Local projection type stabilisation applied to inf-sup stable discretisations of the Oseen problem

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December 5, 2007

The local projection method is applied to inf-sup stable discretisations of the Oseen problem. Error bounds of order r are proven for known inf-sup stable pairs of finite element spaces which approximate velocity and pressure by elements of order r and $r-1$, respectively. In case of a positive reaction coefficient, the error constants are robust with respect to the viscosity but depend on the positive lower bound of the reaction coefficient. Using enriched velocity spaces, error estimates of order r are established which are also robust when both the viscosity and the reaction coefficient tend to zero. Moreover, for certain velocity and pressure approximations by elements of order r, the discrete inf-sup condition holds and a robust error estimate of improved order $r+1/2$ is shown. Numerical results confirm the theoretical convergence results.

MSC 2000. 65N12, 65N30, 65N15

Keywords. Stabilised finite elements, Navier–Stokes equations, inf-sup stable elements, local projection

1 Introduction

The Oseen problem occurs as an important subproblem during the iterative solution of the stationary and instationary Navier–Stokes equations. When solving the Oseen equations by the standard Galerkin finite element method, one is faced with two problems: the problem is generally convection dominated and a compatibility between the approximation spaces for velocity and pressure is necessary.

The streamline-upwind Petrov–Galerkin method (SUPG), introduced in [10], and the pressure-stabilisation Petrov–Galerkin method (PSPG), introduced in [19, 21], allow to treat both problems within a single framework. Moreover, an additional elementwise stabilisation of the divergence constraint, further denoted as grad-div stabilisation, is important for the robustness, see [13, 18, 34]. A fundamental drawback of residual based stabilisation methods is that various terms need to be added to the weak formulation to guarantee the consistency of the method in a strong way. Using inf-sup stable pairs of finite element spaces for approximating velocity and pressure, we can skip the PSPG term to obtain a so-called reduced stabilised scheme [15, 25]. Nevertheless, an additional coupling term between velocity and pressure makes their analysis difficult and the grad-div stabilisation seems to be even more important [11, 15, 25, 30]. In recent years, several approaches have been developed to relax the strong coupling in the SUPG/PSPG type stabilisation and to introduce symmetric stabilising terms, for an overview see [7, 22, 23].

The local projection method has been designed for equal order interpolation and allows a separate stabilisation of velocity, pressure, and incompressibility constraint. It has been introduced for the Stokes equations in [3], extended to the transport problem in [4], and analysed for the Oseen problem with equal order interpolation in [6, 27]. Originally, the local projection technique was proposed as a two-level method where the quantities of interest (e.g. derivatives in streamline direction) are locally projected onto a discontinuos finite element space living on a coarser mesh. Unfortunately, this approach leads to a discretisation stencil being less compact than for the SUPG/PSPG type stabilisation. The general approach given in [14, 27] allows to construct local projection methods with non-increasing discretisation stencil by an appropriate enrichment of standard finite element spaces. In this paper, we will concentrate on the enrichment variant of the local projection method.

The idea of using inf-sup stable finite element pairs is driven by the observation that the flow problem is often part of a coupled flow-transport problem and the mass conservation of the transport equation depends on the properties of the discrete velocity, see [29]. Unfortunately, the property of the velocity field to be discretely divergence-free is disturbed by stabilising the pressure. For inf-sup stable finite element pairs, this pressure stabilisation is not needed and we are only faced with the instability caused by dominated convection.

The main objective of the paper is to analyse convergence properties of the enrichment approach of the local projection stabilisation applied to inf-sup stable discretisations of the Oseen problem. We will consider two different stabilisation terms: one is controlling fluctuations of the derivative in streamline direction and the fluctuation of the divergence separately while the other gives control over fluctuations of the whole gradient. An interesting point is that for inf-sup stable finite element pairs we do not need an H^1 stable interpolation operator with additional orthogonality properties for proving the stability of the discrete problem. This is different for equal order interpolation, see [27, Lemma 2.6]. As a consequence, there is much more flexibility for choosing the approximation and projection spaces. Most of the known inf-sup stable finite element pairs approximate the velocity components by elements of order r and the pressure by elements of order $r-1$ which results in error estimates of order r. Moreover, we propose new inf-sup stable finite element pairs approximating both velocity and pressure by elements of order r. In contrast to the 'classical' equal order interpolation, the velocity components and the pressure are discretised by different elements. We show the discrete inf-sup condition for these finite element spaces und prove an error estimate of order $r+1/2$ uniformly in the viscosity and the reaction coefficient. In case of discontinuous pressure approximations, we add an additional term controlling the jumps of the pressure over inner cell faces. To our knowlegde, estimates of order $r+1/2$ have been known up to now only for discretisations with 'classical' equal order interpolation, see [6, 27].

The plan of this paper is as follows. Sect. 2 states the Oseen problem and its weak formulation. The local projection stabilisation with two different stabilisation terms will be introduced. Sect. 3 considers the solvability and the stability of the stabilised discrete problem. Moreover, the consistency error is analysed. The convergence of the local projection method is investigated in Sect. 4. After considering known inf-sup stable finite element pairs of order r and $r-1$, we study enriched velocity spaces and show error bounds of order r which are uniform in the viscosity also for vanishing reaction coefficient. Moreover, new pairs of inf-sup stable finite element spaces are proposed approximating velocity and pressure by different elements which are both of order r. For these pairs, we will prove a convergence order $r+1/2$. Sect. 5 gives finally some numerical tests which confirm the theoretical results.

Notation. Throughout this paper, C will denote a generic positive constant which is independent of the viscosity parameter and the mesh. Subscripted constants such as C_1 are also independent of the viscosity and the mesh but have a fixed value. We will write shortly $\alpha \lesssim \beta$ if there exists a positive constant C such that $\alpha \leq C\beta$ holds. If $\alpha \lesssim \beta$ and $\beta \lesssim \alpha$, we will write $\alpha \sim \beta$. The Oseen problem will be considered in the domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, which is assumed to be a polygonal or polyhedral domain with boundary $\partial\Omega$. For a measurable d-dimensional subset G of Ω, the usual Sobolev spaces $W^{m,p}(G)$ with norm $\|\cdot\|_{m,p,G}$ and semi-norm $|\cdot|_{m,p,G}$ are used. In the case $p = 2$, we have $H^m(G) = W^{m,2}(G)$ and the index p will be omitted. The L^2 inner product on G is denoted by $(\cdot,\cdot)_G$. Note that the index G will be omitted for $G = \Omega$. This notation of norms, semi-norms, and inner products is also used for the vector-valued and tensor-valued case. For a sufficiently smooth $(d-1)$-dimensional manifold $E \subset \partial G$, the L^2 inner product will be denoted by $\langle\cdot,\cdot\rangle_E$.

2 Oseen problem and its discretisation

2.1 Weak formulation

We consider the Oseen problem

$$\left.\begin{aligned} -\nu\triangle u + \big(b\cdot\nabla\big)u + \sigma u + \nabla p &= f && \text{in } \Omega,\\ \operatorname{div} u &= 0 && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1}$$

where $\nu > 0$ and $\sigma \geq 0$ are constants and $b \in \big(W^{1,\infty}(\Omega)\big)^d$ with $\operatorname{div} b = 0$ is a given velocity field. The Oseen problem can be considered as a linearisation of the steady ($\sigma = 0$) and non-steady ($\sigma > 0$) time-discretised Navier–Stokes equations, respectively.

Let $V := \big(H_0^1(\Omega)\big)^d$ and $Q := L_0^2(\Omega)$. Then, a weak formulation of (1) reads

Find $(u, p) \in V \times Q$ such that

$$\left.\begin{aligned} \nu(\nabla u, \nabla v) + \big((b\cdot\nabla)u, v\big) + \sigma(u, v) - (p, \operatorname{div} v) &= (f, v) \qquad \forall v \in V, \\ (q, \operatorname{div} u) &= 0 \qquad \forall q \in Q. \end{aligned}\right\} \tag{2}$$

Note that $\operatorname{div} b = 0$ implies

$$\big((b\cdot\nabla)u, v\big) = -\big((b\cdot\nabla)v, u\big) \qquad \forall u, v \in V, \tag{3}$$

in particular,

$$\big((b\cdot\nabla)v, v\big) = 0 \qquad \forall v \in V. \tag{4}$$

Thus, applying the Lax–Milgram Lemma in the subspace of divergence-free functions we establish the unique velocity field u. The unique pressure $p \in Q$ such that (u, p) solves (2) follows from the Babuška–Brezzi condition for the pair V/Q, see [16].

2.2 Discrete problem and stabilised formulation

We are given a family $\{\mathcal{T}_h\}$ of shape-regular decompositions of Ω into d-simplices, quadrilaterals, or hexahedra. The diameter of a cell T is denoted by h_T. The mesh parameter h describes the maximum diameter of the cells $T \in \mathcal{T}_h$. The set of all inner element faces $E \not\subset \partial\Omega$ will be denoted by $\mathcal{E}_h$. The diameter of a face $E \in \mathcal{E}_h$ is given by h_E. Each face $E \in \mathcal{E}_h$ is associated with an arbitrary but fixed unit normal vector n_E. Furthermore, let T_E be a fixed element from $\mathcal{T}_h$ such that $E \subset \partial T_E$. Let $T_1, T_2 \in \mathcal{T}_h$ denote two different cells from $\mathcal{T}_h$ which share a common face $E = \partial T_1 \cap \partial T_2$. We define for a piecewise smooth function r_h its jump over the face E as

$$[r_h]_E = (r_h|_{T_1})\big|_E - (r_h|_{T_2})\big|_E$$

where n_E is directed from T_1 into T_2.

Let $Y_h \subset H_0^1(\Omega)$ be a scalar finite element space of continuous, piecewise mapped polynomial functions over $\mathcal{T}_h$. The finite element space V_h for approximating the velocity field is given by $V_h := Y_h^d$. The pressure is discretised using a finite element space $Q_h \subset Q$ of continuous or discontinuous functions with respect to $\mathcal{T}_h$. We will consider inf-sup stable pairs V_h/Q_h throughout this paper.

Assumption 1. *The pair V_h/Q_h fulfils the discrete inf-sup condition, i.e., there exists a positive constant β_0 such that*

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{(q_h, \operatorname{div} v_h)}{|v_h|_1 \, \|q_h\|_0} \geq \beta_0 > 0 \tag{5}$$

uniformly in h.

The standard Galerkin discretisation of (2) in $V_h \times Q_h$ reads

Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$\left.\begin{aligned} \nu(\nabla u_h, \nabla v_h) + \big((b\cdot\nabla)u_h, v_h\big) + \sigma(u_h, v_h) - (p_h, \operatorname{div} v_h) &= (f, v_h) \qquad \forall v_h \in V_h, \\ (q_h, \operatorname{div} u_h) &= 0 \qquad \forall q_h \in Q_h. \end{aligned}\right\} \tag{6}$$

In general, problem (6) lacks stability for $\nu \ll 1$ due to dominating convection. To overcome this problem, we consider the stabilisation by local projection and introduce some additional notations. Let $D_h^i(T)$, $i = 1, 2, 3$, be finite dimensional spaces on the cell $T \in \mathcal{T}_h$ and $\pi_T^i : L^2(T) \to D_h^i(T)$ the associated local L^2 projections into $D_h^i(T)$. The global projection spaces D_h^i are defined by

$$D_h^i := \bigoplus_{T\in\mathcal{T}_h} D_h^i(T), \qquad i = 1, 2, 3.$$

Note that these spaces are discontinuous with respect to $\mathcal{T}_h$. For each $i = 1, 2, 3$, the mapping $\pi_h^i : L^2(\Omega) \to D_h^i$ defined by $(\pi_h^i v)|_T := \pi_T^i(v|_T)$ for all $T \in \mathcal{T}_h$ is the L^2 projection into the projection space D_h^i. Associated with π_h^i, $i = 1, 2, 3$, are the fluctuation operators $\kappa_h^i := id - \pi_h^i$ where $id : L^2(\Omega) \to L^2(\Omega)$ denotes the identity mapping on $L^2(\Omega)$. Note that we allow also $D_h^i = \{0\}$ which means that κ_h^i is the identity. Furthermore, the operators π_h^i and κ_h^i are applied componentwise to vector-valued and tensor-values arguments.

Now we are able to introduce the stabilising terms

$$S_h^a(u, v) := \sum_{T\in\mathcal{T}_h} \Big(\tau_T \big(\kappa_h^1(b \cdot \nabla)u, \kappa_h^1(b \cdot \nabla)v\big)_T + \gamma_T \big(\kappa_h^2 \operatorname{div} u, \kappa_h^2 \operatorname{div} v\big)_T \Big), \tag{7}$$

$$S_h^b(u, v) := \sum_{T\in\mathcal{T}_h} \mu_T \big(\kappa_h^3 \nabla u, \kappa_h^3 \nabla v\big)_T. \tag{8}$$

The term S_h^a introduces control over the fluctuations of the derivatives in streamline direction and over the fluctuations of the divergence separately whereas S_h^b controls the fluctuations of the gradients. We define on the product space $V \times Q$ the bilinear forms

$$\begin{aligned} A_h^i\big((u, p); (v, q)\big) := \nu(\nabla u, \nabla v) + \big((b \cdot \nabla)u, v\big) + \sigma(u, v) \\ + S_h^i(u, v) - (p, \operatorname{div} v) + (q, \operatorname{div} u), \end{aligned} \qquad i \in \{a, b\} \tag{9}$$

and the mesh-dependent norms

$$\big|\!\big|\!\big|(v, q)\big|\!\big|\!\big|_i := \big(\nu|v|_1^2 + \sigma\|v\|_0^2 + (\nu + \sigma)\|q\|_0^2 + S_h^i(v, v)\big)^{1/2}, \qquad i \in \{a, b\}. \tag{10}$$

We will omit the index i in the notations S_h^i, A_h^i, and $|\!|\!|(\cdot, \cdot)|\!|\!|_i$, respectively, if the corresponding statement hold for $i = a$ and $i = b$.

Now, our stabilised discrete problems read

Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$A_h^i\big((u_h, p_h), (v_h, q_h)\big) = (f, v_h) \qquad \forall(v_h, q_h) \in V_h \times Q_h. \tag{11}$$

Existence, uniqueness, and convergence properties of solutions $(u_h, p_h) \in V_h \times Q_h$ will be studied in the next sections.

3 Stability

We start with the solvability of the discrete problem (11).

Lemma 1. *Let* $\max(\nu, \sigma, \tau_T, \gamma_T, \mu_T) \le C$. *Then, there exists a positive constant* β *independent of* ν, σ, *and* h *such that*

$$\inf_{(v_h,q_h)\in V_h\times Q_h} \sup_{(w_h,r_h)\in V_h\times Q_h} \frac{A_h\big((v_h, q_h); (w_h, r_h)\big)}{|\!|\!|(v_h, q_h)|\!|\!| \; |\!|\!|(w_h, r_h)|\!|\!|} \ge \beta > 0 \tag{12}$$

holds true.

Proof. Let (v_h, q_h) be an arbitrary element of $V_h \times Q_h$. We obtain

$$A_h\big((v_h, q_h); (v_h, q_h)\big) = \nu|v_h|_1^2 + \sigma\|v_h\|_0^2 + S_h(v_h, v_h)$$

where property (4) was used.

The discrete inf-sup condition (5) ensures that there exists for all $q_h \in Q_h$ a $z_h = z_h(q_h) \in V_h$ such that

$$(\operatorname{div} z_h, q_h) = -\|q_h\|_0^2, \qquad \|z_h\|_1 \le C_1\|q_h\|_0 \tag{13}$$

holds where C_1 depends only on the inf-sup constant β_0 and the Friedrichs constant for the domain Ω. We get

$$A_h\big((v_h, q_h); (z_h, 0)\big) = \nu(\nabla v_h, \nabla z_h) + \big((b\cdot\nabla)v_h, z_h\big) + \sigma(v_h, z_h) + S_h(v_h, z_h) + \|q_h\|_0^2 \tag{14}$$

by using the first property from (13). We will estimate the first four terms of (14). The first and the third term can be estimated in a standard way. Using the assumption $\nu, \sigma \le C$, we get

$$\begin{aligned}\big|\nu(\nabla v_h, \nabla z_h) + \sigma(v_h, z_h)\big| &\le \nu|v_h|_1\,|z_h|_1 + \sigma\|v_h\|_0\,\|z_h\|_0\\ &\le C\big(\nu|v_h|_1^2 + \sigma\|v_h\|_0^2\big)^{1/2}\|q_h\|_0\\ &\le \frac{\|q_h\|_0^2}{6} + C\big(\nu|v_h|_1^2 + \sigma\|v_h\|_0^2\big)\end{aligned}$$

by using the second property from (13). After an integration by parts, the second term of (14) can be estimates as

$$\big|\big((b\cdot\nabla)v_h, z_h\big)\big| = \big|\big((b\cdot\nabla)z_h, v_h\big)\big| \le C|z_h|_1\|v_h\|_0 \le \frac{\|q_h\|_0^2}{6} + C\|v_h\|_0^2$$

where the boundedness of b and (13) were applied. It remains to consider the stabilising term S_h. Since π_h is the L^2 projection onto the discontinuous finite element space D_h, the corresponding fluctuation operator κ_h is locally L^2 stable. Thus, we get with the boundedness of the user chosen parameters τ_T, γ_T, μ_T, and the boundedness of b in case of considering S_h^a that

$$\big|S_h(v_h, z_h)\big| \le \big(S_h(v_h, v_h)\big)^{1/2}\big(S_h(z_h, z_h)\big)^{1/2} \le C\big(S_h(v_h, v_h)\big)^{1/2}\,|z_h|_1 \le \frac{\|q_h\|_0^2}{6} + CS_h(v_h, v_h).$$

Putting together the above estimates, we obtain

$$A_h\big((v_h, q_h); (z_h, 0)\big) \ge \frac{\|q_h\|_0^2}{2} - C\big(\nu|v_h|_1^2 + \sigma\|v_h\|_0^2 + S_h(v_h, v_h)\big) - C\|v_h\|_0^2. \tag{15}$$

After multiplying this inequality by $2(\nu+\sigma)$ and using the Friedrichs inequality for estimating

$$2(\nu+\sigma)\|v_h\|_0^2 \le C\big(\nu|v_h|_1^2 + \sigma\|v_h\|_0^2\big),$$

we end up with

$$A_h\big((v_h, q_h); 2(\nu+\sigma)(z_h, 0)\big) \ge (\nu+\sigma)\|q_h\|_0^2 - C_2\big(\nu|v_h|_1^2 + \sigma\|v_h\|_0^2 + S_h(v_h, v_h)\big) \tag{16}$$

with a suitable constant C_2. We define for $(v_h, q_h) \in V_h \times Q_h$ the pair $(w_h, r_h) \in V_h \times Q_h$ by

$$(w_h, r_h) := (v_h, q_h) + \frac{2(\nu+\sigma)}{1+C_2}(z_h, 0).$$

Then, we obtain

$$\begin{aligned} A_h\big((v_h,q_h);(w_h,r_h)\big) &\geq \frac{\nu+\sigma}{1+C_2}\|q_h\|_0^2 + \left(1-\frac{C_2}{1+C_2}\right)\big(\nu|v_h|_1^2+\sigma\|v_h\|_0^2+S_h(v_h,v_h)\big) \\ &\geq \frac{1}{1+C_2}|\!|\!|(v_h,q_h)|\!|\!|^2. \end{aligned} \tag{17}$$

It remains to show that $|\!|\!|(w_h,r_h)|\!|\!| \leq C|\!|\!|(v_h,q_h)|\!|\!|$. To this end, we estimate

$$\begin{aligned} |\!|\!|(w_h,r_h)|\!|\!| &\leq |\!|\!|(v_h,q_h)|\!|\!| + \frac{2(\nu+\sigma)}{1+C_2}|\!|\!|(z_h,0)|\!|\!| \leq |\!|\!|(v_h,q_h)|\!|\!| + \frac{2(\nu+\sigma)}{1+C_2}\,C\|z_h\|_1 \\ &\leq |\!|\!|(v_h,q_h)|\!|\!| + C(\nu+\sigma)\|q_h\|_0 \leq C_3|\!|\!|(v_h,q_h)|\!|\!|. \end{aligned}$$

Hence, the stated inf-sup condition holds with the constant $\beta = 1/\big(C_2(1+C_2)\big)$. □

Remark 2. *Lemma 1 gives stability and unique solvability of the discrete problem* (11)*. Note that the mapping $w \mapsto \|\kappa_T w\|_{0,T}$ vanishes on the local projection space $D_h(T)$. Thus, the stability of the discrete problem increases when the dimension of the projection space decreases since the triple norm becomes stronger. In other words, we can control the stability of the discrete problem by choosing appropriate projection spaces.*

Next we will study the consistency error caused by adding the stabilising terms to the standard Galerkin discretisation.

Assumption 2. *The fluctuation operators κ_h^1 and κ_h^3 provide the local approximation properties of order s_1 and s_3, i.e.,*

$$\|\kappa_h^i w\|_{0,T} \leq C h_T^{s_i}|w|_{s_i,T} \qquad \forall w \in H^{s_i}(T),\ \forall T \in \mathcal{T}_h,\ i \in \{1,3\}.$$

Note that assumption A2 is always satisfied for $s_i = 0$ since $(\kappa_h^i w)|_T = w|_T - \pi_T^i(w|_T)$ and π_T^i is the L^2 projection on $D_h^i(T)$. The assumption A2 is fulfilled for $s_i > 0$, for example, if $D_h^i(T) \subset P_{s_i-1}(T)$. This follows from the Bramble–Hilbert lemma.

Lemma 3. *Let $(u,p) \in V \times Q$ and $(u_h,p_h) \in V_h \times Q_h$ be the solutions of* (2) *and* (11)*, respectively. Furthermore, assume that $u \in H^{s+1}(\Omega)^d$ for some integer $s \in [0,r]$. Suppose the fluctuation operator κ_h^1 fulfils assumption A2 with $s_1 = s$ and $b|_T \in W^{s_1,\infty}(T)$ with $\max_T \|b\|_{s_1,\infty,T} \leq C$. Then, we have*

$$\big|A_h^a\big((u-u_h,p-p_h);(v_h,q_h)\big)\big| \leq C\left(\sum_{T\in\mathcal{T}_h}\tau_T\,h_T^{2s_1}\|u\|_{s_1+1,T}^2\right)^{1/2}|\!|\!|(v_h,q_h)|\!|\!|_a \tag{18}$$

for all $(v_h,q_h) \in V_h \times Q_h$. Similarly, if the fluctuation operator κ_h^3 fulfils assumption A2 with $s_3 = s$, the estimate

$$\big|A_h^b\big((u-u_h,p-p_h);(v_h,q_h)\big)\big| \leq C\left(\sum_{T\in\mathcal{T}_h}\mu_T\,h_T^{2s_3}\|u\|_{s_3+1,T}^2\right)^{1/2}|\!|\!|(v_h,q_h)|\!|\!|_b \tag{19}$$

holds for all $(v_h,q_h) \in V_h \times Q_h$.

Proof. Using (11) and

$$A_h\big((u,p);(v_h,q_h)\big) = S_h(u,v_h) + (f,v_h) \quad \forall v_h \in V_h,$$

we see that only $S_h(u,v_h)$ has to be estimated. The definition (7) of the stabilising term gives

$$\big|S_h^a(u,v_h)\big| \le \big(S_h^a(u,u)\big)^{1/2}\big(S_h^a(v_h,v_h)\big)^{1/2} \le \big(S_h^a(u,u)\big)^{1/2}\big\|\!\big|(v_h,q_h)\big|\!\big\|_a.$$

Employing the properties of the fluctuation operator κ_h^1 and the boundedness of $\max_T \|b\|_{s_1,\infty,T}$, we obtain with $\operatorname{div} u = 0$

$$S_h^a(u,u) \le C \sum_{T\in\mathcal{T}_h} h_K^{2s_1}\tau_T\big|(b\cdot\nabla)u\big|^2_{s_1,T} \le C \sum_{T\in\mathcal{T}_h} \tau_T h_T^{2s_1}\|u\|^2_{s_1+1,T}.$$

In a similar way, we can estimate S_h^b where no assumption on b is required. $\square$

4 Convergence

In order to study the convergence order, we characterise the approximation properties of the spaces V_h and Q_h by the existence of corresponding interpolation operators. First, we study the case of usual inf-sup stable pairs V_h/Q_h which approximate the velocity components and the pressure by elements of order r and $r-1$, respectively. In general, the constant in the error estimate is independent of ν and the mesh size h, but depends on σ. Then, we show that under additional assumptions interpolation operators can be constructed which satisfy certain orthogonality properties. These interpolation operators allow us to establish estimates with error constants independent of the data ν, σ, and h. Finally, we turn over to the case of inf-sup stable pairs V_h/Q_h approximating both the velocity components and the pressure by elements of order r. An example for the lowest order case ($r = 1$) with continuous pressure approximation will be the Mini-Element [2, 9]. We give for all considered cases several examples of approximation spaces V_h, Q_h and projection spaces D_h^i, $i = 1,2,3$, such that all assumptions needed in our convergence theory are satisfied.

4.1 Methods of convergence order r in the case $\sigma > 0$

We start considering inf-sup stable pairs V_h/Q_h of finite element spaces of order r and $r-1$, respectively. We assume for this subsection that the polynomial order satisfies $r \ge 2$.

Assumption 3. *There are interpolation operators $j_h : V \cap H^2(\Omega)^d \to V_h$ and $i_h : Q \cap H^2(\Omega) \to Q_h$ with*

$$\|w - j_h w\|_{0,T} + h_T|w - j_h w|_{1,T} \le C h_T^\ell \|w\|_{\ell,T} \quad \forall w \in H^\ell(T)^d,\ 2 \le \ell \le r+1,\ \forall T \in \mathcal{T}_h, \tag{20}$$

$$\|q - i_h q\|_{0,T} + h_T|q - i_h q|_{1,T} \le C h_T^\ell \|q\|_{\ell,T} \quad \forall q \in H^\ell(T),\ 2 \le \ell \le r,\ \forall T \in \mathcal{T}_h. \tag{21}$$

Furthermore, let the pressure interpolation i_h satisfy the orthogonality assumption

$$(q - i_h q, r_h) = 0 \qquad \forall r_h \in D_h^i,\ \forall q \in Q \cap H^2(\Omega),\ i \in \{2,3\}. \tag{22}$$

We consider first the discrete problem with the stabilising term S_h^a defined in (7).

Theorem 4. *Suppose that the spaces V_h, Q_h satisfy A1, A3. The function b satisfies the regularity assumption of Lemma 3. The projection space D_h^1 is chosen such that the associated fluctuation operator κ_h^1 fulfils assumption A2 with some integer $s_1 \in [0, r]$. Let the user chosen parameters satisfy $\gamma_T \sim 1$, $\tau_T \lesssim h_T^{2(r-s_1)}$. Let $(u, p) \in \big(V \cap H^{r+1}(\Omega)^d\big) \times \big(Q \cap H^r(\Omega)\big)$ be the solution of (2) and $(u_h, p_h) \in V_h \times Q_h$ the solution of (11) where the stabilising term S_h^a has been used. Then, there exists for each $\sigma > 0$ a positive constant C_σ independent of ν and h such that the error estimate*

$$|||(u - u_h, p - p_h)|||_a \le C_\sigma \left(\sum_{T \in \mathcal{T}_h} h_T^{2r} \big(\|u\|_{r+1,T}^2 + \|p\|_{r,T}^2 \big) \right)^{1/2} \tag{23}$$

holds true.

Proof. Using Lemma 1, we can estimate

$$\begin{aligned}
|||(j_h u - u_h, i_h p - p_h)|||_a &\le \frac{1}{\beta} \sup_{(w_h, r_h) \in V_h \times Q_h} \frac{A_h^a\big((j_h u - u_h, i_h p - p_h); (w_h, r_h)\big)}{|||(w_h, r_h)|||_a} \\
&\le \frac{1}{\beta} \sup_{(w_h, r_h) \in V_h \times Q_h} \frac{A_h^a\big((u - u_h, p - p_h); (w_h, r_h)\big)}{|||(w_h, r_h)|||_a} \\
&\quad + \frac{1}{\beta} \sup_{(w_h, r_h) \in V_h \times Q_h} \frac{A_h^a\big((j_h u - u, i_h p - p); (w_h, r_h)\big)}{|||(w_h, r_h)|||_a}.
\end{aligned}$$

Applying Lemma 3, we can estimate the consistency error

$$\sup_{(w_h, r_h) \in V_h \times Q_h} \frac{A_h^a\big((u - u_h, p - p_h); (w_h, r_h)\big)}{|||(w_h, r_h)|||_a} \le C \left(\sum_{T \in \mathcal{T}_h} \tau_T\, h_T^{2s_1} \|u\|_{s_1+1,T}^2 \right)^{1/2}.$$

The terms in $A_h^a\big((j_h u - u, i_h p - p); (w_h, r_h)\big)$ will be estimated individually. For the stabilising term S_h^a, we obtain

$$\begin{aligned}
S_h^a(j_h u - u, w_h) &\le \big(S_h^a(j_h u - u, j_h u - u)\big)^{1/2} \big(S_h^a(w_h, w_h)\big)^{1/2} \\
&\le C \left(\sum_{T \in \mathcal{T}_h} (\tau_T + \gamma_T) h_T^{2r} \|u\|_{r+1,T}^2 \right)^{1/2} |||(w_h, r_h)|||_a
\end{aligned}$$

where the L^2 stability of the fluctuation operators κ_h^i, $i \in \{1, 2\}$, the boundedness of b, and the interpolation properties of j_h were used. Furthermore, we get

$$\begin{aligned}
&\big|\nu\big(\nabla(j_h u - u), \nabla w_h\big) + \sigma(j_h u - u, w_h)\big| \\
&\qquad \le \big(\nu |j_h u - u|_1^2 + \sigma \|j_h u - u\|_0^2\big)^{1/2} \big(\nu |w_h|_1^2 + \sigma \|w_h\|_0^2\big)^{1/2} \\
&\qquad \le C \left(\sum_{T \in \mathcal{T}_h} (\nu + \sigma h_T^2) h_T^{2r} \|u\|_{r+1,T}^2 \right)^{1/2} |||(w_h, r_h)|||_a
\end{aligned}$$

by employing the Cauchy–Schwarz inequality and the interpolation properties of j_h.

We consider next the terms which involve the pressure. We have

$$\big(r_h, \operatorname{div}(j_h u - u)\big) \le \|r_h\|_0 \, \|\operatorname{div}(j_h u - u)\|_0$$

$$\leq C \left(\sum_{T\in\mathcal{T}_h} \frac{h_T^{2r}}{\nu+\sigma} \|u\|_{r+1,T}^2 \right)^{1/2} \left|\!\left|\!\left| (w_h, r_h) \right|\!\right|\!\right|_a \tag{24}$$

and, furthermore, by (22)

$$(p - i_h p, \operatorname{div} w_h) = (p - i_h p, \kappa_h^2 \operatorname{div} w_h) \leq C \left(\sum_{T\in\mathcal{T}_h} \gamma_T^{-1} h_T^{2r} \|p\|_{r,T}^2 \right)^{1/2} \left|\!\left|\!\left| (w_h, r_h) \right|\!\right|\!\right|_a. \tag{25}$$

A standard estimate of the convective term is

$$\begin{aligned} \left| \big((b\cdot\nabla)(j_h u - u), w_h \big) \right| &\leq C \left(\sum_{T\in\mathcal{T}_h} h_T^{2r} \|u\|_{r+1,T}^2 \right)^{1/2} \|w_h\|_0 \\ &\leq C \left(\sum_{T\in\mathcal{T}_h} \frac{h_T^{2r}}{\nu+\sigma} \|u\|_{r+1,T}^2 \right)^{1/2} \left|\!\left|\!\left| (w_h, r_h) \right|\!\right|\!\right|_a \end{aligned} \tag{26}$$

where the boundedness of b and in the last step the Friedrichs inequality have been used. Putting all estimates together and using $\tau_T \lesssim h_T^{2(r-s_1)}$, $\gamma_T \sim 1$, $\max(\nu,\sigma) \leq C$, we obtain

$$\left|\!\left|\!\left| (j_h u - u_h, i_h p - p_h) \right|\!\right|\!\right|_a \leq C \left[\sum_{T\in\mathcal{T}_h} h_T^{2r} \left(\|u\|_{r+1,T}^2 + \|p\|_{r,T}^2 \right) \right]^{1/2}. \tag{27}$$

The interpolation properties of j_h, i_h and the asymptotic behaviour of τ_T, γ_T yield

$$\left|\!\left|\!\left| (u - j_h u, p - i_h p) \right|\!\right|\!\right|_a \leq C \left(\sum_{T\in\mathcal{T}_h} h_T^{2r} \left(\|u\|_{r+1,T}^2 + \|p\|_{r,T}^2 \right) \right)^{1/2} \tag{28}$$

The triangle inequality

$$\left|\!\left|\!\left| (u - u_h, p - p_h) \right|\!\right|\!\right|_a \leq \left|\!\left|\!\left| (u - j_h u, p - i_h p) \right|\!\right|\!\right|_a + \left|\!\left|\!\left| (j_h u - u_h, i_h p - p_h) \right|\!\right|\!\right|_a$$

gives the statement of the theorem. □

Next we give — without trying to be complete — examples for approximation spaces V_h, Q_h and projection spaces D_h^1, D_h^2 which satisfy all assumptions of Theorem 4. For a simplex $T \in \mathcal{T}_h$, let $\widehat{T}$ denote the reference unit simplex in $\mathbb{R}^d$. For a quadrilateral/hexahedron T, let $\widehat{T}$ be the unit cube $(-1,1)^d$. The reference mapping $F_T : \widehat{T} \to T$ is affine for simplices and generally non-affine for quadrilaterals and hexahedra. Let $P_k(\widehat{T})$, $k \geq 0$, denote the space of polynomials with degree less than or equal to k while $Q_k(\widehat{T})$, $k \geq 0$, is the space of polynomials of degree less than or equal to k is each variable separately. For convenience, we set $P_{-k}(\widehat{T}) = Q_{-k}(\widehat{T}) = \{0\}$ for all positive integers k. Furthermore, we define for the reference simplex $\widehat{T}$ the spaces

$$P_k^+(\widehat{T}) := P_k(\widehat{T}) + \hat{b} \cdot P_{k-2}(\widehat{T}), \qquad P_k^{++}(\widehat{T}) := P_k(\widehat{T}) + \hat{b} \cdot P_{k-1}(\widehat{T}),$$

where $\hat{b} \in P_{d+1}(\widehat{T})$ denotes a bubble function vanishing on the boundary $\partial\widehat{T}$. We set

$$Q_k^+(\widehat{T}) := Q_k(\widehat{T}) + \hat{b} \cdot \operatorname{span}\{x_i^{k-1}, \ , i = 1, \ldots, d\}$$

on the reference cube $\widehat{T}$ where $\hat{b} \in Q_2(\widehat{T})$ is a bubble function vanishing on $\partial\widehat{T}$. Using these spaces on the reference cells, we will define mapped finite element spaces. Let

$$P_r^{\text{disc}} := \{v \in L^2(\Omega) \,:\, v|_T \circ F_T \in P_r(\widehat{T}),\, \forall T \in \mathcal{T}_h\}, \qquad P_r := P_r^{\text{disc}} \cap H^1(\Omega),$$
$$Q_r^{\text{disc}} := \{v \in L^2(\Omega) \,:\, v|_T \circ F_T \in Q_r(\widehat{T}),\, \forall T \in \mathcal{T}_h\}, \qquad Q_r := Q_r^{\text{disc}} \cap H^1(\Omega).$$

and

$$P_r^{+} := \{v \in H^1(\Omega) \,:\, v|_T \circ F_T \in P_r^{+}(\widehat{T}),\, \forall T \in \mathcal{T}_h\},$$
$$P_r^{++} := \{v \in H^1(\Omega) \,:\, v|_T \circ F_T \in P_r^{++}(\widehat{T}),\, \forall T \in \mathcal{T}_h\},$$
$$Q_r^{+} := \{v \in H^1(\Omega) \,:\, v|_T \circ F_T \in Q_r^{+}(\widehat{T}),\, \forall T \in \mathcal{T}_h\}.$$

As usual, we will write shortly $V_h = Q_k$ and $Q_h = P_k$ instead of $V_h = \big(Q_k \cap H_0^1(\Omega)\big)^d$ and $Q_h = P_k \cap L_0^2(\Omega)$. The mapped spaces P_r^{disc} are used later also on quadrilaterals and hexahedra for pressure spaces and for projection spaces. Note that these spaces do not admit the usual approximation properties on arbitrary families of meshes. However, the usual approximation properties known for unmapped finite elements still hold on families of successively refined meshes which are often used in practise. For details, we refer to [1, 24, 26].

Concerning the construction of pressure interpolations satisfying (22), the following lemmata will be useful. We start with continuous pressure approximations and introduce the notations

$$Q_h(T) := \{q_h|_T \,:\, q_h \in Q_h + \operatorname{span}(1)\}, \qquad \widetilde{Q}_h(T) := \{q_h \,:\, b_T \cdot q_h \in Q_h(T)\}$$

where b_T denotes the mapped bubble function of lowest polynomial degree, i.e. $b_T \in P_{d+1}(T)$ for simplices in $\mathbb{R}^d$ and $b_T \in Q_2(T)$ for quadrilaterals/hexahedra, respectively.

Lemma 5. *Suppose there exists an interpolation operator $i_h^* : Q \cap H^2(\Omega) \to Q_h \subset H^1(\Omega)$ satisfying the approximation property* (21). *Moreover, let the projection spaces D_h^i, $i \in \{2,3\}$, satisfy $D_h^i(T) \subset \widetilde{Q}_h(T)$ for all $T \in \mathcal{T}_h$. Then, there exists an interpolation operator $i_h : Q \cap H^2(\Omega) \to Q_h$ satisfying the approximation property* (21) *and the orthogonality condition* (22).

Proof. We modify i_h^* by setting $i_h q := i_h^* q + d_h(q)$ with $d_h(q)|_T := b_T \cdot \tilde{d}_T$ where $\tilde{d}_T \in \widetilde{Q}_h(T)$ is locally defined by

$$(d_h(q), r_h)_T = (b_T \cdot \tilde{d}_T, r_h)_T = (q - i_h^* q, r_h)_T \qquad \forall r_h \in \widetilde{Q}_h(T),\, \forall T \in \mathcal{T}_h. \tag{29}$$

The unique solution $\tilde{d}_T \in \widetilde{Q}_h(T)$ follows from the observation that $(d, r) \mapsto (b_T \cdot d, r)_T$ is a weighted L^2 inner product on $\widetilde{Q}_h(T)$. Since the bubble function b_T vanishes on the boundary ∂T of each cell, the interpolant $i_h q := i_h^* q + d_h(q)$ belongs to $Q_h \subset Q \cap H^1(\Omega)$ and preserves locally polynomials of degree less than or equal to r. The Bramble–Hilbert lemma gives (21) for simplicial finite elements. In case of quadrilateral and hexahedral finite elements, we restrict to sucessively refined meshes and use the results of [1, 24, 26]. Furthermore, we conclude from (29) that the error $q - i_h q$ is perpendicular to the projection spaces D_h^i, $i \in \{2,3\}$. Hence, the orthogonality property (22) holds. $\square$

The version related to discontinuous pressure approximations reads as follows.

Lemma 6. *Let $Q_h = P_{r-1}^{disc}$ or $Q_h = Q_{r-1}^{disc}$. Suppose $D_h^2 \subset Q_h + \operatorname{span}(1)$ and $D_h^3 \subset Q_h + \operatorname{span}(1)$, respectively. Then, the L^2 projection $i_h : L^2(\Omega) \to Q_h$ satisfies the approximation property* (21) *and the orthogonality condition* (22). *Further,* $\operatorname{div} V_h \subset Q_h + \operatorname{span}(1)$ *yields $(q - i_h q, \operatorname{div} w_h) = 0$ for all $w_h \in V_h$ independent of the choice of D_h^2 and D_h^3, respectively.*

Proof. The discontinuity of the pressure space Q_h implies that the L^2 projection can be localised. Hence, the approximation property (21) follows from the Bramble–Hilbert lemma for simplicial finite elements in the usual way. In case of quadrilateral and hexahedral finite elements, we restrict to sucessively refined meshes and use the results of [1, 24, 26]. Furthermore, we have

$$(q - i_h q, r_h) = 0 \qquad \forall r_h \in Q_h + \mathrm{span}(1).$$

Thus, for $D_h^2 \subset Q_h + \mathrm{span}(1)$ and $D_h^3 \subset Q_h + \mathrm{span}(1)$, respectively, we conclude (22). In case of $\operatorname{div} V_h \subset Q_h + \mathrm{span}(1)$, we can set $r_h = \operatorname{div} w_h$ and get $(q - i_h q, \operatorname{div} w_h) = 0$ for all $w_h \in V_h$ independent of the choice of D_h^2 and D_h^3, respectively. □

We turn now over to concrete examples and start with continuous pressure approximations, see Table 1. The assumptions A1 and A3 for the Taylor-Hood families on simplices and quadrilaterals/hexahedra are clearly satisfied. Indeed, the additional orthogonality assumption (22) for the pressure interpolation can be fulfilled by using a projection space D_h^2 being small enough. The choices $P_{-1}^{\mathrm{disc}} = Q_{-1}^{\mathrm{disc}} = \{0\}$ always satisfy (22). According to Lemma 5, the largest possible projection space D_h^2 such that (22) still holds is given by the bubble part $\widetilde{Q}_h(T)$ of P_{r-1} and Q_{r-1}, respectively. The bubble parts correspond to $P_{r-d-2}^{\mathrm{disc}}$ for simplicial elements and to Q_{r-3}^{disc} for quadrilateral/hexahedral elements. Finally, we mention that the fluctuation operator κ_h^1 satisfies assumption A2 with $s_1 = s$ for all choices of D_h^1 given in Table 1.

Table 1: Taylor-Hood families with stabilisation term S_h^a.

V_h	Q_h	D_h^1	D_h^2	τ_T	γ_T	r	s	t	$\lVert\!\vert \cdot \vert\!\rVert_a$
P_r	P_{r-1}	P_{s-1}^{disc}	P_{t-1}^{disc}	$\lesssim h_T^{2(r-s)}$	~ 1	$r \geq 2$	$s \leq r$	$t \leq r-d-1$	$\mathcal{O}(h^r)$
Q_r	Q_{r-1}	Q_{s-1}^{disc}	Q_{t-1}^{disc}	$\lesssim h_T^{2(r-s)}$	~ 1	$r \geq 2$	$s \leq r$	$t \leq r-2$	$\mathcal{O}(h^r)$

Remark 7. *A careful inspection of the proof of Theorem 4 shows that we cannot replace the stabilisation term S_h^a by S_h^b for continuous pressure approximations. Indeed, an estimate like* (25) *for simplicial elements would require $D_h^3 \subset P_{r-d-2}^{disc}$ and the lower bound $\mu_T \geq C > 0$. Then, for getting a consistency error $\mathcal{O}(h^r)$ in Lemma 3, the fluctuation operator κ_h^3 should satisfy assumption A2 with $s_3 = r$ which means that $D_h^3 \supset P_{r-1}^{disc}$ in contrast to $D_h^3 \subset P_{r-d-2}^{disc}$. A similar argument is true for quadrilateral/hexahedral elements.*

Let us now discuss examples of inf-sup stable finite element pairs V_h/Q_h with discontinuous pressure approximations. The inf-sup stability and the approximation properties of the elements given in Table 2 follows from [12, 16, 28]. The additional orthogonality assumption in A3 is

Table 2: Families with discontinuous pressure approximations and S_h^a.

V_h	Q_h	D_h^1	D_h^2	τ_T	γ_T	r	s	t	$\lVert\!\vert \cdot \vert\!\rVert_a$
P_r^+	P_{r-1}^{disc}	P_{s-1}^{disc}	P_{t-1}^{disc}	$\lesssim h_T^{2(r-s)}$	~ 1	$r \geq 2$	$s \leq r$	$t \leq r$	$\mathcal{O}(h^r)$
Q_r	P_{r-1}^{disc}	P_{s-1}^{disc}	P_{t-1}^{disc}	$\lesssim h_T^{2(r-s)}$	~ 1	$r \geq 2$	$s \leq r$	$t \leq r$	$\mathcal{O}(h^r)$

satisfied for $D_h^2 \subset Q_h + \mathrm{span}(1)$ when using the local L^2 projection as pressure interpolation (see Lemma 6).

Remark 8. *If $D_h^2 \subset Q_h + \mathrm{span}(1)$ then the L^2 projection of a discretely divergence-free function w_h is zero due to*

$$(\pi_h^2 \operatorname{div} w_h, r_h) = (\operatorname{div} w_h, r_h) = 0 \qquad \forall r_h \in D_h^2.$$

This happens for all families listed in Table 2. As a consequence, the discrete solution u_h does not depend on the choice of the projection space. However, the algebraic properties do depend on the choice of D_h^2.

Compared to continuous pressure approximations, the orthogonality property (22) of the pressure interpolation i_h hold for discontinuous pressure approximation with respect to a larger space. This allows us to replace the stabilisation term S_h^a by S_h^b and similar results can be formulated. In this way, we can relax the smoothness conditions on b but lose the flexibility for choosing the projection spaces D_h^1 and D_h^2.

Theorem 9. *Suppose that the spaces V_h, Q_h satisfy A1, A3. The projection space D_h^3 is chosen such that the associated fluctuation operator κ_h^3 fulfils assumption A2 with $s_3 = r$. Let the user chosen parameter satisfy $\mu_T \sim 1$. Let $(u, p) \in \big(V \cap H^{r+1}(\Omega)^d\big) \times \big(Q \cap H^r(\Omega)\big)$ be the solution of (2) and $(u_h, p_h) \in V_h \times Q_h$ the solution of (11) where the stabilising term S_h^b has been used. Then, there exists for each $\sigma > 0$ a positive constant C_σ independent of ν and h such that the error estimate*

$$\left|\!\left|\!\left| (u - u_h, p - p_h) \right|\!\right|\!\right|_b \le C_\sigma \left(\sum_{T \in \mathcal{T}_h} h_T^{2r} \big(\|u\|_{r+1,T}^2 + \|p\|_{r,T}^2 \big) \right)^{1/2} \tag{30}$$

holds true.

Proof. Most of the terms can be estimated as in the proof of Theorem 4. We discuss only the term which needs a different handling. Since $\|\kappa_h^3 \operatorname{div} w_h\|_{0,T} \le C \|\kappa_h^3 \nabla w_h\|_{0,T}$, we have

$$(p - i_h p, \operatorname{div} w_h) = (p - i_h p, \kappa_h^3 \operatorname{div} w_h) \le C \left(\sum_{T \in \mathcal{T}_h} \mu_T^{-1} h_T^{2r+2} \|p\|_{r+1,T}^2 \right)^{1/2} \left|\!\left|\!\left| (w_h, r_h) \right|\!\right|\!\right|_b.$$

Now follow the lines of proof of Theorem 4. Note the the stabilisation term S_h^b can be estimated by using the Cauchy–Schwarz inequality. ☐

Possible classes of methods satisfying all assumptions of Theorem 9 are listed in Table 3. As above, the spaces V_h, Q_h satisfy A1 and A3 since $D_h^3 \subset Q_h + \mathrm{span}(1)$. In order to satisfy A2 the projection space has to be large enough, in particular, $D_h^3 \supset P_{r-1}^{\mathrm{disc}}$. Therefore, we set $D_h^3 := P_{r-1}^{\mathrm{disc}}$.

Remark 10. *The enrichment of P_r in the first rows of Tables 2 and 3 is needed only to guarantee the inf-sup condition on arbitrary shape regular meshes. If we restrict to families of meshes generated by dividing a d-simplex into $(d+1)$ simplices by connecting the barycentre with the vertices, the inf-sup condition holds for $r \ge d$, see [31, 32, 35]. Thus, we can replace in this case P_r^+ by P_r. The pair P_r/P_{r-1} is known as Scott–Vogelius element, see [32].*

Table 3: Families with discontinuous pressure approximations and S_h^b.

V_h	Q_h	D_h^3	μ_T	r	$\lVert\!\vert \cdot \vert\!\rVert_b$
P_r^+	P_{r-1}^{disc}	P_{r-1}^{disc}	~ 1	$r \geq 2$	$\mathcal{O}(h^r)$
Q_r	P_{r-1}^{disc}	P_{r-1}^{disc}	~ 1	$r \geq 2$	$\mathcal{O}(h^r)$

4.2 Methods of convergence order r in the case $\sigma \geq 0$.

A careful inspection of the proof of Theorem 4 shows that the error constant in (23) is for $\sigma = 0$ no longer uniformly bounded for $\nu \to 0$ due to the estimates (24) and (26). We will see in the following that we can get error estimates which hold uniformly for all $\sigma \geq 0$ by choosing a special interpolant $j_h : V \cap H^2(\Omega)^d \to V_h$. The polynomial degree r in this subsection is assumed to fulfil $r \geq 2$.

In order to handle both continuous and discontinuous pressure approximations, we modify our discrete problem by introducing an additional stabilising term J_h which is given by

$$J_h(p,q) := \sum_{E \in \mathcal{E}_h} \alpha_E \big\langle [p]_E, [q]_E \big\rangle_E \tag{31}$$

where α_E are user-chosen parameters. We extend the definition of the bilinear form A_h^i and of the mesh-dependent norm $\lVert\!\vert \cdot \vert\!\rVert_i$ to $i = c$ by

$$A_h^c\big((u,p);(v,q)\big) := A_h^a\big((u,p);(v,q)\big) + J_h(p,q), \tag{32}$$

$$\lVert\!\vert (v,q) \vert\!\rVert_c := \Big(\lVert\!\vert (v,q) \vert\!\rVert_a^2 + J_h(q,q) \Big)^{1/2}. \tag{33}$$

Note that this modification does not cause an additional consistency error since we have for smooth solutions $p \in H^1(\Omega)$ that $[p]_E = 0$ on all $E \in \mathcal{E}_h$ where $\mathcal{E}_h$ is the set of all inner faces.

We start with a quasi-local interpolation operator preserving the discrete divergence [17] and modify it such that the interpolation error becomes orthogonal to the projection space [27].

Assumption 4. *There exists an operator $j_h^* : V \to V_h$ satisfying*

$$(q_h, \operatorname{div}(w - j_h^* w)) = 0 \qquad \forall w \in V,\ \forall q_h \in Q_h, \tag{34}$$

$$|v - j_h^* v|_{m,T} \leq C\, h_T^{\ell - m} |v|_{\ell,\omega(T)} \qquad \forall v \in V \cap H^\ell(\Omega)^d,\ \forall T \in \mathcal{T}_h, \tag{35}$$

for $0 \leq m \leq 1$, $1 \leq \ell \leq r+1$, where $\omega(T)$ denotes a local neighbourhood of T. Moreover, let the local inf-sup condition

$$\exists \beta_1 > 0\ \forall h > 0\ \forall T \in \mathcal{T}_h : \qquad \inf_{q_h \in D_h^1(T)} \sup_{v_h \in Y_h(T)} \frac{(v_h, q_h)_T}{\|v_h\|_{0,T}\, \|q_h\|_{0,T}} \geq \beta_1 > 0 \tag{36}$$

be satisfied where $Y_h(T) := \{v_h|_T \ :\ v_h \in Y_h,\ v_h = 0 \text{ on } \Omega \backslash T\}$ is the local bubble part of the scalar finite element space Y_h.

Remark 11. *The existence of quasi-local interpolation operators j_h^* satisfying* (34) *and* (35) *has been established for a wide family of pairs V_h/Q_h in [17]. Concerning* (36)*, we mention that $Y_h(T)$ — compared to $D_h^1(T)$ — has to be rich enough. In particular, a necessary requirement is* $\dim Y_h(T) \geq \dim D_h^1(T)$. *Examples of spaces Y_h, D_h^1 satisfying* (36) *have been given in [14, 27].*

Lemma 12. *Let A4 be satisfied. Then, there exists an interpolation operator $j_h : V \to V_h$ satisfying the following orthogonality and approximation properties:*

$$(w - j_h w, q_h) = 0 \qquad \forall q_h \in (D_h^1)^d,\ \forall w \in V, \tag{37}$$
$$|v - j_h v|_{m,T} \le C\, h_T^{\ell-m} |v|_{\ell,\omega(T)} \qquad \forall v \in V \cap H^\ell(\Omega)^d,\ \forall T \in \mathcal{T}_h \tag{38}$$

for $0 \le m \le 1$, $1 \le \ell \le r+1$. If additionally $\nabla Q_h \subset (D_h^1)^d$ then the estimate

$$|(r_h, \operatorname{div}(w - j_h w))| \le C \left(\sum_{E\in\mathcal{E}_h} \alpha_E^{-1} h_{T_E}^{2r+1} |w|^2_{r+1,\omega(T_E)} \right)^{1/2} \Big(J_h(r_h, r_h) \Big)^{1/2} \tag{39}$$

holds true for all $r_h \in Q_h$ and for all $w \in V \cap H^{r+1}(\Omega)^d$.

Proof. It has been shown in [27, Theorem 2.2] that there exists under the assumptions (35) and (36) an interpolation operator j_h satisfying (37) and (38). The operator has been constructed by setting $j_h w := j_h^* w + z_h(w)$ where $z_h(w)|_T \in V_h(T) := Y_h(T)^d$ is locally defined by

$$(z_h(w), q_h)_T = (w - j_h^* w, q_h)_T \qquad \forall q_h \in \big(D_h^1(T)\big)^d$$

which guarantees (37). Further, the local bound

$$\|z_h(w)\|_{0,T} \le \frac{1}{\beta_1} \|w - j_h^* w\|_{0,T}$$

has been proven from which (38) follows by using (35) and an inverse inequality. It remains to show (39). From the representation $j_h w = j_h^* w + z_h(w)$, we get for $r_h \in Q_h$ and $w \in V \cap H^{r+1}(\Omega)^d$

$$\begin{aligned}(r_h, \operatorname{div}(w - j_h w)) &= -(r_h, \operatorname{div} z_h(w)) = \sum_{T\in\mathcal{T}_h} (\nabla r_h, z_h(w))_T = \sum_{T\in\mathcal{T}_h} (\nabla r_h, w - j_h^* w)_T \\ &= -\sum_{T\in\mathcal{T}_h} (r_h, \operatorname{div}(w - j_h^* w))_T + \sum_{E\in\mathcal{E}_h} \big\langle [r_h]_E, (w - j_h^* w)\cdot n_E \big\rangle_E.\end{aligned}$$

Here, we have used (34), $z_h(w) = 0$ on ∂T for all $T \in \mathcal{T}_h$, $\nabla(r_h|_T) \in \big(D_h^1(T)\big)^d$ and $w - j_h^* w = 0$ on $\partial\Omega$. The first term on the right hand side vanishes due to (34). The estimate for the interpolation error on cell boundaries $E \in \mathcal{E}_h$ follows from the scaled trace inequality

$$\|v\|_{0,E} \le C \left(h_{T_E}^{-1/2} \|v\|_{0,T_E} + h_{T_E}^{1/2} |v|_{1,T_E} \right) \qquad \forall v \in H^1(T_E)$$

which yields

$$\|w - j_h^* w\|_{0,E} \le C \left(h_{T_E}^{-1/2} h_{T_E}^{r+1} |w|_{r+1,\omega(T_E)} + h_{T_E}^{1/2} h_{T_E}^{r} |w|_{r+1,\omega(T_E)} \right) \le C h_{T_E}^{r+1/2} |w|_{r+1,\omega(T_E)}$$

by applying (35). The estimate (39) follows now by using Cauchy–Schwarz inequality. $\square$

Remark 13. *Estimate* (39) *implies that the special interpolant j_h preserves the discrete divergence for continuous pressure approximations since $J_h(r_h, r_h) = 0$ for $r_h \in H^1(\Omega)$. A simple example for spaces satisfying A4 is the "extended Mini element family" which is given by $V_h = P_r^{++}$, $Q_h = P_r$, and $D_h^1 = P_r^{disc}$.*

Theorem 14. *Suppose that the spaces V_h, Q_h satisfy A1, A3, A4. The function b satisfies the regularity assumption of Lemma 3. The projection space D_h^1 is chosen such that the associated fluctuation operator κ_h^1 fulfils assumption A2 with $s_1 \in \{r-1, r\}$. Let the user chosen parameters satisfy $\gamma_T \sim 1$, $\alpha_E \sim h_E$. We assume $\tau_T \sim h_T^2$ for $s_1 = r-1$ and $h_T^2 \lesssim \tau_T \lesssim 1$ for $s_1 = r$. Let $(u,p) \in \big(V \cap H^{r+1}(\Omega)^d\big) \times \big(Q \cap H^r(\Omega)\big)$ be the solution of (2) and $(u_h, p_h) \in V_h \times Q_h$ the solution of (11) with $i = c$ where the stabilising terms (7) and (31) have been used. Then, there exists a positive constant C independent of ν, σ, and h such that the error estimate*

$$\left\|\!\left|(u-u_h, p-p_h)\right|\!\right\|_c \le C \left(\sum_{T\in\mathcal{T}_h} h_T^{2r} \big(\|u\|_{r+1,T}^2 + \|p\|_{r,T}^2\big) \right)^{1/2} \tag{40}$$

holds true.

Proof. First, a look into the proof of Lemma 1 shows that it still holds for A_h^c since

$$A_h^c\big((v_h,q_h);(z_h,0)\big) = A_h^a\big((v_h,q_h);(z_h,0)\big).$$

Now we follow the lines in the proof of Theorem 4 and discuss only the necessary modifications. The estimation of the additional term which appears only for discontinuous pressure approximations is standard:

$$\begin{aligned} |J_h(i_h p - p, r_h)| &\le \left(J_h(i_h p - p, i_h p - p)\right)^{1/2} \left(J_h(r_h,r_h)\right)^{1/2} \\ &\le C \left(\sum_{T\in\mathcal{T}_h} h_T^{2r} \|p\|_{r,T}^2 \right)^{1/2} \left\|\!\left|(w_h, r_h)\right|\!\right\|_c \end{aligned}$$

where we used $h_E \sim h_T$ for $E \subset \partial T$ and the same ideas as in the proof of Lemma 12 to estimate the interpolation error on cell boundaries. It remains to replace the estimates (24) and (26). Using (39) and $\alpha_E \sim h_E$, we get

$$|(r_h, \operatorname{div}(u - j_h u))| \le C \left(\sum_{T\in\mathcal{T}_h} h_T^{2r} \|u\|_{r+1,T}^2 \right)^{1/2} \left\|\!\left|(w_h, r_h)\right|\!\right\|_c$$

for all $\sigma \ge 0$. Since the velocity interpolant satisfies the additional orthogonality (37) with respect to D_h^1, we can alternatively estimate the convection term after an integration by parts as follows

$$\begin{aligned} \big|\big((b\cdot\nabla)(j_h u - u), w_h\big)\big| &= \big|\big(j_h u - u, (b\cdot\nabla) w_h\big)\big| = \big|\big(j_h u - u, \kappa_h^1 (b\cdot\nabla) w_h\big)\big| \\ &\le C \left(\sum_{T\in\mathcal{T}_h} h_T^{2(r+1)} \tau_T^{-1} \|u\|_{r+1,T}^2 \right)^{1/2} \left\|\!\left|(w_h, r_h)\right|\!\right\|_c . \end{aligned} \tag{41}$$

The statement follows with $\tau_T^{-1} \lesssim h_T^{-2}$. □

Examples satisfying all assumptions of Theorem 14 are given in Tables 4 and 5. Since the pairs P_r/P_{r-1} and $Q_r/P_{r-1}^{\text{disc}}$ satisfy the inf-sup condition A1, the enriched versions of the pairs given in Tables 4 and 5 satisfy A1 too. The enrichments have been chosen large enough to satisfy (36) of A4 which guarantees the orthogonality property of the velocity interpolation (37) for the given projection space D_h^1. See [14, 27] for a proof of (36). Finally, the largest possible projection space D_h^2 for continuous pressure approximations results from the bubble part of the pressure space Q_h.

Table 4: Families with continuous pressure approximations and J_h^a.

V_h	Q_h	D_h^1	D_h^2	τ_T	γ_T	r	t	$\lVert\!\vert \cdot \vert\!\rVert_a$
P_r^+	P_{r-1}	P_{r-2}^{disc}	P_{t-1}^{disc}	$\sim h_T^2$	~ 1	$r \geq 2$	$t \leq r-1-d$	$\mathcal{O}(h^r)$
P_r^{++}	P_{r-1}	P_{r-1}^{disc}	P_{t-1}^{disc}	$h_T^2 \lesssim \tau_T \lesssim 1$	~ 1	$r \geq 2$	$t \leq r-1-d$	$\mathcal{O}(h^r)$
Q_r	Q_{r-1}	Q_{r-2}^{disc}	Q_{t-1}^{disc}	$\sim h_T^2$	~ 1	$r \geq 2$	$t \leq r-2$	$\mathcal{O}(h^r)$

Table 5: Families with discontinuous pressure approximations and J_h^c.

V_h	Q_h	D_h^1	D_h^2	τ_T	γ_T	α_E	r	t	$\lVert\!\vert \cdot \vert\!\rVert_c$
Q_r	P_{r-1}^{disc}	Q_{r-2}^{disc}	P_{t-1}^{disc}	$\sim h_T^2$	~ 1	$\sim h_E$	$r \geq 2$	$t \leq r$	$\mathcal{O}(h^r)$
Q_r^+	P_{r-1}^{disc}	P_{r-1}^{disc}	P_{t-1}^{disc}	$h_T^2 \lesssim \tau_T \lesssim 1$	~ 1	$\sim h_E$	$r \geq 2$	$t \leq r$	$\mathcal{O}(h^r)$

Remark 15. *One could also think to replace S_h^a by S_h^b but the improved estimate of the convection term is more tricky, since — in general — we do not have $\|\kappa_h(b\cdot\nabla)w_h\|_{0,T} \leq C\|\kappa\nabla w_h\|_{0,T}$. Let $\overline{b}$ denote the piecewise constant approximation of b. Then, we have*

$$\begin{aligned}\|\kappa_h(b\cdot\nabla)w_h\|_{0,T} &\leq \|\kappa_h((b-\overline{b})\cdot\nabla)w_h\|_{0,T} + \|\kappa_h(\overline{b}\cdot\nabla)w_h\|_{0,T}\\ &\leq Ch_T|b|_{1,\infty,T}\|\nabla w_h\|_{0,T} + \|b\|_{0,\infty,T}\|\kappa_h\nabla w_h\|_{0,T}\\ &\leq C(\|w_h\|_{0,T} + \|\kappa_h(\nabla w_h)\|_{0,T})\end{aligned}$$

from which

$$|(j_h u - u, \kappa_h(b\cdot\nabla)w_h)| \leq C\left(\sum_{T\in\mathcal{T}_h}\Big((\nu+\sigma)^{-1}h_T^2 + \mu_T^{-1}h_T^2\Big)h_T^{2r}\|u\|_{r+1,T}^2\right)^{1/2} \lVert\!\vert (w_h, r_h) \vert\!\rVert_b$$

follows. Since this estimate is for $\sigma = 0$ not uniformly in $\nu > 0$, we skip this option here.

4.3 Methods of convergence order $r + 1/2$

For equal order interpolations where $V_h = (Y_h \cap H_0^1(\Omega))^d$ and $Q_h = Y_h \cap Q$, error estimates of order $\mathcal{O}((\nu^{1/2} + h^{1/2})h^r)$ have been established in [6, 27]. Unfortunately, these pairs of finite elements are not inf-sup stable and an additional pressure stabilisation, called pressure stabilised Petrov–Galerkin (PSPG) [33], becomes necessary. However, a careful investigation of the proof of Theorem 4 shows that the critical term limiting the convergence order to r is $(p - i_h p, \operatorname{div} w_h)$ estimated in (25). Thus, an improved approximation of the pressure seems to be needed for getting an improved error estimate. Here, we consider inf-sup stable pairs V_h/Q_h of finite element space approximating velocity and pressure by elements of order r.

In order to get error bounds uniformly with respect to $\nu > 0$ for all $\sigma \geq 0$, we restrict ourselves to the case of the stabilising term S_h^c (cf. Remark 15). We consider the two families of spaces given in Table 6.

We show first that the pairs V_h/Q_h given in Table 6 are inf-sup stable, i.e., assumption A1 is satisfied.

Table 6: Families of order $r+1/2$ and J_h^c.

V_h	Q_h	D_h^1	D_h^2	τ_T	γ_T	α_E	r	t	$\lVert\!\vert \cdot \vert\!\rVert_c$
P_r^{++}	P_r	P_{r-1}^{disc}	P_{t-1}^{disc}	$\sim h_T$	$\sim h_T$	$=0$	$r \geq 1$	$t \leq r-d$	$\mathcal{O}(h^{r+1/2})$
Q_r^+	P_r^{disc}	P_{r-1}^{disc}	P_{t-1}^{disc}	$\sim h_T$	$\sim h_T$	~ 1	$r \geq 2$	$t \leq r+1$	$\mathcal{O}(h^{r+1/2})$

Lemma 16. *The discrete inf-sup condition* (5) *is satisfied for the pairs* $V_h/Q_h = P_r^{++}/P_r$ *and* $V_h/Q_h = Q_r^+/P_r^{disc}$.

Proof. The proof for the extended Mini element family $V_h/Q_h = P_r^{++}/P_r$ is based on the construction of a Fortin operator $\Pi_h : V \to V_h$ satisfying

$$\big(r_h, \operatorname{div}(\Pi_h v - v)\big) = 0 \qquad \forall r_h \in Q_h,\ v \in V, \tag{42}$$

$$\|\Pi_h v\|_1 \leq C\|v\|_1 \qquad \forall v \in V. \tag{43}$$

This is equivalent to establish assumption A1, see [16, Chapter II, Lemma 1.1]. Since the bubble part of $(\Pi_h v)|_T$ belongs to $\big(b_T \cdot P_{r-1}(T)\big)^d$, we can fix the bubble part of $(\Pi_h v)|_T$ by

$$\big((\Pi_h v - v), q_h\big)_T = 0 \qquad \forall q_h \in \big(P_{r-1}(T)\big)^d.$$

An integration by parts shows that (42) is fulfilled due to $(\nabla r_h)|_T \in \big(P_{r-1}(T)\big)^d$. The remaining degrees of freedom for Π_h are regularised nodal functionals living at the boundary of each cell. They guarantee the continuity across the cell interfaces and ensure that the domain of definition of Π_h becomes V. The detailed proof for $r=1$ can be found in [16].

We observe for the pair Q_r^+/P_r^{disc}, $r \geq 2$, that $Q_2 \subset Q_r^+$. Since the pair Q_2/Q_0 is inf-sup stable, we can apply the technique by Boland/Nicolaides [5, 16]. Hence, we have to show on each cell $T \in \mathcal{T}_h$ only a local inf-sup condition between the bubble part of $Q_r^+(T)$ and $P_r(T) \cap L_0^2(T)$. The essential point of the proof is the inclusion $b_T \cdot P_{r-1}(T) \subset Q_r^+(T)$ which is satisfied by construction of the enriched space Q_r^+. Then, we get the inf-sup condition for the pair Q_r^+/P_r^{disc} by following the lines of the proof of [28, Theorem 8]. ☐

Theorem 17. *Suppose that the spaces* V_h, Q_h, D_h^1, D_h^2 *and the parameters* τ_T, γ_T, α_E *are chosen as in Table 6. Assume further that the function* b *satisfies the regularity assumption of Lemma 3. Let* $(u,p) \in \big(V \cap H^{r+1}(\Omega)^d\big) \times \big(Q \cap H^{r+1}(\Omega)\big)$ *be the solution of* (2) *and* $(u_h, p_h) \in V_h \times Q_h$ *the solution of* (11) *with* $i = c$ *where the stabilising terms* S_h^a *and* J_h *have been used. Then, there exists a positive constant* C *independent of* ν, σ, *and* h *such that the error estimate*

$$\lVert\!\vert (u-u_h, p-p_h) \vert\!\rVert_c \leq C \left(\sum_{T \in \mathcal{T}_h} h_T^{2r+1} \big(\|u\|_{r+1,T}^2 + \|p\|_{r+1,T}^2 \big) \right)^{1/2} \tag{44}$$

holds true.

Proof. Assumption A1 follows from Lemma 16. Furthermore, the choice $D_h^1 = P_{r-1}^{\text{disc}}$ guarantees assumption A2 with $s_1 = r$ and the consistency error becomes of order $r+1/2$ for $\tau_T \lesssim h_T$.

Assumption A4 is satisfied for the pairs $P_r^{++}/P_{r-1}^{\text{disc}}$ and $Q_r^+/P_{r-1}^{\text{disc}}$, as shown in [27]. Therefore, we can use the improved estimate (41) of the convection term. Moreover, we note that the upper bounds for the sizes of the projection spaces D_h^2 result from the size of the bubble parts of the pressure spaces P_r on simplices and P_r^{disc} on quadrilaterals and hexahedra, respectively. The choice $D_h^2 = P_{t-1}^{\text{disc}}$ allows us to apply Lemmata 5 and 6 such that assumption A3 is satisfied.

Due to the choice of the pressure space to be either P_r or P_r^{disc}, we have the following better estimate for interpolation error in the pressure space

$$\|q - i_h q\|_{0,T} + h_T |q - i_h q|_{1,T} \le C h_T^\ell \|q\|_{\ell,T} \qquad \forall q \in H^\ell(T),\ 2 \le \ell \le r+1,\ \forall T \in \mathcal{T}_h.$$

As a consequence, the estimate (25) can be improved. We obtain

$$\begin{aligned}(p - i_h p, \operatorname{div} w_h) = (p - i_h p, \kappa_h^2 \operatorname{div} w_h) &\le C \left(\sum_{T \in \mathcal{T}_h} \gamma_T^{-1} h_T^{2r+2} \|p\|_{r+1,T}^2 \right)^{1/2} \left|\!\left|\!\left| (w_h, r_h) \right|\!\right|\!\right|_c \\ &\le C \left(\sum_{T \in \mathcal{T}_h} h_T^{2r+1} \|p\|_{r+1,T}^2 \right)^{1/2} \left|\!\left|\!\left| (w_h, r_h) \right|\!\right|\!\right|_c\end{aligned}$$

where $\gamma_T \sim h_T$ was used. Hence, the convergence order $r + 1/2$ is proven. $\square$

5 Numerical results

This section presents numerical results for solving the Oseen problem with inf-sup stable pairs of finite element spaces where the discretisation is stabilised by the local projection method. All calculations were performed with the code MooNMD [20].

Let $\Omega = (0,1)^2$. We consider the Oseen problem

$$-\nu \triangle u + (b \cdot \nabla) u + \sigma u + \nabla p = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \quad u = g \quad \text{on } \Gamma,$$

where the right hand side f and the inhomogeneous Dirichlet boundary condition g have been chosen such that

$$u = \big(\sin(x)\sin(y), \cos(x)\cos(y)\big)^T, \quad p = 2\cos(x)\sin(y) - 2\sin(1)\big(1 - \cos(1)\big)$$

is the solution for the case $\nu = 10^{-8}$, $b = u$, and $\sigma = 1$. This special solution was taken from [8].

We have performed calculations on triangular and quadrilateral meshes which were obtained by successive regular refinement of initial coarse grids. The coarsest mesh (level 0) consists of either two triangle or a single quadrilateral. The meshes on level 1 are shown in Fig. 1.

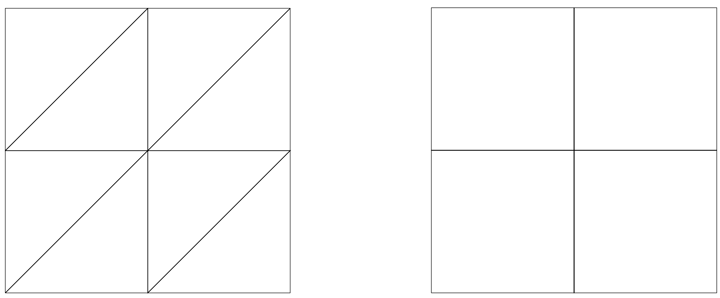

Figure 1: Meshes on level 1 for triangles (left) and quadrilaterals (right).

Our first example is the third-order Taylor–Hood element on triangles, i.e., $V_h/Q_h = P_3/P_2$. Since $d = 2$ and $r = 3$ in this case, Tab. 1 gives that the projection space D_h^1 can be chosen as $\{0\}$, P_0^{disc}, P_1^{disc}, or P_2^{disc} while the projection space D_h^2 has to be $\{0\}$ which means that no projection of the divergence takes place. Table 7 shows for different choices of projection

Table 7: Error and convergence order in the local projection norm $|||\cdot|||_a$ for $V_h/Q_h = P_3/P_2$.

D_h^1	D_h^2	τ_T	γ_T	error	order
P_2^{disc}	$\{0\}$	1	1	7.911-08	2.98
P_1^{disc}	$\{0\}$	h_T^2	1	7.694-08	2.99
P_0^{disc}	$\{0\}$	h_T^4	1	7.690-08	3.00
$\{0\}$	$\{0\}$	h_T^6	1	7.673-08	2.98
P_2^{disc}	P_0^{disc}	1	1	3.890-07	2.08

spaces D_h^1, D_h^2 and the stabilisation parameters τ_T, γ_T the error on level 6 (37,249 unknowns for each velocity component, 16,641 pressure unknowns) and the convergence order in the local projection norm $|||\cdot|||_a$ which was obtained from the results on levels 5 and 6. The setting for the first four data sets is in agreement with Tab. 1. The results are almost identical. Moreover, the obtained convergence orders of 3 confirm the theoretical result given in Thm. 4. The last setting in Tab. 7 violates the condition for the choice of D_h^2. We clearly see that convergence reduces to second order. This is caused by the fact that the term $(p - i_h p, \operatorname{div} w_h)$ can't be handled as in (25), cf. proof of Thm. 4. In order to get a ν-uniform estimate of this term, we obtain by (21) and an inverse inequality

$$
\begin{aligned}
(p - i_h p, \operatorname{div} w_h) &= \sum_{T\in\mathcal{T}_h} (p - i_h p, \operatorname{div} w_h)_T \le \sum_{T\in\mathcal{T}_h} \|p - i_h p\|_{0,T}\, \|\operatorname{div} w_h\|_{0,T} \\
&\le C \sum_{T\in\mathcal{T}_h} h_T^r \|p\|_{r,T}\, |w_h|_{1,T} \le C \sum_{T\in\mathcal{T}_h} h_T^{r-1} \|p\|_{r,T}\, \|w_h\|_{0,T} \\
&\le C_\sigma \left(\sum_{T\in\mathcal{T}_h} h_T^{2(r-1)} \|p\|_{r,T}^2 \right)^{1/2} |||(w_h, r_h)|||_a .
\end{aligned}
$$

Hence, the convergence order reduces from r to $r - 1$.

Our second example considers the pair $V_h/Q_h = Q_3/P_2^{\text{disc}}$ on quadrilaterals. We can use for this situation both stabilising terms S_h^a and S_h^b. According to Tab. 2, the projection spaces D_h^1 and D_h^2 can be independently chosen to be $\{0\}$, P_0^{disc}, P_1^{disc}, or P_2^{disc}. Furthermore, Tab. 3 gives for the stabilising term S_h^b the only choice $D_h^3 = P_2^{\text{disc}}$ with $\mu_T \sim 1$. The errors presented in Tab. 8 were obtained on level 6 (37,249 unknowns for each velocity component, 24,576 pressure unknowns) while the convergence order was calculated from the results on levels 5 and 6. Tab. 8 shows that the results for all parameter choices differ only slightly. Moreover, the optimal convergence order of 3 is achieved. Note that we have chosen in our test $D_h^1 = D_h^2$ since the choice of D_h^2 has no influence on the discrete solution, cf. Remark 8.

Our final test example is the Mini-element P_1^{++}/P_1 on triangles. According to Tab. 6, we have $D_h^1 = P_0^{\text{disc}}$ and $D_h^2 = \{0\}$. Tab. 9 shows the results of our numerical calculations on

Table 8: Error and convergence order in the local projection norm $|||\cdot|||$ for $V_h/Q_h = Q_3/P_2^{\mathrm{disc}}$.

D_h^1	D_h^2	τ_T	γ_T	error	order
$\{0\}$	$\{0\}$	h_T^6	1	9.202-08	3.00
P_0^{disc}	P_0^{disc}	h_T^4	1	9.202-08	3.00
P_1^{disc}	P_1^{disc}	h_T^2	1	9.252-08	3.00
P_2^{disc}	P_2^{disc}	1	1	8.696-08	3.00
$D_h^3 = P_2^{\mathrm{disc}}$		$\gamma_T = 1$		1.028-07	3.00

Table 9: Error and convergence order in the local projection norm $|||\cdot|||_a$ for $V_h/Q_h = P_1^{++}/P_1$.

D_h^1	D_h^2	τ_T	γ_T	error	order
P_0^{disc}	$\{0\}$	h_T	h_T	2.929-04	1.51
P_0^{disc}	P_0^{disc}	h_T	h_T	1.610-04	1.51

level 7 (49,409 unknowns for each velocity component, 16,641 pressure unknowns) where the convergence order was obtained from the results on levels 6 and 7. Note that also the choice $D_h^1 = D_h^2 = P_0^{\mathrm{disc}}$ is considered. Although this choice is not covered by our theory, the optimal convergence order 3/2 is achieved in both cases. Furthermore, the results of both choices differ only be a factor of 1.8.

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