \frac{h_{K,y}}{2}\hat{y} \right)^T \tag{11}$$

be the reference transformation which is a simple affine mapping. Our finite element space V^N is defined as

$$V^N := \left\{ v \in C(\overline{\Omega}) \;:\; v|_K \in V(K)\; \forall K \in \mathcal{T}^N,\; v = 0 \text{ on } \partial\Omega \right\} \tag{12}$$

where

$$V(K) := \left\{ v \;:\; v \circ F_K \in \widehat{V} \right\}$$

is a finite dimensional function space on K. The space V^N will be a non-standard one, see Sect. 3.2.

Using the finite element space V^N, we can state the standard Galerkin discretisation of (10) which reads

Find $\tilde{u}^N \in V^N$ such that

$$a(\tilde{u}^N, v^N) = (f, v^N) \qquad \forall v^V \in V^N. \tag{13}$$

Note that the discrete problem (13) is uniquely solvable due to (3).

2.4 Local projection stabilisation

We proceed by introducing some more notation which will be used for defining the local projection method.

Let π_K denote the $L^2(K)$-projection into $P_{r-1}(K)$. The fluctuation operator $\kappa_K : L^2(K) \to L^2(K)$ is given as $\kappa_K := id_K - \pi_K$ where id_K is the identity mapping on $L^2(K)$.

The fundamental approximation property of the fluctuation operator κ_K is stated in the following lemma.

Lemma 3. *For $0 \le s \le r$, the fluctuation operator κ_K fulfils*

$$\|\kappa_K w\|_{0,K} \le C\, h_K^s |w|_{s,K} \qquad \forall w \in H^s(K)$$

for all $K \in \mathcal{T}^N$.

Proof. The assertion is a consequence of the Bramble–Hilbert lemma. □

Since we are interested in an additional control on the streamline derivative, we introduce the following stabilisation term

$$s^N(v,w) := \sum_{K \in \mathcal{T}^N} \tau_K \big(\kappa_K(b \cdot \nabla v), \kappa_K(b \cdot \nabla w) \big)_K$$

with the cell-dependent parameters τ_K, $K \in \mathcal{T}^N$. Note that

$$\left|s^N(v,w)\right| \le \left(s^N(v,v)\right)^{1/2}\left(s^N(w,w)\right)^{1/2} \qquad \forall v,w \in H^1(\Omega) \tag{14}$$

holds.

The stabilisation parameters τ_K, $K \in \mathcal{T}^N$, are chosen as

$$\tau_K := \begin{cases} C_1\,N^{-2}, & K \subset \Omega_{11}, \\ 0, & \text{otherwise}, \end{cases} \tag{15}$$

with a suitable constant C_1 which is independent of ε and N. As for the SDFEM on Shishkin meshes considered in [21, 22], the stabilisation acts only on the coarse subdomain Ω_{11}. However, the stabilisation parameters δ_K, $K \subset \Omega_{11}$, used in [21, 22] were chosen to be $C_1\,N^{-1}$ for the case $\varepsilon \le N^{-1}$.

The stabilised bilinear form a^N is defined via

$$a^N(u,v) := a(u,v) + s^N(u,v).$$

The stabilised discrete problem reads

Find $u^N \in V^N$ such that

$$a^N(u^N, v^N) = (f, v^N) \qquad \forall v^N \in V^N. \tag{16}$$

We will use the norms

$$\|v\|_{LP} := \left(\varepsilon\,|v|_1^2 + c_0\,\|v\|_0^2 + s^N(v,v)\right)^{1/2}, \qquad \|v\|_{1,\varepsilon} := \left(\varepsilon\,|v|_1^2 + c_0\,\|v\|_0^2\right)^{1/2}$$

in our analysis.

3 Enriched finite elements and anisotropic error estimates

3.1 Family of enriched finite elements

Now we will introduce for each $r \ge 2$ a new finite element which will be used later on for our local projection method. To this end, we start with defining the finite element $(\widehat{K}, \widehat{V}_r, \widehat{\mathcal{N}}_r)$ on the reference cell $\widehat{K} = (-1,1)^2$.

Let L_i, $i \ge 0$, denote the one-dimensional Legendre polynomials normalised such that $L_i(+1) = 1$. The Legendre polynomials are orthogonal with respect to the L^2-inner product on the interval $(-1,+1)$, i.e.,

$$\int_{-1}^{1} L_i(\hat{s})\,L_j(\hat{s})\,d\hat{s} = \eta_i\,\delta_{ij}, \qquad i,j \ge 0,$$

where δ_{ij} denotes the Kronecker Delta and $\eta_i := 2/(2i+1)$.

Let $r \ge 2$ be an integer. We define

$$\begin{aligned}\widehat{V}_r := Q_r(\widehat{K}) &\oplus \operatorname{span}\big((1-\hat{x}^2)(1-\hat{y}^2)\,\hat{x}^{r-1},\ (1-\hat{x}^2)(1-\hat{y}^2)\,\hat{y}^{r-1}\big)\\ &\oplus \operatorname{span}\big((1+\hat{y})(1-\hat{x}^2)\,L_{r-1}(\hat{x}),\ (1-\hat{y})(1-\hat{x}^2)\,L_{r-1}(\hat{x}),\\ &\qquad (1+\hat{x})(1-\hat{y}^2)\,L_{r-1}(\hat{y}),\ (1-\hat{x})(1-\hat{y}^2)\,L_{r-1}(\hat{y})\big).\end{aligned} \tag{17}$$

A careful inspection shows that

$$\dim \widehat{V}_r = r^2 + 2r + 7$$

and

$$P_{r+1}(\widehat{K}) \subset \widehat{V}_r \subset Q_{r+1}(\widehat{K}).$$

Note that $\widehat{V}_r$ contains exactly 6 additional functionals compared to the standard Q_r-element, independent of $r \ge 2$. We define the nodal functionals

$$N_1 : \hat{\varphi} \mapsto \hat{\varphi}(-1,-1), \quad N_2 : \hat{\varphi} \mapsto \hat{\varphi}(+1,-1), \quad N_3 : \hat{\varphi} \mapsto \hat{\varphi}(+1,+1), \quad N_4 : \hat{\varphi} \mapsto \hat{\varphi}(-1,+1),$$

which correspond to the values at the vertices of $\widehat{K}$. For $i \ge 0$, the moments along the edges of $\widehat{K}$ are given by

$$N_{-,i} : \hat{\varphi} \mapsto \int_{-1}^{+1} \hat{y}^i\, \hat{\varphi}(-1,\hat{y})\, d\hat{y}, \qquad N_{+,i} : \hat{\varphi} \mapsto \int_{-1}^{+1} \hat{y}^i\, \hat{\varphi}(+1,\hat{y})\, d\hat{y},$$

$$N_{i,-} : \hat{\varphi} \mapsto \int_{-1}^{+1} \hat{x}^i\, \hat{\varphi}(\hat{x},-1)\, d\hat{x}, \qquad N_{i,+} : \hat{\varphi} \mapsto \int_{-1}^{+1} \hat{x}^i\, \hat{\varphi}(\hat{x},+1)\, d\hat{x}.$$

We define for $i, j \ge 0$ the cell moments by

$$N_{i,j} : \hat{\varphi} \mapsto \int_{\widehat{K}} \hat{x}^i\, \hat{y}^j\, \hat{\varphi}(\hat{x},\hat{y})\, d\hat{y}\, d\hat{x}.$$

The set $\widehat{\mathcal{N}}_r$ of nodal functionals is defined as

$$\begin{aligned}\widehat{\mathcal{N}}_r := \big\{N_1, N_2, N_3, N_4\big\} &\cup \big\{N_{-,i}, N_{+,i}, N_{i,-}, N_{i,+} \ :\ i = 0, \dots, r-1\big\}\\ &\cup \big\{N_{i,j} \ :\ 0 \le i, j \le r-2\big\} \cup \big\{N_{r-1,0},\ N_{0,r-1}\big\}.\end{aligned}$$

A simple counting shows that $\widehat{\mathcal{N}}_r$ contains exactly $r^2 + 2r + 7$ nodal functionals. Hence, the number of nodal functionals in $\widehat{\mathcal{N}}_r$ equals the dimension of $\widehat{V}_r$.

We prove now that the set $\widehat{\mathcal{N}}_r$ of nodal functionals is unisolvent on the function space $\widehat{V}_r$.

Lemma 4. *The set $\widehat{\mathcal{N}}_r$ of nodal functionals is unisolvent with respect to the function space $\widehat{V}_r$.*

Proof. Since the number of nodal functionals in $\widehat{\mathcal{N}}_r$ and the dimension of $\widehat{V}_r$ coincide, it suffices to show that $\hat{v} \in \widehat{V}_r$ with vanishing nodal functionals will result in $\hat{v} \equiv 0$.

The definition (17) of $\widehat{V}_r$ shows that we have for each edge $\widehat{E}$ of $\widehat{K}$ that $\hat{v}|_{\widehat{E}} \in P_{r+1}(\widehat{E})$. Using that the moments up to order $r-1$ vanish along the edge $\widehat{E}$, we get from the orthogonality of the Legendre polynomials that

$$\hat{v}|_{\widehat{E}} = \alpha\, L_r(\hat{s}) + \beta\, L_{r+1}(\hat{s}), \qquad \alpha, \beta \in \mathbb{R},$$

where $\hat{s} \in (-1,1)$ is the variable along the edge $\widehat{E}$. Since $\hat{v}$ is zero at vertices (which correspond to $\hat{s} = \pm 1$) and $L_i(\pm 1) = (\pm 1)^i$, $i \geq 0$, we conclude that $\hat{v}|_{\widehat{E}} = 0$ for all edges $\widehat{E}$ of $\widehat{K}$. Hence, $\hat{v}|_{\partial\widehat{K}} = 0$ and $\hat{v}$ can be written as

$$\hat{v} = (1-\hat{x}^2)(1-\hat{y}^2)\,\hat{q}, \qquad \hat{q} \in Q_{r-2}(\widehat{K}) \oplus \operatorname{span}(\hat{x}^{r-1}, \hat{y}^{r-1}),$$

due to (17). The vanishing cell moments result in

$$0 = (\hat{q}, \hat{v})_{\widehat{K}} = \big(\hat{q}, (1-\hat{x}^2)(1-\hat{y}^2)\,\hat{q}\big)_{\widehat{K}}.$$

Since the factor $(1-\hat{x}^2)(1-\hat{y}^2)$ is positive inside $\widehat{K}$, we obtain $\hat{q} = 0$. Hence, $\hat{v} \equiv 0$ and the lemma is proved. □

3.2 Properties of the interpolation operator

The unisolvence given in Lemma 4 allows us to define the interpolation operator $\widehat{I} : C(\overline{\widehat{K}}) \to \widehat{V}$ by

$$\widehat{N}(\widehat{I}\hat{v}) = \widehat{N}(\hat{v}) \qquad \forall \widehat{N} \in \widehat{\mathcal{N}}_r.$$

Using $\widehat{I}$, the interpolation operator $I_K : C(\overline{K}) \to V(K)$ is defined as

$$(I_K v) := \big(\widehat{I}(v \circ F_K)\big) \circ F_K^{-1}.$$

Due to the definition of $\widehat{I}$, the restriction of $I_K v$ onto an edge $E \subset \partial K$ depends only on the restriction of v onto E. Hence, the local interpolation operators I_K, $K \in \mathcal{T}^N$, can be put together to form the global interpolation operator $I^N : C(\overline{\Omega}) \to V^N$ which is given by

$$(I^N v)|_K := I_K(v|_K) \qquad \forall K \in \mathcal{T}^N,\ v \in C(\overline{\Omega}). \tag{18}$$

Due to the definition of V^N given in (12), the standard Q_r-space is a subspace of V^N.

We proceed with giving some fundamental properties of the interpolation operator I^N.

Lemma 5. *For the interpolation operator $I^N : C(\overline{\Omega}) \to V^N$ defined in* (18)*, the orthogonality property*

$$(w - I^N w, q)_K = 0 \qquad \forall K \in \mathcal{T}^N,\ \forall q \in P_{r-1}(K),\ \forall w \in C(\overline{K}).$$

is fulfilled.

Proof. We transform the integral to the reference cell $\widehat{K}$ and obtain

$$(w - I^N w, q)_K = h_{K,x}\, h_{K,y}\, \big(w \circ F_K - (I^N w) \circ F_K, q \circ F_K\big)_{\widehat{K}}.$$

Since F_K defined in (11) is an affine mapping, we have $q \circ F_K \in P_{r-1}(\widehat{K})$. The definition of I^N yields $(I^N w) \circ F_K = \widehat{I}(w \circ F_K)$. Due to the definition of $\widehat{I}$, the above integral vanishes since all $P_{r-1}(\widehat{K})$-moments are included in the set of nodal functionals. □

We will have now a look at L^∞-estimates of the interpolation.

Lemma 6. *Let $K \in \mathcal{T}^N$. Then, the estimate*

$$\|I^N v\|_{0,\infty,K} \le C\|v\|_{0,\infty,K} \qquad \forall v \in C(\overline{K})$$

holds true where C is independent of K and N.

Proof. We start the proof by showing

$$\|\widehat{I}\hat{v}\|_{0,\infty,\widehat{K}} \le C\|\hat{v}\|_{0,\infty,\widehat{K}}$$

where $\hat{v} := v \circ F_K \in C(\overline{\widehat{K}})$. The desired estimate follows by transforming this estimate from $\widehat{K}$ to K.

We set $t := \dim \widehat{V}_r$. Let $\{\hat{\varphi}_i \,:\, i = 1,\dots,t\}$ be the basis of $\widehat{V}_r$ which is dual with respect to the set $\widehat{\mathcal{N}}_r = \{N_i \,:\, i = 1,\dots,t\}$ of nodal functionals, i.e., $N_i(\hat{\varphi}_j) = \delta_{ij}$, $i, j = 1,\dots,t$. Then, we have

$$\|\widehat{I}\hat{v}\|_{0,\infty,\widehat{K}} = \left\| \sum_{i=1}^{t} N_i(\hat{v})\hat{\varphi}_i \right\|_{0,\infty,\widehat{K}} \le \sum_{i=1}^{t} |N_i(\hat{v})|\, \|\hat{\varphi}_i\|_{0,\infty,\widehat{K}}$$

by the triangle inequality. Due to the definition of the nodal functionals via point values and integrals, we have that

$$|N_i(\hat{v})| \le C\|\hat{v}\|_{0,\infty,\widehat{K}}, \qquad i = 1,\dots,t.$$

Since $\|\hat{\varphi}_i\|_{0,\infty,\widehat{K}} \le C$, $i = 1,\dots,t$, we conclude

$$\|\widehat{I}\hat{v}\|_{0,\infty,\widehat{K}} \le C\|\hat{v}\|_{0,\infty,\widehat{K}}$$

and the assertion of the lemma follows by transforming this estimate from $\widehat{K}$ to K. $\square$

3.3 Anisotropic error estimates

The first derivatives of the interpolation error can be estimated by the following lemma.

Lemma 7. *Let s be an integer satisfying $1 \le s \le r+1$. Then, the estimates*

$$\big\|(v - I^N v)_x\big\|_{0,K} \le C \sum_{i+j=s} h_{K,x}^i\, h_{K,y}^j \left\| \frac{\partial^{s+1} v}{\partial x^{i+1}\,\partial y^j} \right\|_{0,K} \qquad \forall v \in C(\overline{K}) \text{ with } \partial_x v \in H^s(K) \quad (19)$$

and

$$\big\|(v - I^N v)_y\big\|_{0,K} \le C \sum_{i+j=s} h_{K,x}^i\, h_{K,y}^j \left\| \frac{\partial^{s+1} v}{\partial x^{i}\,\partial y^{j+1}} \right\|_{0,K} \qquad \forall v \in C(\overline{K}) \text{ with } \partial_y v \in H^s(K) \quad (20)$$

hold true where the constant C is independent of N and $K \in \mathcal{T}^N$.

Proof. We restrict ourselves to the proof of estimate (19) since the proof of (20) follows the same lines. Provided

$$\big\|(\hat v - \widehat I\hat v)_{\hat x}\|_{0,K} \le C\big|\hat v_{\hat x}\big|_{s,K} \qquad \forall \hat v \in C(\overline{\widehat K}) \text{ with } \partial_{\hat x} \in H^s(\widehat K) \tag{21}$$

holds true, the estimate (19) follows immediately by transforming the estimate (21) from $\widehat K$ to K and using the relation between $\widehat I$ and I^N. Hence, it remains to show (21).

To this end, we will use the techniques by Apel and Dobrowolski [2], see also [1, Chapter 2]. Let $\widehat W$ denote the space which is obtained by the differentiation of $\widehat V_r$ with respect to $\hat x$, i.e., $\widehat W := \partial_{\hat x}\widehat V_r$. We set $t := \dim \widehat W$. From the definition of $\widehat V_r$, we conclude that $t = r^2 + r + 5$. To apply [2, Lemma 3], we have to find a set $\widehat{\mathcal F}_r$ of t linear functionals which have the following properties:

(i) $F \in \big(H^s(\widehat K)\big)' \ \forall F \in \widehat{\mathcal F}_r$,

(ii) $F\big(\partial_{\hat x}(\hat v - \widehat I\hat v)\big) = 0 \ \forall F \in \widehat{\mathcal F}_r,\ \forall \hat v \in C(\overline{\widehat K})$ such that $\partial_{\hat x}\hat v \in H^s(\widehat K)$,

(iii) the set $\widehat{\mathcal F}_r$ is unisolvent on $\widehat W$.

We define for non-negative integers i, j, k the linear functionals

$$F_i : \hat v \mapsto \int_{\widehat K} \hat x^i\, \hat v(\hat x, \hat y)\, d\hat y\, d\hat x, \qquad F_{j,k} : \hat v \mapsto \int_{\widehat K} \hat x^j\, \hat y^k\, \frac{\partial \hat v}{\partial \hat y}\, d\hat y\, d\hat x.$$

The set $\widehat{\mathcal F}_r$ is given as

$$\widehat{\mathcal F}_r := \{F_i \ :\ i = 0, \dots, r\} \cup \{F_{j,k} \ :\ 0 \le j, k \le r-1\} \cup \{F_{r,0}, F_{r,1}, F_{0,r}, F_{1,r}\}.$$

Note that $\widehat{\mathcal F}_r$ contains exactly $r^2 + r + 5$ linear functionals. Due to the definition, we have for $F \in \widehat{\mathcal F}_r$ that $F \in \big(H^s(\widehat K)\big)'$ for all $s \ge 1$. Hence, condition (i) is fulfilled. To show condition (ii), let $\hat v \in C(\overline{\widehat K})$ such that $\partial_{\hat x}\hat v \in H^s(\widehat K)$. We start with the nodal functionals F_i, $0 \le i \le r$, which contain no derivatives. An integration by parts gives

$$F_i\big(\partial_{\hat x}(\hat v - \widehat I\hat v)\big) = \int_{\widehat K} \hat x^i\, \partial_{\hat x}(\hat v - \widehat I\hat v)\, d\hat y\, d\hat x = -i \int_{\widehat K} \hat x^{i-1}\, (\hat v - \widehat I\hat v)\, d\hat y\, d\hat x + \int_{\partial\widehat K} \hat x^i\, \hat n_1\, (\hat v - \widehat I\hat v)\, d\gamma$$

where $\hat n = (\hat n_1, \hat n_2)^T$ denotes for unit normal vector on $\partial\widehat K$. The first integral vanishes since $i - 1 \le r - 1$. The boundary integral is a sum of the integrals on those edges where $\hat n_1$ is non-zero. These are just the edges where $\hat x$ is constant. Hence, the boundary integral equals to zero since the edge moments with a constant test function vanish.

We proceed with the nodal functionals which involve a derivative. We have for the indices j, k either $0 \le j, k \le r-1$ or one index is equal to r and the other index is 0 or 1. We obtain by an integration by parts

$$F_{j,k}\big(\partial_{\hat x}(\hat v - \widehat I\hat v)\big) = \int_{\widehat K} \hat x^j\, \hat y^k\, \frac{\partial^2}{\partial\hat x\partial\hat y}(\hat v - \widehat I\hat v)\, d\hat y\, d\hat x$$

$$= -k \int_{\widehat{K}} \hat{x}^j\, \hat{y}^{k-1}\, \partial_{\hat{x}}(\hat{v} - \widehat{I}\hat{v})\, d\hat{y}\, d\hat{x} + \int_{\partial\widehat{K}} \hat{x}^j\, \hat{y}^k\, \hat{n}_2\, \partial_{\hat{x}}(\hat{v} - \widehat{I}\hat{v})\, d\gamma. \tag{22}$$

The boundary integral reduces to a sum of integrals over the edges where $\hat{n}_2$ is non-zero. These are the edges with $\hat{y} = \pm 1$, i.e., $\hat{y}$ is constant there. Ignoring the sign, the integrals along these edges have the form

$$\int_{-1}^{1} \hat{x}^j \partial_{\hat{x}}(\hat{v} - \widehat{I}\hat{v})(\hat{x}, \pm 1)\, d\hat{x} = -j \int_{-1}^{1} \hat{x}^{j-1}\, (\hat{v} - \widehat{I}\hat{v})(\hat{x}, \pm 1)\, d\hat{x} + \left[\hat{x}^j \left(\hat{v}(\hat{x}, \pm 1) - (\widehat{I}\hat{v})(\hat{x}, \pm 1)\right)\right]\Big|_{\hat{x}=-1}^{+1}.$$

The integral gives zero since the edge moments vanish due to $j - 1 \le r - 1$. The occurring difference term is identically zero due to the used point values. Hence, the boundary integral in (22) is zero. The remaining integral in (22) is again integrated by parts to obtain

$$\int_{\widehat{K}} \hat{x}^j\, \hat{y}^{k-1}\, \partial_{\hat{x}}(\hat{v} - \widehat{I}\hat{v})\, d\hat{y}\, d\hat{x} = -j \int_{\widehat{K}} \hat{x}^{j-1}\, \hat{y}^{k-1}\, (\hat{v} - \widehat{I}\hat{v})\, d\hat{y}\, d\hat{x} + \int_{\partial\widehat{K}} \hat{x}^j\, \hat{y}^{k-1}\, \hat{n}_1\, (\hat{v} - \widehat{I}\hat{v})\, d\gamma.$$

The boundary integral reduces to a sum of integrals over those edges where $\hat{n}_1$ is non-zero. These are the edges with $\hat{x} = \pm 1$. Hence, $\hat{y}$ is the variable along these edges and the corresponding integrals vanish since edge moments with $k - 1 \le r - 1$ occur. The integral over $\widehat{K}$ vanishes due to the ranges of j and k. Hence, the expression in (22) gives zero. Summarising, the condition (ii) is fulfilled.

It remains to show condition (iii), the unisolvence of $\widehat{\mathcal{F}}_r$ on $\widehat{W}$. To this end, let $\{\hat{w}_k : k = 1, \ldots, t\}$ be a basis of $\widehat{W}$. We define the matrix $A = (a_{jk}) \in \mathbb{R}^{t \times t}$ by $a_{jk} = F_j(\widehat{w}_k)$, $j, k = 1, \ldots, t$. The unisolvence of $\widehat{\mathcal{F}}_r$ on $\widehat{W}$ is equivalent to the unique solvability of the linear system with matrix A. This is given provided the system $A\beta = \alpha$ has for each right-hand side $\alpha \in \mathbb{R}^t$ exactly one solution $\beta \in \mathbb{R}^t$. Since the linear system is finite dimensional, the unisolvence is also equivalent to the property that for each right-hand side $\alpha \in \mathbb{R}^t$ there exists at least one function $\hat{w} = \sum_{k=1}^{t} \beta_k \hat{w}_k \in \widehat{W}$ with $F_j(\hat{w}) = \alpha_j$, $j = 1, \ldots, t$. Let α be an arbitrary vector of $\mathbb{R}^t$. Since all nodal functionals in $\widehat{\mathcal{F}}_r$ are linearly independent, there exists a function $\hat{\varphi} \in C^\infty(\overline{\widehat{K}})$ such that $F_j(\partial_{\hat{x}}\hat{\varphi}) = \alpha_j$, $j = 1, \ldots, t$. Let $\hat{w} := \partial_{\hat{x}}\widehat{I}\hat{\varphi} \in \widehat{W}$. Then, condition (ii) gives that $F_j(\hat{w}) = \alpha_j$, $j = 1, \ldots, t$. Hence, condition (iii) is fulfilled.

Since the conditions (i), (ii), and (iii) are satisfied, Lemma 3 in [2] ensures that estimate (21) holds true. The transformation of (21) from $\widehat{K}$ to K shows that the interpolation operator I^N satisfies the anisotropic interpolation error estimate (19). As already said, (20) can be shown in a similar way. □

Remark 8. *The enriched finite element spaces $Q_r^{bubble,1}$ introduced in [14] fails to satisfy anisotropic error estimates as given in Lemma 7. Due to [2, Lemma 4], a necessary condition for the existence of an anisotropic error estimate is that a function $\hat{\varphi}$ on the reference element $\widehat{K}$ which depends only on $\hat{x}$ has an interpolant $\widehat{I}\hat{\varphi}$ which depends also only on $\hat{x}$. If we consider $Q_2^{bubble,1}$ from [14] with the vertex-edge-cell interpolation then the interpolant of $\hat{\varphi} := L_3(\hat{x}) - L_1(\hat{x}) = 5/2(\hat{x}^3 - \hat{x})$ has the interpolant $-15/4(1 - \hat{x}^2)(1 - \hat{y}^2)\hat{x}$ which clearly depends also on $\hat{y}$. Hence, the necessary condition fails and the desired anisotropic interpolation error estimates are not available.*

Now we have a look at the interpolation error.

Lemma 9. *Let $q \in [1,\infty]$ and $2 \le s \le r+2$. Then, there exists a constant C independent of N and $K \in \mathcal{T}^N$ such that*

$$\|v - I^N v\|_{0,q,K} \le C \sum_{i+j=s} h_{K,x}^i \, h_{K,y}^j \left\| \frac{\partial^s v}{\partial x^i \, \partial y^j} \right\|_{0,q,K}$$

holds true for all $v \in W^{s,q}(K)$.

Proof. To prove this lemma, we use again the technique provided in [2]. Since this time no derivatives are involved, the conditions (i)–(iii) in the proof of Lemma 7 are automatically fulfilled if one sets $\widehat{\mathcal{F}}_r := \widehat{\mathcal{N}}_r$. The bound $s \ge 2$ is caused by the fact that the nodal functionals from $\widehat{\mathcal{N}}_r$ are defined for continuous functions. Note that the assumptions on s and q ensure that $W^{s,q}(\widehat{K}) \subset C(\overline{\widehat{K}})$. Again, the estimate obtained on the reference cell $\widehat{K}$ is transformed to $K \in \mathcal{T}^N$ to get the assertion of this lemma. □

4 Analysis of enrichment approach

In this section, we will analyse the stabilised discrete problem (16) and prove in the local projection norm $\|\cdot\|_{LP}$ an estimate between the discrete solution u^N and the interpolant $I^N u$ of the solution u of the weak formulation (10). Furthermore, an estimate for $\|u - u^N\|_{1,\varepsilon}$ will be shown. Note that the constants in both estimates will be independent of N and the diffusion parameter ε.

4.1 Interpolation error estimates for solution

We will start with estimates for the solution u of (10) where decomposition due to Assumption 1 will be used.

Lemma 10. *Let* (6), (7), *and* (8) *be fulfilled. Then, the estimates*

$$\begin{aligned}
\|I^N E_{12}\|_{0,\infty,\Omega_{11}\cup\Omega_{21}} &\le C\|E_{12}\|_{0,\infty,\Omega_{11}\cup\Omega_{21}} \le C\,N^{-(r+2)}, \\
\|I^N E_{21}\|_{0,\infty,\Omega_{11}\cup\Omega_{12}} &\le C\|E_{21}\|_{0,\infty,\Omega_{11}\cup\Omega_{12}} \le C\,N^{-(r+2)}, \\
\|I^N E_{22}\|_{0,\infty,\Omega\setminus\Omega_{22}} &\le C\|E_{22}\|_{0,\infty,\Omega\setminus\Omega_{22}} \le C\,N^{-(r+2)}
\end{aligned}$$

hold true.

Proof. Using Lemma 6, we have

$$\|I^N E_{12}\|_{0,\infty,\Omega_{11}\cup\Omega_{21}} \le C\|E_{12}\|_{0,\infty,\Omega_{11}\cup\Omega_{21}} \le C\,N^{-(r+2)}$$

where (6) and the choice of the transition point λ_y were used. The remaining two estimate follow in the same way by using Lemma 6, the bounds by the exponential functions, and the choice of the transition points λ_x and λ_y. □

Lemma 11. *Let u denote the solution of* (10) *and let Assumption 1 be fulfilled. Then, the estimate*

$$\big|(I^N u - u)(x,y)\big| \le \begin{cases} C\,N^{-(r+2)}, & (x,y)\in\Omega_{11},\\ C\,(N^{-1}\ln N)^{r+2}, & \textit{otherwise}, \end{cases}$$

is fulfilled. Furthermore, we have that

$$\|I^N u - u\|_0 \le C(N^{-1}\ln N)^{r+2}, \qquad \|I^N u - u\|_{0,\Omega_{11}} \le CN^{-(r+2)}$$

hold true.

Proof. To prove this lemma, the decomposition (4) will be used. We have

$$I^N u - u = (I^N S - S) + (I^N E_{12} - E_{12}) + (I^N E_{21} - E_{21}) + (I^N E_{22} - E_{22}).$$

The S-term is estimated by using Lemma 9 with $q=\infty$ and $s=r+2$. We obtain for all $(x,y)\in\Omega$ by exploiting (5) that

$$|(I^N S - S)(x,y)| \le CN^{-(r+2)}\|S\|_{r+2,\infty} \le CN^{-(r+2)}.$$

For E_{12} and $(x,y)\in\Omega_{12}\cup\Omega_{22}$, we get by similar arguments that

$$|(I^N E_{12} - E_{12})(x,y)| \le C\sum_{i+j=r+2} N^{-i}(\varepsilon N^{-1}\ln N)^j\varepsilon^{-j} \le C(N^{-1}\ln N)^{r+2}$$

where (6) was exploited. To estimate E_{12} on $\Omega_{11}\cup\Omega_{21}$, we obtain

$$\|(I^N E_{12} - E_{12})\|_{0,\infty,\Omega_{11}\cup\Omega_{21}} \le C\|E_{12}\|_{0,\infty,\Omega_{11}\cup\Omega_{21}} \le CN^{-(r+2)}$$

by applying Lemma 6 and Lemma 10. The estimate for E_{21} is obtained by similar arguments on the subdomains $\Omega_{21}\cup\Omega_{22}$ and $\Omega_{11}\cup\Omega_{12}$, respectively. For estimating the E_{22}, we apply Lemma 9 with $s=r+2$ and $q=\infty$ on Ω_{22} and use (8). The estimates on the other subdomains follow from Lemma 6 and Lemma 10.

The logarithmic factor in the estimate on $\Omega\setminus\Omega_{11}$ is caused by the bounds for the layer terms. All four terms have no logarithmic factor on Ω_{11}.

The estimates of the L^2-norm are a simple consequence of the just proven pointwise bounds. □

We proceed with estimating the gradient of the interpolation error.

Lemma 12. *Let u be the solution of* (10) *and Assumption 1 be fulfilled. Then, the estimate*

$$\varepsilon^{1/2}\|\nabla(I^N u - u)\|_0 \le C\,(N^{-1}\ln N)^{r+1}$$

holds where C is independent of ε and N.

Proof. In the following, we present only the estimates for the term which involves the x-derivative since the term with the y-derivative can be estimated by using the same ideas.

Applying the decomposition (4) of the solution u of (10), we obtain with the triangle inequality that

$$\begin{aligned}\|(I^N u - u)_x\|_0 \le & \|(I^N S - S)_x\|_0 + \|(I^N E_{12} - E_{12})_x\|_0 \\ & + \|(I^N E_{21} - E_{21})_x\|_0 + \|(I^N E_{22} - E_{22})_x\|_0.\end{aligned}$$

Now, each term will be estimated separately.

We start with the S-term. Using Lemma 7 with $s = r+1$, we obtain

$$\|(I^N S - S)_x\|_0 \le C \sum_{i+j=r+1} h_{K,x}^i \, h_{K,y}^j \left\| \frac{\partial^{r+2} S}{\partial x^{r+1}\, \partial y^j} \right\|_{0,K} \le C N^{-(r+1)} \|S\|_{r+2} \le C N^{-(r+1)}$$

where $h_{K,x}, h_{k,y} \le C N^{-1}$ and (9) were used.

For estimating the E_{21}-term on $\Omega_{21} \cup \Omega_{22}$, we consider an arbitrary $K \subset \Omega_{21} \cup \Omega_{22}$ and use Lemma 7 with $s = r+1$. We obtain by using the bound (7)

$$\begin{aligned}\left\|(I^N E_{21} - E_{21})_x\right\|_{0,K} &\le C \sum_{i+j=r+1} h_{K,x}^i \, h_{K,y}^j \left\| \frac{\partial^{r+2} E_{21}}{\partial x^{i+1} \partial y^j} \right\|_{0,K} \\ &\le C \sum_{i+j=r+1} (\varepsilon N^{-1} \ln N)^i \, N^{-j} \, \varepsilon^{-(i+1)} \| \exp(-\beta_1(1-x)/\varepsilon)\|_{0,K} \\ &\le C \varepsilon^{-1} \, (N^{-1} \ln N)^{r+1} \| \exp(-\beta_1(1-x)/\varepsilon)\|_{0,K}.\end{aligned}$$

Putting together the estimates for all $K \subset \Omega_{21} \cup \Omega_{22}$, one gets

$$\begin{aligned}\left\|(I^N E_{21} - E_{21})_x\right\|_{0,\Omega_{21}\cup\Omega_{22}}^2 &= \sum_{K \subset \Omega_{21}\cup\Omega_{22}} \left\|(I^N E_{21} - E_{21})_x\right\|_{0,K}^2 \\ &\le C \varepsilon^{-2} \, (N^{-1} \ln N)^{2(r+1)} \| \exp(-\beta_1(1-x)/\varepsilon)\|_{0,\Omega_{21}\cup\Omega_{22}}^2.\end{aligned}$$

Applying Lemma 2, we obtain

$$\| \exp(-\beta_1(1-x)/\varepsilon)\|_{0,\Omega_{21}\cup\Omega_{22}}^2 = \int_{y=0}^{1} \int_{x=1-\lambda_x}^{1} \exp(-2\beta_1(1-x)/\varepsilon)\, dx\, dy \le C\, \varepsilon.$$

Hence, the estimate

$$\varepsilon^{1/2} \left\|(I^N E_{21} - E_{21})_x\right\|_{0,\Omega_{21}\cup\Omega_{22}} \le C (N^{-1} \ln N)^{r+1}$$

holds true. To estimate the E_{21}-term on $\Omega_{11} \cup \Omega_{12}$, we apply the triangle inequality to split

$$\left\|(I^N E_{21} - E_{21})_x\right\|_{0,\Omega_{11}\cup\Omega_{12}} \le \left\|(I^N E_{21})_x\right\|_{0,\Omega_{11}\cup\Omega_{12}} + \left\|(E_{21})_x\right\|_{0,\Omega_{11}\cup\Omega_{12}}.$$

Using the bound (7), we end up with

$$\left\|(E_{21})_x\right\|_{0,\Omega_{11}\cup\Omega_{12}}^2 \le C\, \varepsilon^{-2} \int_{y=0}^{1} \int_{x=0}^{1-\lambda_x} \exp(-2\beta_1(1-x)/\varepsilon)\, dx\, dy \le C \varepsilon^{-1} \, N^{-2(r+2)}$$

where Lemma 2 was used. By an inverse inequality, we obtain

$$\left\|(I^N E_{21})_x\right\|_{0,\Omega_{11}\cup\Omega_{12}} \leq CN\|I^N E_{21}\|_{0,\Omega_{11}\cup\Omega_{12}}.$$

Using Lemma 6 and the behaviour of E_{21} due to (7), we have

$$\begin{aligned}
\|I^N E_{21}\|^2_{0,\Omega_{11}\cup\Omega_{12}} &= \sum_{i=1}^{N/2} \int_{x_{i-1}}^{x_i} \int_0^1 |I^N E_{21}(x,y)|^2 \, dy \, dx \\
&\leq C \sum_{i=1}^{N/2} \int_{x_{i-1}}^{x_i} e^{-2\beta_1(1-x_i)/\varepsilon} \, dx \\
&\leq C \sum_{i=1}^{N/2-1} \int_{x_i}^{x_{i+1}} e^{-2\beta_1(1-x_i)/\varepsilon} \, dx + C \int_{x_{N/2-1}}^{x_{N/2}} e^{-2\beta_1(1-x_i)/\varepsilon} \, dx \\
&\leq C(\varepsilon + N^{-1}) N^{-(2r+4)}
\end{aligned}$$

where Lemma 2 and $x_{N/2} = 1 - \lambda_x$ were exploited. Hence, we obtain

$$\varepsilon^{1/2}\left\|(I^N E_{21} - E_{21})_x\right\|_0 \leq C(N^{-1}\ln N)^{r+1}$$

by collecting the above bounds.

For estimating the E_{12}-term on $\Omega_{12} \cup \Omega_{22}$, we start with the application of Lemma 7 with $s = r + 1$ on $K \subset \Omega_{12} \cup \Omega_{22}$. We get by using the bound (6) that

$$\begin{aligned}
\left\|(I^N E_{12} - E_{12})_x\right\|_{0,K} &\leq C \sum_{i+j=r+1} h^i_{K,x}\, h^j_{K,y} \left\| \frac{\partial^{r+2} E_{12}}{\partial x^{i+1} \partial y^j} \right\|_{0,K} \\
&\leq C \sum_{i+j=r+1} N^{-i} \, (\varepsilon N^{-1} \ln N)^j \, \varepsilon^{-j} \| \exp(-\beta_2(1-y)/\varepsilon)\|_{0,K} \\
&\leq C(N^{-1}\ln N)^{r+1} \| \exp(-\beta_2(1-y)/\varepsilon)\|_{0,K}.
\end{aligned}$$

Combining the estimates for all $K \subset \Omega_{12} \cup \Omega_{22}$ results in

$$\begin{aligned}
\varepsilon\left\|(I^N E_{12} - E_{12})_x\right\|^2_{0,\Omega_{12}\cup\Omega_{22}} &= \sum_{K\subset\Omega_{12}\cup\Omega_{22}} \varepsilon \left\|(I^N E_{12} - E_{12})_x\right\|^2_{0,K} \\
&\leq C\varepsilon \, (N^{-1}\ln N)^{2(r+1)} \| \exp(-\beta_2(1-y)/\varepsilon)\|^2_{0,\Omega_{12}\cup\Omega_{22}} \\
&\leq C(N^{-1}\ln N)^{2(r+1)}
\end{aligned}$$

where the last inequality was obtained by applying Lemma 2. To estimate the E_{12}-term on $\Omega_{11} \cup \Omega_{21}$, we apply Lemma 7 with $s = 1$. One obtains on each $K \subset \Omega_{11} \cup \Omega_{21}$ that

$$\left\|(I^N E_{12} - E_{12})_x\right\|_{0,K} \leq CN^{-1}\Big(\left\|(E_{12})_{xx}\right\|_{0,K} + \left\|(E_{12})_{xy}\right\|_{0,K}\Big).$$

Hence, we have on $\Omega_{11} \cup \Omega_{21}$ the estimate

$$\left\|(I^N E_{12} - E_{12})_x\right\|^2_{0,\Omega_{11}\cup\Omega_{21}} \leq CN^{-2}\Big(\left\|(E_{12})_{xx}\right\|^2_{0,\Omega_{11}\cup\Omega_{21}} + \left\|(E_{12})_{xy}\right\|^2_{0,\Omega_{11}\cup\Omega_{21}}\Big)$$

$$
\begin{aligned}
&\le CN^{-2} \int_0^1 \int_0^{1-\lambda_y} \varepsilon^{-2} \exp(-2\beta_2(1-y)/\varepsilon)\, dy\, dx \\
&\le \varepsilon^{-1} N^{-2r+6}
\end{aligned}
$$

due to (6) where again Lemma 2 was applied. Hence, we obtain

$$
\varepsilon^{1/2} \big\| (I^N E_{12} - E_{12})_x \big\|_0 \le C\, (N^{-1} \ln N)^{r+1}
$$

by collecting the above bounds.

The first step to estimate the E_{22}-term on Ω_{22} is the use of Lemma 7 with $s = r+1$. We get on each $K \subset \Omega_{22}$ that

$$
\begin{aligned}
\big\| (I^N E_{22} - E_{22})_x \big\|_{0,K} &\le C \sum_{i+j=r+1} h_{K,x}^i\, h_{K,y}^j \left\| \frac{\partial^{r+2} E_{22}}{\partial x^{i+1} \partial y^j} \right\|_{0,K} \\
&\le C \sum_{i+j=r+1} (\varepsilon N^{-1} \ln N)^i\, (\varepsilon N^{-1} \ln N)^j\, \varepsilon^{-(i+j+1)} \, * \\
&\qquad\qquad * \, \| \exp\big(-(\beta_1(1-x) + \beta_2(1-y))/\varepsilon \big) \|_{0,K} \\
&\le C\varepsilon^{-1}\, (N^{-1} \ln N)^{r+1} \| \exp\big(-(\beta_1(1-x) + \beta_2(1-y))/\varepsilon \big) \|_{0,K}
\end{aligned}
$$

holds where the bound (8) was exploited. Hence, the estimate

$$
\begin{aligned}
\big\| (I^N E_{22} - E_{22})_x \big\|_{0,\Omega_{22}}^2 &= \sum_{K \subset \Omega_{22}} \big\| (I^N E_{22} - E_{22})_x \big\|_{0,K}^2 \\
&\le C\varepsilon^{-2}\, (N^{-1} \ln N)^{2(r+1)} \| \exp\big(-(\beta_1(1-x) + \beta_2(1-y))/\varepsilon \big) \|_{0,\Omega_{22}}^2 \\
&\le C (N^{-1} \ln N)^{2(r+1)}
\end{aligned}
$$

is obtained by using Lemma 2 for handling the exponential term in the following form

$$
\begin{aligned}
\| \exp\big(-(\beta_1(1-x) + \beta_2(1-y))/\varepsilon \big) \|_{0,\Omega_{22}}^2 &= \int_{1-\lambda_x}^1 \int_{1-\lambda_y}^1 e^{-2(\beta_1(1-x)+\beta_2(1-y))/\varepsilon}\, dy\, dx \\
&= \left(\int_{1-\lambda_x}^1 e^{-2(\beta_1(1-x))/\varepsilon}\, dx \right) \left(\int_{1-\lambda_y}^1 e^{-2(\beta_2(1-y))/\varepsilon}\, dy \right) \\
&\le C\varepsilon^2.
\end{aligned}
$$

It remains to estimate the interpolation error of E_{22} on $\Omega \setminus \Omega_{22}$. On $\Omega_{11} \cup \Omega_{12}$, one can use the same techniques as for estimating the E_{21}-term on these subdomains. Compared to E_{21}, there is an additional y-dependent factor in the upper bound for E_{22} which can be bounded by one. The estimation of the E_{22}-term on Ω_{21} follows the lines of the estimate of the E_{21}-term on this subdomain. The additional factor is again bounded by one.

Putting together all above estimates, the assertion of the lemma follows. □

4.2 Error analysis

We start the error analysis with showing the unique solvability of the stabilised discrete problem (16) which is a consequence of the following coercivity of the bilinear form a^N.

Lemma 13. *The stabilised bilinear form a^N fulfils*

$$a^N(v,v) \geq \|v\|_{LP}^2 \qquad \forall v \in V,$$

i.e., a^N is coercive on V.

Proof. Using the definition of a^N, we obtain after an integration by parts that

$$\begin{aligned} a^N(v,v) &= \varepsilon(\nabla v, \nabla v) + (b \cdot \nabla v + c\, v, v) + s^N(v,v) \\ &= \varepsilon\, |v|_1^2 + \left(c - \frac{1}{2}\operatorname{div} b, v^2\right) + s^N(v,v) \geq \|v\|_{LP}^2 \end{aligned}$$

due to assumption (3) and the definition of $\|\cdot\|_{LP}$. □

Since the local projection methods is only weakly consistent, the appearing consistency error has to be considered.

Lemma 14. *Let u and u^N denote the solution of the weak formulation* (10) *and the stabilised discrete problem* (16)*, respectively. Then, we have*

$$a^N(u - u^N, w^N) = s^N(u, w^N)$$

for all $w^N \in V^N$.

Proof. Subtracting the stabilised discrete problem (16) from the weak formulation (10) yields

$$\begin{aligned} a^N(u - u^N, w^N) &= a(u, w^N) + s^N(u, w^N) - a^N(u^N, w^N) \\ &= (f, w^N) + s^N(u, w^N) - (f, w^N) = s^N(u, w^N) \qquad \forall w^N \in V^N \end{aligned}$$

and the lemma is proved. □

We will now estimate the stabilisation term s^N in the case of some special arguments.

Lemma 15. *Let u be the solution of the variational problem* (10)*. Furthermore, let Assumption 1 be fulfilled. Then, the estimate*

$$\left|s^N(I^N u, w^N)\right| \leq C N^{-(r+1)}\, \|w^N\|_{LP}$$

holds for all $w^N \in V^N$ where I^N is the interpolation operator defined in (18)*. The constant C is independent of N and ε.*

Proof. Using the decomposition (4) of the solution u, we obtain

$$\begin{aligned} s^N(I^N u, w^N) &= s^N(I^N S, w^N) + \sum_{ij} s^N(I^N E_{ij}, w^N) \\ &= s^N(I^N S - S, w^N) + s^N(S, w^N) + \sum_{ij} s^N(I^N E_{ij}, w^N) \end{aligned}$$

where the ij-sum runs through $\{12, 21, 22\}$. Each term will be estimated separately.

We start with the term which involves only S and get after applying inequality (14) that

$$\begin{aligned} \left|s^N(S, w^N)\right| &\le \left(s^N(S,S)\right)^{1/2}\left(s^N(w^N,w^N)\right)^{1/2} \le \left(\sum_{K\subset\Omega_{11}} \tau_K \|\kappa_K(b\cdot\nabla S)\|_{0,K}^2\right)^{1/2} \|w^N\|_{LP} \\ &\le C\left(\sum_{K\subset\Omega_{11}} \tau_K\, N^{-2r} |b\cdot\nabla S|_{r,K}^2\right)^{1/2} \|w^N\|_{LP} \\ &\le C\left(\sum_{K\subset\Omega_{11}} \tau_K\, N^{-2r} \|S\|_{r+1,K}^2\right)^{1/2} \|w^N\|_{LP} \le C N^{-(r+1)} \|w^N\|_{LP} \end{aligned}$$

where the smoothness of b, Lemma 3 with $s = r$, the choice (15) for τ_K, and the bound (9) were exploited.

We proceed with the term which contains the difference $I^N S - S$ and obtain by applying inequality (14) that

$$\begin{aligned} \left|s^N(I^N S - S, w^N)\right| &\le \left(\sum_{K\subset\Omega_{11}} \tau_K \|\kappa_K(b\cdot\nabla(I^N S - S))\|_{0,K}^2\right)^{1/2} \left(s^N(w^N,w^N)\right)^{1/2} \\ &\le C\left(\sum_{K\subset\Omega_{11}} \tau_K \|\nabla(I^N S - S)\|_{0,K}^2\right)^{1/2} \|w^N\|_{LP} \\ &\le C\left(\sum_{K\subset\Omega_{11}} \tau_K N^{-2(r+1)} \|S\|_{r+1,K}^2\right)^{1/2} \|w^N\|_{LP} \le C N^{-(r+2)} \|w^N\|_{LP} \end{aligned}$$

where the Lemma 3 with $s = 0$, the smoothness of b, Lemma 7 with $s = r+1$ for $\|\nabla(I^N S - S)\|_{0,\Omega_{11}}$ as in the proof of Lemma 12, the bound (9), and the choice (15) for τ_K were used.

Finally, we investigate the exponential terms E_{ij}. Let E denote one the three layer functions. Using Lemma 3 with $s = 0$, the smoothness of b, and an inverse inequality, we get on $K \subset \Omega_{11}$ that

$$\begin{aligned} \|\kappa_K(b\cdot\nabla I^N E)\|_{0,K} &\le C\|b\cdot\nabla I^N E\|_{0,K} \le C|I^N E|_{1,K} \le C\,N\,\|I^N E\|_{0,K} \\ &\le C\,N\,|K|^{1/2}\|I^N E\|_{0,\infty,K}. \end{aligned}$$

Hence, we obtain

$$\left|s^N(I^N E, w^N)\right| \le \left(\sum_{K\subset\Omega_{11}} \tau_K \|\kappa_K(b\cdot\nabla I^N E)\|_{0,K}^2\right)^{1/2} \|w^N\|_{LP}$$

$$
\begin{aligned}
&\leq C\left(\sum_{K\subset\Omega_{11}} \tau_K\, N^2\,|K|\,\|I^N E\|^2_{0,\infty,K}\right)^{1/2} \|w^N\|_{LP}\\
&\leq C\left(\sum_{K\subset\Omega_{11}} \tau_K\, N^2\,|K|\right)^{1/2} \|I^N E\|_{0,\infty,\Omega_{11}}\,\|w^N\|_{LP}\\
&\leq CN^{-(r+2)}\,\|w^N\|_{LP}
\end{aligned}
$$

where the choice (15) for τ_K and Lemma 10 for bounding $\|I^N E\|_{0,\infty,\Omega_{11}}$ were applied. Putting together all above estimates, the assertion of the lemma follows. □

We proceed with the consideration of the convective and reactive terms.

Lemma 16. *Let u be the solution of* (10) *and Assumption 1 be fulfilled. Then, the estimate*

$$
\left|(b\cdot\nabla(I^N u-u)+c(I^N u-u),w^N)\right| \leq C\,N^{-(r+1)}\,\|w^N\|_{LP}
$$

holds for all $w^N\in V^N$ where C is independent of ε and N.

Proof. An integration by parts of the convective term yields

$$
\begin{aligned}
(b\cdot\nabla(I^N u-u)+c(I^N u-u),w^N) =&(w^N(c-\operatorname{div} b),I^N u-u)\\
&-(I^N u-u,b\cdot\nabla w^N)_{\Omega_{11}}-(I^N u-u,b\cdot\nabla w^N)_{\Omega\setminus\Omega_{11}}.
\end{aligned}
$$

Each term will be considered individually.

The first term is estimated as

$$
\left|(w^N(c-\operatorname{div} b),I^N u-u)\right| \leq C\|w^N\|_0\|I^N u-u\|_0 \leq C(N^{-1}\ln N)^{r+2}\,\|w^N\|_{LP}.
$$

which follows from the Cauchy–Schwarz inequality and Lemma 11.

The orthogonality property of I^N given in Lemma 5 yields

$$
\begin{aligned}
\left(I^N u-u,b\cdot\nabla w^N\right)_{\Omega_{11}} &= \sum_{K\subset\Omega_{11}}\left(I^N u-u,b\cdot\nabla w^N-\pi_K(b\cdot\nabla w^N)\right)_K\\
&=\sum_{K\subset\Omega_{11}}\left(I^N u-u,\kappa_K(b\cdot\nabla w^N)\right)_K.
\end{aligned}
$$

Using additionally the choice (15) for τ_K and again Lemma 11, we get

$$
\begin{aligned}
\left|(I^N u-u,b\cdot\nabla w^N)_{\Omega_{11}}\right| &\leq \left(\sum_{K\subset\Omega_{11}}\tau_K^{-1}\|I^N u-u\|^2_{0,K}\right)^{1/2}\left(\sum_{K\subset\Omega_{11}}\tau_K\|\kappa_K(b\cdot\nabla w^N)\|^2_{0,K}\right)^{1/2}\\
&\leq CN^{r+1}\|w^N\|_{LP}.
\end{aligned}
$$

Applying a Hölder inequality, we obtain

$$
\left|(I^N u-u,b\cdot\nabla w^N)_{\Omega\setminus\Omega_{11}}\right| \leq \|I^N u-u\|_{0,\infty,\Omega\setminus\Omega_{11}}\|b\cdot\nabla w^N\|_{0,1,\Omega\setminus\Omega_{11}}
$$

The first factor is estimated by Lemma 11 while the second factor is bounded as follows

$$\|b\cdot\nabla w^N\|_{0,1,\Omega\setminus\Omega_{11}} \le |\Omega\setminus\Omega_{11}|^{1/2}\|b\cdot\nabla w^N\|_{0,\Omega\setminus\Omega_{11}} \le C(\ln N)^{1/2}\varepsilon^{1/2}|w^N|_{1,\Omega\setminus\Omega_{11}}$$

where $|\Omega\setminus\Omega_{11}| \le C\varepsilon\ln N$ and the smoothness of b were used. Hence, we obtain

$$\left|(I^N u - u, b\cdot\nabla w^N)_{\Omega\setminus\Omega_{11}}\right| \le CN^{-(r+2)}(\ln N)^{r+5/2}\|w^N\|_{LP}$$

by combing the various bounds.

Putting together all above estimate, the lemma is proved. □

Now we can state our main result.

Theorem 17. *Let u denote the solution of* (10) *and u^N solution of the stabilised discrete problem* (16)*. Furthermore, let Assumption 1 be fulfilled. Then, the estimates*

$$\|I^N u - u^N\|_{LP} \le C\,(N^{-1}\ln N)^{r+1}$$

and

$$\|u - u^N\|_{1,\varepsilon} \le C\,(N^{-1}\ln N)^{r+1}$$

hold true where the constants are independent of ε and N.

Proof. Using Lemma 13 which gives the coercivity of a^N, we obtain

$$\begin{aligned}\|I^N u - u^N\|_{LP}^2 &\le a^N(I^N u - u^N, I^N u - u^N)\\ &= a^N(I^N u - u, I^N u - u^N) + a^N(u - u^N, I^N u - u^N)\\ &= a(I^N u - u, I^N u - u^N) + s^N(I^N u, I^N u - u^N)\end{aligned} \tag{23}$$

where Lemma 14 and the definition of bilinear form a^N were exploited. The second term in (23) can be estimated by using Lemma 15. We obtain

$$|s^N(I^N u, I^N u - u^N)| \le CN^{-(r+1)}\,\|I^N u - u^N\|_{LP}.$$

For estimating the first term in (23), we use the definition of the bilinear form a and get

$$\begin{aligned}|a(I^N u - u, I^N u - u^N)| &\le |\varepsilon(\nabla(I^N u - u), \nabla(I^N u - u^N))|\\ &\quad + |(b\cdot\nabla(I^N u - u) + c(I^N u - u), I^N u - u^N)|\\ &\le \varepsilon^{1/2}\|\nabla(I^N u - u)\|_0\,\varepsilon^{1/2}\|\nabla(I^N u - u^N)\|_0 + C\,N^{-(r+1)}\,\|I^N u - u^N\|_{LP}\\ &\le C\,(N^{-1}\ln N)^{r+1}\,\|\nabla(I^N u - u^N)\|_{LP}\end{aligned}$$

where Lemma 16 and Lemma 12 were applied. Putting these estimate into (23), the first statement of this theorem follows.

Using the triangle inequality, we obtain

$$\|u - u^N\|_{1,\varepsilon} \le \|u - I^N u\|_{1,\varepsilon} + \|I^N u - u^N\|_{1,\varepsilon}.$$

The first term can be estimated by using the definition of $\|\cdot\|_{1,\varepsilon}$ and the bounds from Lemma 12 and Lemma 11. The second term is bounded by $\|I^N u - u^N\|_{LP}$. Hence, the desired assertion is proven. □

5 Conclusions

We have considered in this paper the local projection stabilisation applied to higher order discretisation of convection-diffusion problems on Shishkin meshes. We proposed and analysed new finite elements which are enrichments of the standard Q_r-elements by 6 functions and which contain the polynomial space P_{r+1}. For the difference between the solution u of the weak formulation (10) and solution u_h of the stabilised discrete problem (16), an ε-uniform convergence rate $\mathcal{O}\big((N^{-1}\ln N)^{r+1}\big)$ in ε-weighted H^1-norm was proven, see Theorem 17. Moreover, the difference between u_h and and the interpolant $I^N u$ of the solution u convergences in the local projection norm $\|\cdot\|_{LP}$ ε-uniformly with order $\mathcal{O}\big((N^{-1}\ln N)^{r+1}\big)$, see Theorem 17.

References

[1] T. Apel, *Anisotropic finite elements. Local estimates and applications*, Advances in Numerical Mathematics, Teubner, Leipzig, 1999.

[2] T. Apel and M. Dobrowolski, *Anisotropic interpolation with applications to the finite element method*, Computing, 47 (1992), pp. 277–293.

[3] R. Becker and M. Braack, *A finite element pressure gradient stabilization for the Stokes equations based on local projections*, Calcolo, 38 (2001), pp. 173–199.

[4] R. Becker and M. Braack, *A two-level stabilization scheme for the Navier-Stokes equations*, in Numerical mathematics and advanced applications, M. Feistauer, V. Dolejší, P. Knobloch, and K. Najzar, eds., Berlin, 2004, Springer-Verlag, pp. 123–130.

[5] M. Braack and E. Burman, *Local projection stabilization for the Oseen problem and its interpretation as a variational multiscale method*, SIAM J. Numer. Anal., 43 (2006), pp. 2544–2566.

[6] K. Eriksson and C. Johnson, *Adaptive streamline diffusion finite element methods for stationary convection-diffusion problems*, Math. Comp., 60 (1993), pp. 167–188.

[7] P. A. Farrell, A. F. Hegarty, J. J. H. Miller, E. O'Riordan, and G. I. Shishkin, *Robust computational techniques for boundary layers*, vol. 16 of Applied Mathematics (Boca Raton), Chapman & Hall/CRC, Boca Raton, FL, 2000.

[8] T. J. R. Hughes and A. Brooks, *A multidimensional upwind scheme with no crosswind diffusion*, in Finite element methods for convection dominated flows (Papers, Winter Ann. Meeting Amer. Soc. Mech. Engrs., New York, 1979), vol. 34 of AMD, Amer. Soc. Mech. Engrs. (ASME), New York, 1979, pp. 19–35.

[9] C. Johnson and J. Saranen, *Streamline diffusion methods for the incompressible Euler and Navier–Stokes equations*, Math. Comp., 47 (1986), pp. 1–18.

[10] C. Johnson, A. Schatz, and L. Wahlbin, *Crosswind smear and pointwise errors in the streamline diffusion finite element methods*, Math. Comp., 49 (1987), pp. 25–38.

[11] T. Linss, *Analysis of a Galerkin finite element method on a Bakhvalov-Shishkin mesh for a linear convection-diffusion problem*, IMA J. Numer. Anal., 20 (2000), pp. 621–632.

[12] T. Linss and M. Stynes, *Asymptotic analysis and Shishkin-type decomposition for an elliptic convection-diffusion problem*, J. Math. Anal. Appl., 261 (2001), pp. 604–632.

[13] T. Linss and M. Stynes, *Numerical methods on Shishkin meshes for linear convection-diffusion problems.*, Comput. Methods Appl. Mech. Eng., 190 (2001), pp. 3527–3542.

[14] G. Matthies, P. Skrzypacz, and L. Tobiska, *A unified convergence analysis for local projection stabilisations applied to the Oseen problem*, Preprint 44/2006, Fakultät für Mathematik, Otto-von-Guericke-Universität Magdeburg, 2006.

[15] J. J. H. Miller, E. O'Riordan, and G. I. Shishkin, *Fitted numerical methods for singular perturbation problems: Error estimates in the maximum norm for linear problems in one and two dimensions*, World Scientific Publishing Co. Inc., River Edge, NJ, 1996.

[16] U. Nävert, *A finite element method for convection-diffusion problems*, PhD thesis, Chalmers University of Technology, Göteborg, 1982.

[17] H.-G. Roos, *Layer-adapted grids for singular perturbation problems*, ZAMM Z. Angew. Math. Mech., 78 (1998), pp. 291–309.

[18] H.-G. Roos, M. Stynes, and L. Tobiska, *Numerical methods for singularly perturbed differential equations. Convection–diffusion and flow problems*, no. 24 in Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1996.

[19] M. Stynes, *Steady-state convection-diffusion problems*, Acta Numer., 14 (2005), pp. 445–508.

[20] M. Stynes and E. O'Riordan, *A uniformly convergent Galerkin method on a Shishkin mesh for a convection-diffusion problem*, J. Math. Anal. Appl., 214 (1997), pp. 36–54.

[21] M. Stynes and L. Tobiska, *The SDFEM for a convection-diffusion problem with a boundary layer: optimal error analysis and enhancement of accuracy*, SIAM J. Numer. Anal., 41 (2003), pp. 1620–1642.

[22] M. Stynes and L. Tobiska, *Using rectangular Q_p elements in the SDFEM for a convection-diffusion problem with a boundary layer*, Preprint 08/2006, Fakultät für Mathematik, Otto-von-Guericke-Universität Magdeburg, 2006. accepted for Appl. Numer. Math.