Local projection methods on layer-adapted meshes for higher order discretisations of convection-diffusion problems

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We consider singularly perturbed convection-diffusion problems in the unit square where the solutions show the typical exponential layers. Layer-adapted meshes (Shishkin and Bakhvalov–Shishkin meshes) and the local projection method are used to stabilise the discretised problem. Using enriched Q_r-elements on the coarse part of the mesh and standard Q_r-elements on the remaining parts of the mesh, we show that the difference between the solution of the stabilised discrete problem and a special interpolant of the solution of the continuous problem convergences ε-uniformly with order $\mathcal{O}(N^{-(r+1/2)})$. Moreover, an ε-uniform convergence in the ε-weighted H^1-norm with order $\mathcal{O}\big((N^{-1}\ln N)^{-r}\big)$ on Shishkin meshes and with order $\mathcal{O}(N^{-r})$ on Bakhvalov–Shishkin meshes will be proved. Numerical results which support the theory will be presented.

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1 Introduction

Let $\Omega = (0,1)^2$ be the unit square. We consider the singularly perturbed boundary value problem

$$\begin{aligned} -\varepsilon\triangle u + b\cdot\nabla u + cu &= f \qquad \text{in } \Omega,\\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where ε is a small positive parameter while $b : \Omega \to \mathbb{R}^2$, $c : \Omega \to \mathbb{R}$, and $f : \Omega \to \mathbb{R}$ are sufficiently smooth functions satisfying

$$b_1(x,y) \geq \beta_1 > 0, \quad b_2(x,y) \geq \beta_2 > 0, \quad c(x,y) \geq 0 \qquad \forall (x,y) \in \overline{\Omega} \tag{2}$$

and

$$c(x,y) - \frac{1}{2} \operatorname{div} b(x,y) \geq c_0 > 0 \qquad \forall (x,y) \in \overline{\Omega}. \tag{3}$$

These assumptions ensure that (1) has a unique solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$. If the assumptions in (2) are satisfied then condition (3) can be always fulfilled for sufficiently small ε by a change of variables $v(x,y) = e^{\sigma x} u(x,y)$ with a suitable constant σ.

Since β_1 and β_2 are positive, the solution u of (1) shows typical exponential layers near $x = 1$ and $y = 1$. We assume for simplicity of our analysis that neither parabolic nor interior layers are present. The smallness of ε causes global unphysical oscillations if standard discretisation schemes on general meshes are applied. In order to obtain satisfactory discrete solutions with suitable accuracy, stabilisation methods and/or a-priori chosen meshes are often used. For an overview on these techniques, we refer to [25] where also the analytic behaviour of the solution u is discussed.

A-priori adapted meshes can be used if sufficient information on the structure of the solution are available. Early ideas of layer-adapted meshes go back to Bakhvalov [1]. The piecewise uniform Shishkin meshes [21] were proposed originally for finite difference schemes. The first paper which considered Shishkin meshes for finite element methods seems to be [26]. The authors analysed the standard Galerkin method with bilinear finite elements. The combination of the Bakhvalov's idea for using a uniform coarse mesh and a graded fine mesh with the Shishkin's simple choice of the transition point was considered by Linß [15, 16]. Following [16], we will call these meshes Bakhvalov–Shishkin meshes. Details will be given in Sect. 2.1

Even on layer-adapted meshes, the standard Galerkin discretisation lacks stability, see [18] for some numerical results. Moreover, the systems of linear equations which correspond to the standard Galerkin discretisation are hardly to solve by iterative methods [16, 18].

For stabilising convection-diffusion problems, the streamline-diffusion finite element method (SDFEM) which was proposed by Hughes and Brooks [11] is a powerful method which provides good stability properties and high accuracy outside interior and boundary layers. The SDFEM was investigated by many authors, see for example [10, 13, 14, 22]. The disadvantage of the SDFEM, in particular for higher order discretisations, is that several terms which include also second order derivatives have to be added to the weak formulation in order to ensure the strong consistency of the resulting method. However, the SDFEM on Shishkin meshes is much less sensitive to the choice of the transition points as standard Galerkin discretisations, see [23].

Stynes and Tobiska [28] studied the SDFEM with higher order finite elements applied to convection-diffusion equations on Shishkin meshes. Using Lin identities and anisotropic error estimates for a special interpolation operator, they proved for Q_r-elements, $r \geq 2$, that the difference between the SDFEM solution and the special interpolant of the solution of (1) converges in the streamline-diffusion norm with order $\mathcal{O}(N^{-(r+1/2)})$. Furthermore, postprocessing operators which allows to achieve estimates for the error between the weak solution and the discrete solution are suggested in [27, 28].

The local projection stabilisation technique is a different approach for stabilising the standard Galerkin discretisation. Stabilisation of the standard Galerkin method is achieved by adding terms which give a weighted L^2-control on the fluctuations $(id - \pi)$ of the derivatives of the quantity of interest where π is a projection into a discontinuous finite element space. Introduced for the Stokes problem in [2], the local projection stabilisation method has been extended to the transport problem in [3]. An analysis of the local projection stabilisation applied to the Oseen problem can be found in [4, 20]. The local projection stabilisation is only weakly consistent. However, the appearing consistency error can be bounded such that the optimal convergence order is maintained.

Originally, the local projection stabilisation technique was introduced as a two level method where the projection maps into a discontinuous finite element space which lives on patches of elements [2–4]. Note that standard finite element spaces can be used for both the approximation space and the projection space, see [4, 20]. This approach has a severe drawback since the discretisation stencil increases. Moreover, the necessary data structures might not be available in an existing computer code. The key for the analysis of the local projection method is the existence of a special interpolation operator which provides the standard interpolation error estimates and an additional orthogonality property [20]. The abstract setting given in [20] allows to construct the enrichment approach of the local projection method where the approximation space and the projection space are defined on the same mesh. The approximation space is enriched compared to standard finite element spaces. In [20], it was shown that it suffices to enrich the standard quadrilateral Q_r-element, $r \geq 2$, by just two additional functions, independent of r. Hence, the discretisation stencil remains small.

The local projection method on Shishkin meshes was considered in [19]. For arbitrary $r \geq 2$, new finite elements were introduced which are enrichments of the standard Q_r-element by six additional functions leading to an element which contains already the space P_{r+1}. It was shown in [19] that the error between the solution of the stabilised discrete problem and an interpolant of the solution of the continuous problem converges in the local projection norm with order $\mathcal{O}\big((N^{-1} \ln N)^{r+1}\big)$, uniformly in ε. Moreover, the error between the solution of the stabilised discrete problem and the solution of the continuous problem itself has in the global energy norm the convergence order $\mathcal{O}\big((N^{-1} \ln N)^{r+1}\big)$, again uniformly in ε. Both results rely on the fact that the space P_{r+1} is a subspace of the used enriched finite elements since this allows to obtain better interpolation error estimates compared to the Q_r-element.

In contrast to [19], we will use in this paper different finite elements on different regions of the mesh. As in [19, 28], the stabilisation acts only in the coarse part mesh. Hence, we can use standard Q_r-elements in the layer regions and the enriched finite elements introduced in [20] only in the coarse part of the mesh. This will lead to a smaller number of unknowns compared to the method in [19] where elements with larger enrichment were used on all mesh cells.

The main objective of this paper is the proof of an error estimate between the solution of the stabilised discrete solution and a special interpolation of the solution of the continuous problem. We will show that this error convergences ε-uniformly with order $\mathcal{O}(N^{-(r+1/2)})$ on Shishkin meshes and on Bakhvalov–Shishkin meshes. Furthermore, ε-uniform error estimates in the ε-weighted H^1-norm between the stabilised discrete solution and the solution u of (1) will be proved. On Shishkin meshes, the order $\mathcal{O}\big((N^{-1} \ln N)^{r}\big)$ is obtained while the convergence order on Bakhvalov–Shishkin meshes is $\mathcal{O}(N^{-r})$.

This paper is organised as follows. Section 2 describes layer-adapted meshes and their properties. Moreover, the local projection stabilisation is introduced. Auxiliary results are presented in Section 3. Several error estimates and a generalisation of Lin formulas are presented. The convergence proofs will be given in Section 4. Some numerical results are presented in Section 5. The papers ends with concluding remarks in Section 6.

Notation. Throughout this paper, C denotes a generic constant which is independent of the diffusion parameter ε and the mesh parameter N. Although we consider finite element of arbitrary order $r \geq 2$, the dependence of any constant on the order r will not be elaborated.

Let G be an arbitrary measurable two-dimensional subset $G \subset \Omega$. The measure of G is denoted by $|G|$. On G, the usual Sobolev spaces $W^{m,p}(G)$ with norm $\|\cdot\|_{m,p,G}$ and semi-norm $|\cdot|_{m,p,G}$ are used. In the case $p = 2$, we write $H^m(G)$ instead of $W^{m,2}(G)$ and skip the index p in the norm and the semi-norm. The L^2-inner product on G is denoted by $(\cdot,\cdot)_G$. Note that the index G in norms, semi-norms, and inner products is omitted in the case $G = \Omega$. All notation is also used for the vector-valued case.

Let $P_s(K)$ denote the space of all polynomials of total degree less than or equal to s while $Q_s(K)$ is the space of all polynomials of degree less than or equal to s in each variable separately.

2 Layer-adapted meshes and local projection stabilisation

2.1 Layer-adapted meshes and their properties

We will consider in this paper two types of layer-adapted meshes: Shishkin meshes (S-meshes) and Bakhvalov–Shishkin meshes (B-S-meshes). S-meshes are piecewise uniform meshes which are adapted to the boundary layers [21,23,24]. The B-S-meshes were introduced by Linß [15,16]. Compared to S-meshes, the B-S-mesh is uniform only on coarse part of the mesh and graded toward the boundary in the fine parts of the mesh. In contrast to a Bakhvalov mesh [1], the transition points of B-S-meshes are chosen as for S-meshes instead of solving nonlinear scalar equations.

Let N be an even integer. We denote by λ_x and λ_y the transition parameters which indicate where the mesh changes from coarse to fine. These parameters are given by

$$\lambda_x := \min\left(\frac{1}{2}, (r+1)\frac{\varepsilon}{\beta_1}\ln N\right), \qquad \lambda_y := \min\left(\frac{1}{2}, (r+1)\frac{\varepsilon}{\beta_2}\ln N\right).$$

To be precise, we assume that λ_x and λ_y take the second argument inside the corresponding minimum. Otherwise, our analysis could be simplified a lot since N^{-1} would be much smaller than ε. Moreover, we suppose that $\varepsilon \leq N^{-1}$ which is realistic for this type of problems.

Note that in the definition of λ_x and λ_y the factor in front of $\varepsilon \ln(N)/\beta_i$, $i = 1, 2$, has to be large enough. This prevents oscillations and guarantees the optimal order of convergence, see [23].

The domain Ω is divided into four parts as sketched in the left picture of Fig. 1. Let $\overline{\Omega} = \overline{\Omega}_{11} \cup \overline{\Omega}_{12} \cup \overline{\Omega}_{21} \cup \overline{\Omega}_{22}$ where the subdomains are given by

$$\begin{aligned}
\Omega_{11} &:= (0, 1-\lambda_x) \times (0, 1-\lambda_y), \qquad & \Omega_{12} &:= (0, 1-\lambda_x) \times (1-\lambda_y, 1),\\
\Omega_{21} &:= (1-\lambda_x, 1) \times (0, 1-\lambda_y), \qquad & \Omega_{22} &:= (1-\lambda_x, 1) \times (1-\lambda_y, 1).
\end{aligned}$$

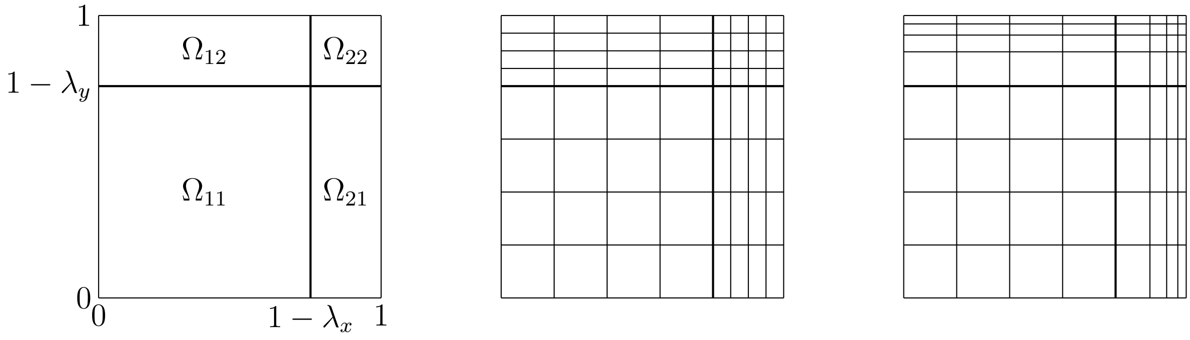

Figure 1: Division of Ω (left), a S-mesh (middle), and a B-S-mesh (right), all for $N = 8$.

Let $\mathcal{T}_x^N := \{(x_{i-1}, x_i) \,:\, i = 1, \dots, N\}$ and $\mathcal{T}_y^N := \{(y_{j-1}, y_j) \,:\, j = 1, \dots, N\}$ be two partitions of the interval $(0, 1)$. We choose

$$x_i := \begin{cases} 2i(1-\lambda_x)/N, & i = 0, \dots, N/2, \\ 1 - 2(N-i)\lambda_x/N, & i = N/2+1, \dots, N, \end{cases}$$

and

$$y_j := \begin{cases} 2j(1-\lambda_y)/N, & j = 0, \dots, N/2, \\ 1 - 2(N-j)\lambda_y/N, & j = N/2+1, \dots, N, \end{cases}$$

for S-meshes. We define

$$x_i := \begin{cases} 2i(1-\lambda_x)/N, & i = 0, \dots, N/2, \\ 1 + \dfrac{(r+1)\varepsilon}{\beta_1} \ln\left(\dfrac{N^2 - 2(N-i)(N-1)}{N^2}\right), & i = N/2+1, \dots, N, \end{cases}$$

and

$$y_j := \begin{cases} 2j(1-\lambda_y)/N, & j = 0, \dots, N/2, \\ 1 + \dfrac{(r+1)\varepsilon}{\beta_2} \ln\left(\dfrac{N^2 - 2(N-j)(N-1)}{N^2}\right), & j = N/2+1, \dots, N, \end{cases}$$

for B-S-meshes, see [16] for the case $r = 1$. For both types of meshes, the points $x_{N/2}$ and $y_{N/2}$ can be calculated by both branches in the case statement.

Let $\mathcal{T}^N$ denote the tensor-product of $\mathcal{T}_x^N$ and $\mathcal{T}_y^N$. Fig. 1 shows a S-mesh (middle picture) and a B-S-mesh (right picture). Each of the four subdomains consists of $N^2/4$ cells. All cells in $\mathcal{T}^N$ are rectangles which are aligned with the coordinate axes. The midpoint of $K \in \mathcal{T}^N$ is denoted by (x_K, y_K) while $h_{K,x}$ and $h_{K,y}$ are the edge sizes of K in x-direction and y-direction, respectively. For both types of meshes, the rectangles in Ω_{11} are of size $\mathcal{O}(N^{-1}) \times \mathcal{O}(N^{-1})$. On S-meshes, the cells in Ω_{22} are of size $\mathcal{O}(\varepsilon N^{-1} \ln N) \times \mathcal{O}(\varepsilon N^{-1} \ln N)$. The cells in $\Omega_{12} \cup \Omega_{21}$ of S-meshes have a long edge of size $\mathcal{O}(N^{-1})$ and a short edge of length $\mathcal{O}(\varepsilon N^{-1} \ln N)$. Properties of B-S-meshes are given in the following lemma whose results are generalisations of the results from [16].

Lemma 1. *Let x_i, $i = N/2, \ldots, N$, be the points for a B-S-mesh. Then, the estimates*

$$x_i - x_{i-1} \le \frac{2(r+1)\varepsilon}{\beta_1(i - N/2)} \le C\,N^{-1}, \qquad i = N/2+1, \ldots, N,$$

and

$$(x_i - x_{i-1})^\alpha \exp\left(-\frac{\beta_1(1-x_i)}{\varepsilon}\right) \le C\varepsilon^\alpha\,N^{-\alpha}, \qquad i = N/2+1, \ldots, N, \quad \alpha \in [0, r+1],$$

hold true. The corresponding results are valid also for y_i, $i = N/2, \ldots, N$, if β_1 is replaced by β_2.

Proof. Our proof is based on arguments given by Linß [16]. To show the first statement, we use for $i = N/2, \ldots, N$ that

$$\begin{aligned} x_i - x_{i-1} &= \frac{(r+1)\varepsilon}{\beta_1} \ln\left(\frac{N^2 - 2(N-i)(N-1)}{N^2 - 2\big(N-(i-1)\big)(N-1)}\right) \\ &= \frac{(r+1)\varepsilon}{\beta_1} \ln\left(\frac{N(2\nu+1) - 2\nu}{N(2\nu-1) - 2(\nu-1)}\right) \end{aligned}$$

where $\nu = i - N/2$. An easy calculation shows that

$$\frac{N(2\nu+1) - 2\nu}{N(2\nu-1) - 2(\nu-1)} \le \frac{2\nu+1}{2\nu-1}.$$

The monotonicity of the logarithm results in

$$x_i - x_{i-1} \le \frac{(r+1)\varepsilon}{\beta_1} \ln \frac{2\nu+1}{2\nu-1}.$$

Furthermore, the Taylor expansion gives the estimate

$$(2\nu-1)\exp(2/\nu) \ge (2\nu-1)(1+2/\nu) = 2\nu + 3 - 2/\nu \ge 2\nu+1$$

where $0 < 1/\nu \le 1$ was used. Hence, we obtain

$$x_i - x_{i-1} \le \frac{(r+1)\varepsilon}{\beta_1} \ln \frac{(2\nu-1)\exp(2/\nu)}{2\nu-1} = \frac{(r+1)\varepsilon}{\beta_1} \cdot \frac{2}{\nu} = \frac{2(r+1)\varepsilon}{\beta_1\nu} \le C\,N^{-1}$$

which is the desired estimate. For proving the second statement, we observe for $\alpha \in [0, r+1]$ that

$$\exp\left(-\frac{\beta_1(1-x_i)}{\varepsilon}\right) = \left(\frac{N^2 - 2(N-i)(N-1)}{N^2}\right)^{r+1} \le \left(\frac{N^2 - 2(N-i)(N-1)}{N^2}\right)^{\alpha}$$

since $N^{-1} \le \big(N^2 - 2(N-i)(N-1)\big)/N^2 \le 1$ for $i = N/2, \ldots, N$. Indeed, the expression $\big(N^2 - 2(N-i)(N-1)\big)/N^2$ is monotonically increasing in i with value N^{-1} for $i = N/2$ and value 1 for $i = N$. Hence, we obtain

$$(x_i - x_{i-1})^\alpha \exp\left(-\frac{\beta_1(1-x_i)}{\varepsilon}\right) \le \left(\frac{2(r+1)\varepsilon}{\beta_1\nu} \cdot \frac{N^2 - 2(N-i)(N-1)}{N^2}\right)^{\alpha}. \tag{4}$$

To proceed, we estimate

$$\frac{N^2 - 2(N-i)(N-1)}{\nu N^2} = \frac{N^2 - (N-2\nu)(N-1)}{\nu N^2} = \frac{2\nu(N-1)}{\nu N^2} + \frac{N}{\nu N^2} \le 3\, N^{-1}$$

where $\nu = i - N/2$ and $1/\nu \le 1$ were used. Putting this into (4), the second statement of this lemma follows. □

2.2 Solution decomposition

The analysis presented in this paper relies on the precise knowledge of the behaviour of the solution u of problem (1). We make the following assumption which is similar to those used in [19, 28].

Assumption 2. *The solution u can be decomposed as*

$$u = S + E_{12} + E_{21} + E_{22}$$

with $S \in C^{r+1}(\Omega)$, $E_{12}, E_{21}, E_{22} \in C^{r+2}(\Omega)$. The smooth part S of the solution u fulfils

$$\left| \frac{\partial^{i+j} S}{\partial x^i \partial y^j}(x,y) \right| \le C, \qquad 0 \le i+j \le r+1, \tag{5}$$

while the layer functions satisfy

$$\left| \frac{\partial^{i+j} E_{12}}{\partial x^i \partial y^j}(x,y) \right| \le C\, \varepsilon^{-j}\, e^{-\beta_2(1-y)/\varepsilon}, \qquad 0 \le i+j \le r+2, \tag{6}$$

$$\left| \frac{\partial^{i+j} E_{21}}{\partial x^i \partial y^j}(x,y) \right| \le C\, \varepsilon^{-i}\, e^{-\beta_1(1-x)/\varepsilon}, \qquad 0 \le i+j \le r+2, \tag{7}$$

$$\left| \frac{\partial^{i+j} E_{22}}{\partial x^i \partial y^j}(x,y) \right| \le C\, \varepsilon^{-(i+j)}\, e^{-[\beta_1(1-x)/\varepsilon + \beta_2(1-y)/\varepsilon]}, \qquad 0 \le i+j \le r+2, \tag{8}$$

for all $(x,y) \in \Omega$. Here, E_{21} and E_{12} are exponential boundary layers along $x = 1$ and $y = 1$, respectively, while E_{22} is an exponential corner layer at the point $(1,1)$.

Note that the bound

$$\|S\|_{r+1} \le C \tag{9}$$

follows directly from the point-wise bounds given in (5).

In [17], conditions on the right-hand side f of problem (1) were given which guarantee a decomposition of the solution into a smooth part and boundary layer parts such that lower order derivatives can be estimates by exponential bounds. The extension of these results to the case of higher order derivatives as needed in our case seems to be possible but tedious. The number of these sufficient conditions will increase rapidly with increasing differentiation order. We refer to [25, Sect. 7] for more details on these compatibility conditions.

We give finally some estimates for integrals which involve exponential functions.

Lemma 3. *Let α, β be positive constants and $\lambda := (r+1)(\varepsilon \ln N)/\beta$. Then, the estimates*

$$\int\limits_0^{1-\lambda} \exp\big(-\alpha\beta(1-z)/\varepsilon\big)\,dz \le C\varepsilon N^{-\alpha(r+1)} \quad \text{and} \quad \int\limits_{1-\lambda}^{1} \exp\big(-\alpha\beta(1-z)/\varepsilon\big)\,dz \le C\varepsilon$$

hold true.

Proof. The assertions of this lemma follow from the properties of exponential functions and the choice of λ. Details can be found in [19, Lemma 2]. $\square$

2.3 Galerkin discretisation

Let $V := H_0^1(\Omega)$. We define the bilinear form

$$a(v,w) := \varepsilon(\nabla v, \nabla w) + (b \cdot \nabla v + cv, w).$$

A weak formulation of the convection-diffusion problem (1) reads

Find $u \in V$ such that

$$a(u,v) = (f,v) \qquad \forall v \in V. \tag{10}$$

Note that the variational formulation (10) has a unique solution due to (3).

In order to discretise the problem, we will introduce the finite element space V^N on $\mathcal{T}^N$. To this end, we start with defining finite elements on the reference cell $\widehat{K} = (-1,+1)^2$. Let $\hat{v}_i$, $i = 1,2,3,4$, and $\widehat{E}_i$, $i = 1,2,3,4$, denote the vertices and the edges of $\widehat{K}$, respectively.

The standard Q_r-element is equipped with the non-standard vertex-edge-cell interpolation operator $\widehat{I}_r : C(\overline{\widehat{K}}) \to Q_r(\widehat{K})$ which is defined by

$$\begin{aligned}
(\widehat{I}_r \hat{w})(\hat{v}_i) &= \hat{w}(\hat{v}_i), && i = 1,2,3,4,\\
\int\limits_{\widehat{E}_i} \widehat{I}_r \hat{w}\,\hat{q}\,d\gamma &= \int\limits_{\widehat{E}_i} \hat{w}\,\hat{q}\,d\gamma, && i = 1,2,3,4,\ q \in P_{r-2}(\widehat{E}_i),\\
\int\limits_{\widehat{K}} \widehat{I}_r \hat{w}\,\hat{q}\,d\hat{y}\,d\hat{x} &= \int\limits_{\widehat{K}} \hat{w}\,\hat{q}\,d\hat{y}\,d\hat{x}, && \hat{q} \in Q_{r-2}(\widehat{K}).
\end{aligned}$$

Furthermore, we define the space $Q_r^+(\widehat{K})$ as an enrichment of the space $Q_r(\widehat{K})$ by

$$Q_r^+(\widehat{K}) := Q_r(\widehat{K}) \oplus \operatorname{span}\Big((1-\hat{x}^2)(1-\hat{y}^2)\hat{x}^{r-1},\ (1-\hat{x}^2)(1-\hat{y}^2)\hat{y}^{r-1}\Big).$$

This enriched space was introduced in [20, Sect. 4.2]. The associated vertex-edge-cell interpolation operator $\widehat{I}_r^+ : C(\overline{\widehat{K}}) \to Q_r^+(\widehat{K})$ is given by

$$(\widehat{I}_r^+ \hat{w})(\hat{v}_i) = \hat{w}(\hat{v}_i), \qquad i = 1,2,3,4,$$

$$\int_{\widehat{E}_i} \widehat{I}_r^+ \hat{w}\,\hat{q}\,d\gamma = \int_{\widehat{E}_i} \hat{w}\,\hat{q}\,d\gamma, \qquad i = 1,2,3,4,\ q \in P_{r-2}(\widehat{E}_i),$$
$$\int_{\widehat{K}} \widehat{I}_r^+ \hat{w}\,\hat{q}\,d\hat{y}\,d\hat{x} = \int_{\widehat{K}} \hat{w}\,\hat{q}\,d\hat{y}\,d\hat{x}, \qquad \hat{q} \in Q_{r-2}(\widehat{K}) \oplus \operatorname{span}\left(\hat{x}^{r-1},\,\hat{y}^{r-1}\right).$$

The unisolvance of both interpolation operators can be checked easily, see [28, Lemma 3] for the proof for $\widehat{I}_r$.

The reference transformation $F_K : \widehat{K} \to K$ with

$$F_K(\hat{x},\hat{y}) = \left(x_K + \frac{h_{K,x}}{2}\hat{x}, y_K + \frac{h_{K,y}}{2}\hat{y}\right)^T$$

is a simple affine mapping. Let

$$V(K) := \begin{cases} \left\{v \ :\ v \circ F_K \in Q_r^+(\widehat{K})\right\}, & K \subset \Omega_{11}, \\ \left\{v \ :\ v \circ F_K \in Q_r(\widehat{K})\right\}, & K \subset \Omega \setminus \Omega_{11}, \end{cases}$$

be a finite dimensional function space on K. The local interpolation operator $I_K : C(\overline{K}) \to V(K)$ is given by

$$I_K v := \begin{cases} \left(\widehat{I}_r^+(v \circ F_K)\right) \circ F_K^{-1}, & K \subset \Omega_{11}, \\ \left(\widehat{I}_r(v \circ F_K)\right) \circ F_K^{-1}, & K \subset \Omega \setminus \Omega_{11}. \end{cases}$$

Remark 4. *Note that we have for all edges $E \subset \partial K$, $K \in \mathcal{T}^N$, that*

$$v_h|_E \in P_r(E), \qquad v_h \in V(K).$$

Furthermore, the restriction of $I_K v$ onto an edge $E \subset \partial K$ depends only on the restriction of v onto E. This follows immediately from the definition of the interpolation operators $\widehat{I}_r$ and $\widehat{I}_r^+$.

Our finite element space V^N is defined as

$$V^N := \left\{v \in C(\overline{\Omega}) \ :\ v|_K \in V(K)\ \forall K \in \mathcal{T}^N,\ v = 0 \text{ on } \partial\Omega\right\}.$$

Note that the space V^N is non-standard since it consists of enriched Q_r-elements on Ω_{11} and standard Q_r-elements on $\Omega \setminus \Omega_{11}$.

We proceed with the global interpolation operator $I^N : C(\overline{\Omega}) \to V^N$. Due to Remark 4, we can define I^N locally by

$$(I^N v)|_K := I_K\big(v|_K\big) \qquad \forall K \in \mathcal{T}^N, v \in C(\overline{\Omega}). \tag{11}$$

Using the finite element space V^N, we can state the standard Galerkin discretisation of (10) which reads

Find $\tilde{u}^N \in V^N$ such that

$$a(\tilde{u}^N, v^N) = (f, v^N) \qquad \forall v^V \in V^N. \tag{12}$$

Note that the discrete problem (12) is uniquely solvable due to (3).

2.4 Local projection stabilisation

We proceed with introducing some more notation which will be used for defining the local projection method.

Let π_K denote the $L^2(K)$-projection into $P_{r-1}(K)$. The fluctuation operator $\kappa_K : L^2(K) \to L^2(K)$ is given as $\kappa_K := id_K - \pi_K$ where id_K is the identity mapping on $L^2(K)$.

The fundamental approximation property of the fluctuation operator κ_K is stated in the following lemma which is a consequence of the Bramble–Hilbert lemma [5].

Lemma 5. *For $0 \le s \le r$, the fluctuation operator κ_K fulfils*

$$\|\kappa_K w\|_{0,K} \le C\, h_K^s |w|_{s,K} \qquad \forall w \in H^s(K)$$

for all $K \in \mathcal{T}^N$.

Since we are interested in an additional control on the derivative in streamline direction, we introduce the following stabilisation term

$$s^N(v,w) := \sum_{K \in \mathcal{T}^N} \tau_K \big(\kappa_K(b \cdot \nabla v), \kappa_K(b \cdot \nabla w)\big)_K$$

with the cell-dependent parameters τ_K, $K \in \mathcal{T}^N$. Note that a Cauchy–Schwarz-like estimate

$$\big|s^N(v,w)\big| \le \big(s^N(v,v)\big)^{1/2} \big(s^N(w,w)\big)^{1/2} \qquad \forall v, w \in H^1(\Omega) \tag{13}$$

holds true due to the structure of s^N.

The stabilisation parameters τ_K, $K \in \mathcal{T}^N$, are chosen as

$$\tau_K := \begin{cases} C_1\, N^{-1}, & K \subset \Omega_{11}, \\ 0, & \text{otherwise}, \end{cases} \tag{14}$$

with a suitable constant C_1 which is independent of ε and N. As for the SDFEM on S-meshes considered in [27, 28], the stabilisation acts only on the coarse subdomain Ω_{11}. Note that the stabilisation parameters δ_K, $K \subset \Omega_{11}$, used in [27, 28] were also chosen to be $C_1\, N^{-1}$ for the case $\varepsilon \le N^{-1}$, in contrast to the analysis presented in [19] where $\tau_K = C_1\, N^{-2}$ for $K \subset \Omega_{11}$ was used.

The stabilised bilinear form a^N is defined via

$$a^N(u,v) := a(u,v) + s^N(u,v) \qquad u, v \in V.$$

The stabilised discrete problem reads

Find $u^N \in V^N$ such that

$$a^N(u^N, v^N) = (f, v^N) \qquad \forall v^N \in V^N. \tag{15}$$

We will use in our analysis the norms

$$\|v\|_{LP} := \big(\varepsilon\, |v|_1^2 + c_0\, \|v\|_0^2 + s^N(v,v)\big)^{1/2}, \qquad \|v\|_{1,\varepsilon} := \big(\varepsilon\, |v|_1^2 + c_0\, \|v\|_0^2\big)^{1/2}$$

Note that the ε-weighted H^1-norm $\|\cdot\|_{1,\varepsilon}$ is part of the local projection norm $\|\cdot\|_{LP}$.

The following orthogonality property plays an important role in the error analysis.

Lemma 6. *Let $K \subset \Omega_{11}$. Then, the orthogonality property*

$$(w - I^N w, q)_K = 0 \qquad \forall q \in P_{r-1}(K),\, \forall w \in C(\overline{K}).$$

is fulfilled for the interpolation operator I^N defined in (11).

Proof. Since the reference mapping F_K is affine, we obtain by transforming the integral from K to $\widehat{K}$ that

$$(w - I^N w, q)_K = |K| \left(w \circ F_K - (I^N w) \circ F_K, q \circ F_K \right)_{\widehat{K}}.$$

The definition of I^N on $K \subset \Omega_{11}$ gives that $(I^N w) \circ F_K = \widehat{I}_r^+(w \circ F_K)$. Furthermore, we have that $q \circ F_K \in P_{r-1}(\widehat{K})$ due to the affine mapping F_K. The definition of $\widehat{I}_r^+$ yields immediately that the integral on $\widehat{K}$ vanishes. Hence, the orthogonality is proved. □

3 Auxiliary results

This section will provide auxiliary estimates which will be used later on in the error analysis presented in Sect. 4

The finite element space V^N consists of the enriched finite element Q_r^+ on Ω_{11} and of the standard Q_r-element on $\Omega \setminus \Omega_{11}$. The interpolation operator I^N coincides on $\Omega \setminus \Omega_{11}$ with the vertex-edge-cell interpolation operator which was already used in [28] where the whole finite element space was based in the standard Q_r-element. Hence, we can adapt some local results from [28] and apply them to mesh cells $K \subset \Omega \setminus \Omega_{11}$.

Following [28, Lemma 4], we can state a generalisation of the Lin formula.

Lemma 7. *Let $K \in \mathcal{T}^N$ with $K \subset \Omega \setminus \Omega_{11}$. Then, there exists a constant C which is independent of N and K such that*

$$\left| ((I_K w - w)_x, q_x^N)_K \right| \le C\, h_{K,y}^{r+1} \left\| \frac{\partial^{r+2} w}{\partial x \partial y^{r+1}} \right\|_{0,K} \left\| q_x^N \right\|_{0,K}$$

and

$$\left| ((I_K w - w)_y, q_y^N)_K \right| \le C\, h_{K,x}^{r+1} \left\| \frac{\partial^{r+2} w}{\partial x^{r+1} \partial y} \right\|_{0,K} \left\| q_y^N \right\|_{0,K}$$

holds true for all $w \in H^{r+2}(K)$ and $q^N \in Q_r(K)$.

The following error estimate is adapted from [28, Lemma 6].

Lemma 8. *Let s be an integer with $1 \le s \le r$. Furthermore, let $K \subset \Omega \setminus \Omega_{11}$ be a mesh cell from $\mathcal{T}^N$. Then, there exists a constant C independent of N and K such that*

$$\left\| (v - I^N v)_x \right\|_{0,K} \le C \sum_{i+j=s} h_{K,x}^i\, h_{K,y}^j \left\| \frac{\partial^{s+1} v}{\partial x^{i+1} \partial y^j} \right\|_{0,K}$$

and

$$\left\|(v - I^N v)_y\right\|_{0,K} \le C \sum_{i+j=s} h_{K,x}^i \, h_{K,y}^j \left\| \frac{\partial^{s+1} v}{\partial x^i \partial y^{j+1}} \right\|_{0,K}$$

hold true for all $v \in H^{s+1}(K)$.

We proceed with L^q-estimates of the interpolation error.

Lemma 9. *Let $q \in [1, \infty]$ and $2 \le s \le r + 1$. Then, there exists a positive constant C which does not depend on N and $K \in \mathcal{T}^N$ such that the estimate*

$$\|v - I^N v\|_{0,q,K} \le C \sum_{i+j=s} h_{K,x}^i \, h_{K,y}^j \left\| \frac{\partial^s v}{\partial x^i \partial y^j} \right\|_{0,q,K}$$

holds true for all $v \in W^{s,q}(K)$.

Before presenting the proof of this lemma we note that the assumptions on s and q ensure that $W^{s,q}(K) \subset C(\overline{K})$. Hence, the interpolation operator I^N is well-defined.

Proof. By transforming the integrals from K to $\widehat{K}$, we see that it is equivalent to show the estimate

$$\|\hat{v} - \widehat{I}\hat{v}\|_{0,q,\widehat{K}} \le C\,|\hat{v}|_{s,q,\widehat{K}} \qquad \forall \hat{v} \in W^{s,q}(\widehat{K})$$

where $\widehat{I}$ equals to $\widehat{I}_r^+$ for $K \subset \Omega_{11}$ and to $\widehat{I}_r$ for $K \subset \Omega \setminus \Omega_{11}$. The required estimate on $\widehat{K}$ is a direct consequence of the Bramble–Hilbert lemma. □

The next lemma state the L^∞-stability of the interpolation operator I^N on each mesh cell $K \in \mathcal{T}^N$.

Lemma 10. *Let $K \in \mathcal{T}^N$. Then, there exists a positive constant independent of N and K such that the estimate*

$$\|I^N v\|_{0,\infty,K} \le C\,\|v\|_{0,\infty,K} \qquad \forall v \in C(\overline{K})$$

holds true.

Proof. The desired estimate is equivalent to the estimate

$$\|\widehat{I}\hat{v}\|_{0,\infty,\widehat{K}} \le C\,\|\hat{v}\|_{0,\infty,\widehat{K}} \qquad \forall \hat{v} \in C(\overline{\widehat{K}})$$

on the reference square $\widehat{K}$ where $\widehat{I}$ denotes either $\widehat{I}_r^+$ (for $K \subset \Omega_{11}$) or $\widehat{I}_r$ (for $K \subset \Omega \setminus \Omega_{11}$). This estimate holds due to the definition of the interpolation operators on $\widehat{K}$ via the nodal functionals which use values in the vertices, weighted integrals along edges, and weighted cell integrals. Details can be found in [19, 28]. □

We will proceed with estimates for the layer parts of the solution u of (10) where we use the decomposition due to Assumption 2.

Lemma 11. *Let* (6)*,* (7)*, and* (8) *be fulfilled. Then, the estimates*

$$\begin{aligned}
\|I^N E_{12}\|_{0,\infty,\Omega_{11}\cup\Omega_{21}} &\le C\|E_{12}\|_{0,\infty,\Omega_{11}\cup\Omega_{21}} \le C\,N^{-(r+1)},\\
\|I^N E_{21}\|_{0,\infty,\Omega_{11}\cup\Omega_{12}} &\le C\|E_{21}\|_{0,\infty,\Omega_{11}\cup\Omega_{12}} \le C\,N^{-(r+1)},\\
\|I^N E_{22}\|_{0,\infty,\Omega\setminus\Omega_{22}} &\le C\|E_{22}\|_{0,\infty,\Omega\setminus\Omega_{22}} \le C\,N^{-(r+1)}
\end{aligned}$$

hold true.

Proof. These estimates follow immediately from Lemma 10 and the choice of the transition parameters λ_x and λ_y. For details, we refer to [19]. □

For the interpolation error $I^N u - u$ where the solution u of (10) satisfies Assumption 2, we will give point-wise estimates and L^2-estimates.

Lemma 12. *Let the solution u of* (10) *fulfil Assumption 2. Then, there exists a positive constant C which is independent of ε and N such that the estimate*

$$\left|(I^N u - u)(x,y)\right| \le C\,N^{-(r+1)} \qquad \forall (x,y)\in\overline{\Omega}$$

holds true on B-S-meshes while the estimate

$$\left|(I^N u - u)(x,y)\right| \le \begin{cases} C\,N^{-(r+1)}, & (x,y)\in\Omega_{11},\\ C\,(N^{-1}\ln N)^{r+1}, & \textit{otherwise}, \end{cases}$$

is satisfied on S-meshes. Moreover, we have

$$\|I^N u - u\|_0 \le C\,N^{-(r+1)}$$

on B-S-meshes while

$$\|I^N u - u\|_0 \le C\,(N^{-1}\ln N)^{r+1}, \qquad \|I^N u - u\|_{0,\Omega_{11}} \le C\,N^{-(r+1)}$$

hold on S-meshes.

Proof. For proving the estimate on both types of meshes, we start with using the decomposition of u due to Assumption 2. We get

$$I^N u - u = (I^N S - S) + (I^N E_{12} - E_{12}) + (I^N E_{21} - E_{21}) + (I^N E_{22} - E_{22}).$$

The S-term can be estimated by Lemma 9 with $s = r+1$ and $q = \infty$. We obtain for all $(x,y)\in\overline{\Omega}$ the estimate

$$\left|(I^N S - S)(x,y)\right| \le C\,N^{-(r+1)}\,\|S\|_{r+1,\infty} \le C\,N^{-(r+1)}$$

due to (5). To estimate the E_{12}-term on $\Omega_{11}\cup\Omega_{21}$, we use the triangle inequality and Lemma 10 to get

$$\|I^N E_{12} - E_{12}\|_{0,\infty,\Omega_{11}\cup\Omega_{21}} \le C\|E_{12}\|_{0,\infty,\Omega_{11}\cup\Omega_{21}} \le C\,N^{-(r+1)}$$

where Lemma 11 was applied. The estimate on $\Omega_{12} \cup \Omega_{22}$ starts with applying Lemma 9 with $s = r + 1$ and $q = \infty$. We obtain for $(x, y) \in K = (x_{i-1}, x_i) \times (y_{j-1}, y_j)$ that

$$\left|(I^N E_{12} - E_{12})(x, y)\right| \leq C \sum_{a+b=r+1} h_{K,x}^a \, h_{K,y}^b \left\| \frac{\partial^{r+1} E_{12}}{\partial x^a \partial y^b} \right\|_{0,\infty,K}.$$

On S-meshes, we proceed by using $h_{K,x} \leq C\, N^{-1}$, $h_{K,y} \leq C\, \varepsilon\, N^{-1} \ln N$, and (6). We obtain

$$\left|(I^N E_{12} - E_{12})(x, y)\right| \leq C \sum_{a+b=r+1} N^{-a} (\varepsilon N^{-1} \ln N)^b \, \varepsilon^{-b} \leq C (N^{-1} \ln N)^{r+1}$$

for all $(x, y) \in K \subset \Omega_{21} \cup \Omega_{22}$. On B-S-meshes, a refined estimate is needed. Using $h_{K,x} \leq C\, N^{-1}$, Lemma 1, and (6), we get

$$\left|(I^N E_{12} - E_{12})(x, y)\right| \leq C \sum_{a+b=r+1} N^{-a} (y_j - y_{j-1})^b \, \varepsilon^{-b} \exp\left(-\frac{\beta_2 (1 - y_j)}{\varepsilon}\right) \leq C\, N^{-(r+1)}$$

for $(x, y) \in K$.

To estimate the E_{21}-term, we use on $\Omega_{11} \cup \Omega_{12}$ the same technique as for E_{12} on $\Omega_{11} \cup \Omega_{21}$ and on $\Omega_{21} \cup \Omega_{22}$ a similar way as for E_{12} on $\Omega_{12} \cup \Omega_{22}$.

The E_{22}-term on Ω_{22} is bound by using the anisotropic error estimate from Lemma 9. On $\Omega \setminus \Omega_{22}$, we apply the triangle inequality and use Lemma 11.

Note that the logarithmic factor for S-meshes is only present on $\Omega \setminus \Omega_{11}$. Hence, a bound without logarithmic factor is obtained on Ω_{11}.

The L^2-estimates follow immediately from the point-wise estimates. □

We will now prove an estimate for the H^1-seminorm of the interpolation error.

Lemma 13. *Let the solution u of* (10) *satisfy Assumption 2. Then, there exists a constant C independent of ε and N such that the estimate*

$$\varepsilon^{1/2}\, |u - I^N u|_1 \leq \begin{cases} C\,(N^{-1} \ln N)^r, & \text{on S-meshes}, \\ C\, N^{-r}, & \text{on B-S-meshes}, \end{cases}$$

holds true.

Proof. We start with the solution decomposition due to Assumption 2. We obtain

$$|u - I^N u|_1 \leq |S - I^N S|_1 + |E_{12} - I^N E_{12}|_1 + |E_{21} - I^N E_{21}|_1 + |E_{22} - I^N E_{22}|_1$$

by the triangle inequality. Each term will be estimates separately.

Standard interpolation error estimates result in

$$|S - I^N S|_1 \leq C\, N^{-r} \tag{16}$$

on both types of meshes where we have used (9). For the remaining terms, we will discuss only the estimates of the terms with x-derivatives since the terms with y-derivatives can be handled by the same arguments.

We proceed with the E_{12}-term. Let $K = (x_{i-1}, x_i)\times(y_{j-1}, y_j)$ be an arbitrary cells in $\Omega_{12}\cup\Omega_{22}$. We apply the anisotropic error estimate from Lemma 8 and get

$$\|(E_{12} - I^N E_{12})_x\|_{0,K} \le C \sum_{a+b=r} h^a_{K,x} h^b_{K,y} \left\| \frac{\partial^{r+1} E_{12}}{\partial x^{a+1} \partial y^b} \right\|_{0,K}.$$

Using $h_{K,x} \le C\, N^{-1}$, $h_{K,y} \le C\,\varepsilon\, N^{-1} \ln N$ for $K \subset \Omega_{12} \cup \Omega_{22}$ of S-meshes, we obtain

$$\|(E_{12} - I^N E_{12})_x\|_{0,K} \le C\,(N^{-1} \ln N)^r \| \exp(-\beta_2(1-y)/\varepsilon\|_{0,K}$$

where (6) was exploited. Putting together the estimates on $\Omega_{12} \cup \Omega_{22}$, one gets

$$\begin{aligned}\varepsilon^{1/2}\, \|(E_{12} - I^N E_{12})_x\|_{\Omega_{12}\cup\Omega_{22}} &\le C\,\varepsilon^{1/2}\,(N^{-1} \ln N)^r \| \exp(-\beta_2(1-y)/\varepsilon\|_{0,\Omega_{12}\cup\Omega_{22}} \\ &\le C\,\varepsilon\,(N^{-1} \ln N)^r\end{aligned}$$

by using Lemma 3. On B-S-meshes, we start with

$$\left\| \frac{\partial^{r+1} E_{12}}{\partial x^{a+1} \partial y^b} \right\|_{0,K} \le |K|^{1/2} \left\| \frac{\partial^{r+1} E_{12}}{\partial x^{a+1} \partial y^b} \right\|_{0,\infty,K} \le C\, h^{1/2}_{K,x}\, h^{1/2}_{K,y}\, \varepsilon^{-b} \exp(-\beta_2(1-y_j)/\varepsilon)$$

where $|K| = h_{K,x}\, h_{K,y}$ and (6) were used. Hence, we obtain

$$\begin{aligned}\|(E_{12} - I^N E_{12})_x\|_{0,K} &\le C \sum_{a+b=r} h^{a+1/2}_{K,x} h^{b+1/2}_{K,y}\, \varepsilon^{-b} \exp(-\beta_2(1-y_j)/\varepsilon) \\ &\le C \sum_{a+b=r} N^{-(a+1/2)}\, \varepsilon^{-b}\, \varepsilon^{b+1/2}\, N^{-(b+1/2)} \le C\,\varepsilon^{1/2}\, N^{-(r+1)}\end{aligned}$$

by using $h_{K,x} \le C\, N^{-1}$ and Lemma 1. Collecting these estimates for all $K \subset \Omega_{12} \cup \Omega_{22}$, we end up with

$$\varepsilon^{1/2}\, \|(E_{12} - I^N E_{12})_x\|_{0,\Omega_{12}\cup\Omega_{22}} \le C\,\varepsilon\, N^{-r}$$

where we used that $N^2/2$ mesh cells belong to $\Omega_{12}\cup\Omega_{22}$. To estimate the E_{12}-term on $\Omega_{11}\cup\Omega_{21}$, we apply the estimate

$$\|(E_{12} - I^N E_{12})_x\|_{0,K} \le C\, N^{-1} \left[\|(E_{12})_{xx}\|_{0,K} + \|(E_{12})_{xy}\|_{0,K} \right]$$

which follows for $K \subset \Omega_{21}$ from Lemma 8 and for $K \subset \Omega_{11}$ from standard interpolation estimates. Taking into account (6), we see that we have to bound only $(E_{12})_{xy}$. We obtain

$$\varepsilon^{1/2}\, \|(E_{12} - I^N E_{12})_x\|_{0,\Omega_{11}\cup\Omega_{21}} \le C\, N^{-1}\, \varepsilon^{-1/2} \| \exp(-\beta_2(1-y)/\varepsilon)\|_{0,\Omega_{11}\cup\Omega_{21}} \le C\, N^{-(r+1)} \quad (17)$$

due to the choice of the transition point and Lemma 3.

For estimating the E_{21}-term, we start with considering the subdomain $\Omega_{21} \cup \Omega_{22}$. Using the same arguments as for bounding the E_{12}-term on $\Omega_{12} \cup \Omega_{22}$, we obtain

$$\varepsilon^{1/2}\, \|(E_{21} - I^N E_{21})_x\|_{0,\Omega_{21}\cup\Omega_{22}} \le \begin{cases} C\,(N^{-1} \ln N)^r, & \text{on S-meshes,} \\ C\, N^{-r}, & \text{on B-S-meshes,} \end{cases}$$

where (7) was taken into account. To get an estimate for the E_{21}-term on $\Omega_{11} \cup \Omega_{12}$, we apply the triangle inequality and obtain

$$\|(E_{21} - I^N E_{21})_x\|_{0,\Omega_{11}\cup\Omega_{12}} \le \|(E_{21})_x\|_{0,\Omega_{11}\cup\Omega_{12}} + \|(I^N E_{21})_x\|_{0,\Omega_{11}\cup\Omega_{12}}.$$

Using an inverse inequality, we get

$$\|(I^N E_{21})_x\|_{0,\Omega_{11}\cup\Omega_{12}} \le C\,N\,\|I^N E_{21}\|_{0,\Omega_{11}\cup\Omega_{12}} \le C\,N^{-r}$$

due to Lemma 11 and $|\Omega_{11} \cup \Omega_{12}| < 1$. Exploiting (7) and Lemma 3, we end up with

$$\|(E_{21})_x\|_{0,\Omega_{11}\cup\Omega_{12}} \le C\,\varepsilon^{-1/2}\,N^{-(r+1)}.$$

Hence, the estimate

$$\varepsilon^{1/2}\,\|(E_{21} - I^N E_{21})_x\|_{0,\Omega_{11}\cup\Omega_{12}} \le C\big(\varepsilon^{1/2}\,N^{-r} + N^{-(r+1)}\big) \tag{18}$$

is obtained.

To estimate the E_{22}-term on $\Omega_{11} \cup \Omega_{12}$, we apply the same technique as for the E_{21}-term on the same subdomain. On Ω_{22}, the anisotropic error estimate from Lemma 8 is used. For estimating the E_{22}-term on Ω_{21}, we apply the triangle inequality to obtain

$$\|(E_{22} - I^N E_{22})_x\|_{0,\Omega_{21}} \le \|(I^N E_{22})_x\|_{0,\Omega_{21}} + \|(E_{22})_x\|_{0,\Omega_{21}}.$$

Using (8) and Lemma 3 results in

$$\|(E_{22})_x\|_{0,\Omega_{21}} \le C\,N^{-(r+1)}.$$

For $K \subset \Omega_{21}$, we have

$$\|(I^N E_{22})_x\|_{0,K} \le C\,h_{K,x}^{-1}\,\|I^N E_{22}\|_{0,K} \le C\,h_{K,x}^{-1/2}\,h_{K,y}^{1/2}\,\|I^N E_{22}\|_{0,\infty,K}$$

by an inverse inequality and $|K| = h_{K,x} h_{K,y}$. Taking into consideration that $h_{K,y} \le C\,N^{-1}$, we get

$$\|(I^N E_{22})_x\|_{0,\Omega_{21}}^2 \le C \sum_{K\subset\Omega_{21}} h_{K,x}^{-1}\,N^{-1}\,\|I^N E_{22}\|_{0,\infty,K}^2 \le C\,N^{-1}\|I^N E_{22}\|_{0,\infty,\Omega_{21}}^2 \sum_{K\subset\Omega_{21}} h_{K,x}^{-1}.$$

On both types of meshes, one can show that $h_{K,x}^{-1} \le C\,N/\varepsilon$ holds. Hence, we obtain

$$\|(I^N E_{22})_x\|_{0,\Omega_{21}}^2 \le C\,N^{-1}\,\|I^N E_{22}\|_{0,\infty,\Omega_{21}}^2\,N^2\,N/\varepsilon \le C\,\varepsilon^{-1}\,N^{-2r}$$

where we used Lemma 11 and the fact that Ω_{21} contains $N^2/4$ mesh cells. Hence, we get

$$\varepsilon^{1/2}\|(E_{22} - I^N E_{22})_x\|_0 \le C\,N^{-r} \tag{19}$$

by collecting the estimates on the subdomains where $\varepsilon \le N^{-1}$ was used.

Putting together all above estimates, the assertion of this lemma follows immediately. □

4 Error analysis

The first step of our error analysis consists in proving that the stabilised discrete problem (15) is uniquely solvable. To this end, we will show that the stabilised bilinear form a^N is uniformly coercive with respect to the local projection norm $\|\cdot\|_{LP}$.

Lemma 14. *The stabilised bilinear form a^N is uniformly coercive with respect to the local projection norm $\|\cdot\|_{LP}$ since the estimate*

$$a^N(v,v) \ge \|v\|_{LP}^2 \qquad \forall v \in V$$

is satisfied.

Proof. Taking into account the definition of a^N and applying an integration by parts, we get for all $v \in V$ that

$$\begin{aligned} a^N(v,v) &= \varepsilon(\nabla v, \nabla v) + (b \cdot \nabla v + cv, v) + s^N(v,v) \\ &= \varepsilon|v|_1^2 + \left(c - \frac{1}{2}\operatorname{div} b, v^2\right) + s^N(v,v) \ge \|v\|_{LP}^2 \end{aligned}$$

where assumption (3) and the definition of $\|\cdot\|_{LP}$ were used. □

In comparison to residual-based stabilisation techniques like SDFEM, the local projection method is only weakly consistent. Hence, the consistency error has to be estimated.

Lemma 15. *Let u and u^N denote the solutions of* (10) *and* (15)*, respectively. Then, we have*

$$a^N(u - u^N, w^N) = s^N(u, w^N)$$

for all $w^N \in V^N$.

Proof. The statement follows immediately by subtracting the stabilised discrete problem (15) from the weak formulation (10). □

We proceed with showing an estimate for the stabilisation term s^N with special arguments.

Lemma 16. *Let the solution u of* (10) *fulfil Assumption 2. Then, the estimate*

$$\left|s^N(I^N u, w^N)\right| \le C\, N^{-(r+1/2)} \left\|w^N\right\|_{LP} \qquad \forall w^N \in V^N$$

holds where the interpolation operator I^N is defined in (11)*. The constant C does not depend on ε and N.*

Proof. Exploiting Assumption 2, we obtain

$$s^N(I^N u, w^N) = s^N(I^N S - S, w^N) + s^N(S, w^N) + \sum_{ij} s^N(I^N E_{ij}, w^N)$$

where the ij-sum runs through $\{12, 21, 22\}$. Applying estimate (13) to each term separately results always in the factor $\left(s^N(w^N, w^N)\right)^{1/2}$ which is part of the local projection norm $\|w^N\|_{LP}$.

Hence, we have to estimate terms of the form $s^N(w,w)$ with $w = I^N S - S$, $w = S$, or $w = E_{ij}$. Since the stabilisation parameter τ_K is non-zero only on Ω_{11}, the sum in the definition of s^N is just a sum over $K \subset \Omega_{11}$.

We first handle the term which contains S only. We obtain

$$\begin{aligned}\sum_{K\subset\Omega_{11}} \tau_K \|\kappa_K(b\cdot\nabla S)\|_{0,K}^2 &\le C \sum_{K\subset\Omega_{11}} C_1\, N^{-1}\, N^{-2r}\, |b\cdot\nabla S|_{r,K}^2 \\ &\le C \sum_{K\subset\Omega_{11}} N^{-(2r+1)}\, \|S\|_{r+1,K}^2 \le C N^{-(2r+1)}\end{aligned}$$

where the smoothness of b, Lemma 5 with $s = r$, the choice (14) for τ_K, and the bound (9) were used.

The next estimate considers the term with the difference $I^N S - S$. We get

$$\begin{aligned}\sum_{K\subset\Omega_{11}} \tau_K \|\kappa_K(b\cdot\nabla(I^N S - S))\|_{0,K}^2 &\le C \sum_{K\subset\Omega_{11}} C_1\, N^{-1}\, \|\nabla(I^N S - S)\|_{0,K}^2 \\ &\le C \sum_{K\subset\Omega_{11}} N^{-(2r+1)} \|S\|_{r+1,K}^2 \le C\, N^{-(2r+1)}\end{aligned}$$

where Lemma 5 with $s = 0$, the smoothness of b, the choice of τ_K, standard interpolation error estimates, and the bound (9) were used.

We have to estimate finally the terms which contain the layers functions E_{ij}. To this end, let E denote one of the three exponential functions. If one applies Lemma 5 with $s = 0$, one obtains

$$\|\kappa_K(b\cdot\nabla I^N E)\|_{0,K} \le C\, |I^N E|_{1,K} \le C\, N\, \|I^N E\|_{0,K} \le C\, N\, |K|^{1/2}\, \|I^N E\|_{0,\infty,K}$$

where the smoothness of b and an inverse inequality were used. Adding all these estimates for $K \subset \Omega_{11}$, we get

$$\begin{aligned}\sum_{K\subset\Omega_{11}} \tau_K \|\kappa_K(b\cdot\nabla I^N E)\|_{0,K}^2 &\le C \sum_{K\subset\Omega_{11}} N^{-1}\, N^2\, |K|\, \|I^N E\|_{0,\infty,K}^2 \\ &\le C\, N \left(\sum_{K\subset\Omega_{11}} |K|\right) \|I^N E\|_{0,\infty,\Omega_{11}} \le C N^{-(2r+1)}\end{aligned}$$

where the choice of τ_K, $|\Omega_{11}| < 1$, and Lemma 11 were exploited.

Putting together all previous estimates, the statement of this lemma follows. □

The convective and reactive terms in the bilinear form a are estimates by the following lemma.

Lemma 17. *Let the solution u of* (10) *fulfil Assumption 2. Then, there exists a constant C which is independent of ε and N such that the estimate*

$$\left|(b\cdot\nabla(I^N u - u) + c(I^N u - u), w^N)\right| \le C\, N^{-(r+1/2)}\, \|w^N\|_{LP} \qquad \forall w^N \in V^N$$

holds true where the interpolation operator I^N is defined in (11).

Proof. The convective term is integrated by parts to obtain

$$\begin{aligned}(b\cdot\nabla(I^N u-u)+c(I^N u-u),w^N) &= ((c-\operatorname{div} b)w^N, I^N u-u)\\ &\quad-(I^N u-u, b\cdot\nabla w^N)_{\Omega_{11}}-(I^N u-u, b\cdot\nabla w^N)_{\Omega\setminus\Omega_{11}}.\end{aligned}$$

Using the Cauchy–Schwarz inequality, the smoothness of b, c, and Lemma 12, we get

$$\left|((c-\operatorname{div} b)w^N, I^N u-u)\right| \le C\|w^N\|_0\,\|I^N u-u\|_0 \le \begin{cases} C(N^{-1}\ln N)^{r+1}\,\|w^N\|_{LP}, & \text{on S-meshes,}\\ CN^{-(r+1)}\,\|w^N\|_{LP}, & \text{on BS-meshes,}\end{cases}$$

which gives for both types of meshes the estimate

$$\left|((c-\operatorname{div} b)w^N, I^N u-u)\right| \le CN^{-(r+1/2)}\,\|w^N\|_{LP}.$$

To estimate the integral on Ω_{11}, the orthogonality property of I^N given in Lemma 6 is exploited. One obtains

$$\begin{aligned}(I^N u-u, b\cdot\nabla w^N)_{\Omega_{11}} &= \sum_{K\subset\Omega_{11}}\big(I^N u-u, b\cdot\nabla w^N-\pi_K(b\cdot\nabla w^N)\big)_K\\ &= \sum_{K\subset\Omega_{11}}\big(I^N u-u, \kappa_K(b\cdot\nabla w^N)\big)_K.\end{aligned}$$

Using the choice $\tau_K = C_1\,N^{-1}$ and applying Lemma 12 results in

$$\begin{aligned}\left|(I^N u-u, b\cdot\nabla w^N)_{\Omega_{11}}\right| &\le \left(\sum_{K\subset\Omega_{11}}\tau_K^{-1}\|I^N u-u\|_{0,K}^2\right)^{1/2}\left(\sum_{K\subset\Omega_{11}}\tau_K\|\kappa_K(b\cdot\nabla w^N)\|_{0,K}^2\right)^{1/2}\\ &\le CN^{-(r+1/2)}\|w^N\|_{LP}.\end{aligned}$$

Note that this estimate limits the convergence order to $\mathcal{O}(N^{-(r+1/2)})$. It remains to estimate the integral on $\Omega\setminus\Omega_{11}$. To this end, a Hölder inequality yields

$$\left|(I^N u-u, b\cdot\nabla w^N)_{\Omega\setminus\Omega_{11}}\right| \le \|I^N u-u\|_{0,\infty,\Omega\setminus\Omega_{11}}\|b\cdot\nabla w^N\|_{0,1,\Omega\setminus\Omega_{11}}.$$

The first factor can be estimated by using again Lemma 12. To bound the second term, we estimate

$$\|b\cdot\nabla w^N\|_{0,1,\Omega\setminus\Omega_{11}} \le C(\ln N)^{1/2}\,\varepsilon^{1/2}\,|w^N|_{1,\Omega\setminus\Omega_{11}} \le C(\ln N)^{1/2}\,\|w^N\|_{LP}$$

where we used the $|\Omega\setminus\Omega_{11}|\le C\varepsilon\ln N$ and the smoothness of b. Hence, we get

$$\left|(I^N u-u, b\cdot\nabla w^N)_{\Omega\setminus\Omega_{11}}\right| \le C\begin{cases} N^{-(r+1)}(\ln N)^{r+3/2}\,\|w^N\|_{LP}, & \text{on S-meshes,}\\ N^{-(r+1)}(\ln N)^{1/2}\,\|w^N\|_{LP}, & \text{on B-S-meshes.}\end{cases}$$

In both cases, the estimate

$$\left|(I^N u-u, b\cdot\nabla w^N)_{\Omega\setminus\Omega_{11}}\right| \le C\,N^{-(r+1/2)}\,\|w^N\|_{LP}$$

is obtained. Putting together all estimates from above, the statement of the lemma follows. □

The diffusion term of the bilinear form a is handled by the next lemma.

Lemma 18. *Let the solution u of* (10) *satisfy Assumption 2. Then, there exists a constant C which is independent of ε and N such that the estimate*

$$\left|\varepsilon(\nabla(I^N u - u), \nabla w^N)\right| \le C\, N^{-(r+1/2)}\, \|w^N\|_{1,\varepsilon}$$

holds true for all $w^N \in V^N$.

Proof. We will present in this proof only the estimates for terms which contain derivatives in x-direction since the terms with y-derivatives can be handled by the same arguments.

Using Assumption 2, we can write

$$I^N u - u = (I^N S - S) + (I^N E_{12} - E_{12}) + (I^N E_{21} - E_{21}) + (I^N E_{22} - E_{22}).$$

Each term will be estimate separately.

We start with the S-term and obtain

$$\left|\varepsilon((I^N S - S)_x, w_x^N)\right| \le \varepsilon^{1/2} \|(I^N S - S)_x\|_0\, \varepsilon^{1/2} \|w_x^N\|_0 \le C\, N^{-(r+1/2)} \|w^N\|_{1,\varepsilon}$$

by the Cauchy–Schwarz inequality, $\varepsilon \le N^{-1}$, and (16).

To estimate the E_{12}-term, we first consider an arbitrary $K \subset \Omega_{12} \cup \Omega_{22}$ where the size in y-direction is small. Applying Lemma 7, we get

$$\left|\varepsilon((I^N E_{12} - E_{12})_x, w_x^N)_K\right| \le C\, \varepsilon\, h_{K,y}^{r+1} \left\| \frac{\partial^{r+2} E_{12}}{\partial x \partial^{r+1} y} \right\|_{0,K} \|w_x^N\|_{0,K}.$$

On S-meshes, using $h_{K,y} \le C\, \varepsilon\, N^{-1} \ln N$ and (6), we obtain

$$\begin{aligned}
&\left|\varepsilon((I^N E_{12} - E_{12})_x, w_x^N)_{\Omega_{12}\cup\Omega_{22}}\right| \\
&\qquad\qquad \le C\, \varepsilon\, (\varepsilon N^{-1} \ln N)^{r+1} \varepsilon^{-(r+1)}\, \|\exp(-\beta_2(1-y)/\varepsilon)\|_{\Omega_{12}\cup\Omega_{22}}\, \|w_x^N\|_{0,\Omega_{12}\cup\Omega_{22}} \\
&\qquad\qquad \le C\, \varepsilon^{1/2}\, (N^{-1} \ln N)^{r+1}\, \|w^N\|_{1,\varepsilon}.
\end{aligned}$$

On B-S-meshes, we get on $K = (x_{i-1} - x_i) \times (y_{j-1}, y_j)$ by using (6), $|K| \le C\, N^{-2}$, and Lemma 1

$$h_{K,y}^{r+1} \|\exp(-\beta_2(1-y)/\varepsilon)\|_{0,K} \le h_{K,y}^{r+1}\, |K|^{1/2}\, \|\exp(-\beta_2(1-y)/\varepsilon)\|_{0,\infty} \le C\, N^{-(r+2)}.$$

Hence, we obtain

$$\left|\varepsilon((I^N E_{12} - E_{12})_x, w_x^N)_{\Omega_{12}\cup\Omega_{22}}\right| \le C\, \varepsilon^{1/2}\, N^{-(r+1)}\, \|w^N\|_{1,\varepsilon}$$

on B-S-meshes where we used that $\Omega_{12} \cup \Omega_{22}$ contains $N^2/2$ mesh cells. In order to get a bound for the E_{12}-term on $K \subset \Omega_{11} \cup \Omega_{21}$, the Cauchy–Schwarz inequality and (17) are used to obtain

$$\begin{aligned}
\left|\varepsilon((I^N E_{12} - E_{12})_x, w_x^N)_{\Omega_{11}\cup\Omega_{21}}\right| &\le C \varepsilon^{1/2} \|(I^N E_{12} - E_{12})_x\|_{0,\Omega_{11}\cup\Omega_{21}}\, \varepsilon^{1/2} \|w_x^N\|_{0,\Omega_{11}\cup\Omega_{21}} \\
&\le C\, N^{-(r+1)} \|w^N\|_{1,\varepsilon}.
\end{aligned}$$

Putting together all estimates for the E_{12}-term, we get

$$\left|\varepsilon((I^N E_{12} - E_{12})_x, w_x^N)\right| \le C\, N^{-(r+1/2)}\, \|w^N\|_{1,\varepsilon}$$

where $\varepsilon \le N^{-1}$ was exploited.

We continue with estimating the E_{21}-term. On $\Omega_{21} \cup \Omega_{22}$, we use Lemma 7 and the same technique as for estimating the E_{12}-term on $\Omega_{12} \cup \Omega_{22}$ but with exchanging the roles of x and y. In order to estimate the E_{21}-term on $\Omega_{11} \cup \Omega_{12}$, we apply the Cauchy–Schwarz inequality and (18) and get

$$\left|\varepsilon((I^N E_{21} - E_{21})_x, w_x^N)_{\Omega_{11}\cup\Omega_{12}}\right| \le C\, N^{-(r+1/2)} \|w^N\|_{1,\varepsilon}$$

where $\varepsilon \le N^{-1}$ was exploited. Collecting all estimates for the E_{21}-term gives

$$\left|\varepsilon((I^N E_{21} - E_{21})_x, w_x^N)\right| \le C\, N^{-(r+1/2)} \|w^N\|_{1,\varepsilon}.$$

For estimating the E_{22} on $\Omega_{12} \cup \Omega_{22}$, we apply again Lemma 7 as for the E_{12}-term on the same subdomain. On $\Omega_{11} \cup \Omega_{21}$, we use apply the Cauchy–Schwarz inequality and use (19).

All above estimates together give the assertion of this lemma. □

After all these preparations, we can state our main result.

Theorem 19. *Let the solution u of* (10) *fulfil Assumption 2. The solution of the stabilised discrete problem* (15) *is denoted by u^N. Then, there exists a constant C independent of ε and N such that*

$$\|I^N u - u^N\|_{LP} \le C\, N^{-(r+1/2)}$$

and

$$\|u - u^N\|_{1,\varepsilon} \le \begin{cases} C\,(N^{-1}\ln N)^r, & \text{on S-meshes,} \\ C\,N^{-r} & \text{on B-S-meshes,} \end{cases}$$

hold true.

Proof. The coercivity of a^N proved in Lemma 14 gives

$$\begin{aligned} \|I^N u - u^N\|_{LP}^2 &\le a^N(I^N u - u^N, I^N u - u^N) \\ &= a^N(I^N u - u, I^N u - u^N) + a^N(u - u^N, I^N u - u^N) \\ &= a(I^N u - u, I^N u - u^N) + s^N(I^N u, I^N u - u^N) \end{aligned} \tag{20}$$

where Lemma 15 and the definition of bilinear form a^N were exploited. The second term in (20) can be estimated by using Lemma 16. We obtain

$$|s^N(I^N u, I^N u - u^N)| \le C N^{-(r+1/2)}\, \|I^N u - u^N\|_{LP}.$$

For estimating the first term in (20), we use the definition of the bilinear form a and get

$$\begin{aligned} &|a(I^N u - u, I^N u - u^N)| \\ &\qquad \le |\varepsilon(\nabla(I^N u - u), \nabla(I^N u - u^N))| + |(b\cdot\nabla(I^N u - u) + c(I^N u - u), I^N u - u^N)| \\ &\qquad \le C\, N^{-(r+1/2)}\, \|I^N u - u^N\|_{LP} \end{aligned}$$

where Lemma 17 and Lemma 18 were applied. Putting these estimate into (20), the first statement of this theorem follows.

Using the triangle inequality, we obtain

$$\|u - u^N\|_{1,\varepsilon} \le \|I^N u - u^N\|_{1,\varepsilon} + \|u - I^N u\|_{1,\varepsilon}.$$

The first term is bounded by $\|I^N u - u^N\|_{LP}$. The second term can be estimated by using the definition of $\|\cdot\|_{1,\varepsilon}$ and the bounds from Lemma 12 and Lemma 13. Hence, the desired assertion is proved. □

5 Numerical results

We consider the problem

$$\begin{aligned} -\varepsilon\triangle u + (3-x, 4-y)\nabla u + u &= f \qquad \text{in } \Omega = (0,1)^2,\\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where f was chosen such that

$$u(x,y) = \sin(x)\left(1 - e^{-2(1-x)/\varepsilon}\right)\sin(2y)\left(1 - e^{-3(1-y)/\varepsilon}\right)$$

is the exact solution. The function u shows the typical behaviour with boundary layers. Furthermore, Assumption 2 is fulfilled.

The used finite element space V^N is based on Q_3^+-elements on Ω_{11} and Q_3-elements on $\Omega \setminus \Omega_{11}$. Calculations have been performed on S-meshes and B-S-meshes. We have chosen the stabilisation parameter $\tau_K = N^{-1}$ on Ω_{11}.

All calculations were obtained by using the program package MooNMD [12]. The systems of linear equations which correspond to the stabilised discrete problems have been solved directly by using UMFPACK [6–9].

In the following, 'ord' will correspond always to the exponent α in a convergence order of the form $\mathcal{O}(N^{-\alpha})$ while 'ln-ord' denotes the exponent α in a convergence order of the form $\mathcal{O}((N^{-1}\ln N)^\alpha)$.

Table 1 shows for different values of the mesh parameter N the error $u^N - I^N u$ in the local projection norm and the error $u - u^N$ in the ε-weighted H^1-norm. The error $\|u - u^N\|_{1,\varepsilon}$ convergences with the order predicted by Theorem 19. We clearly see the difference between the S-mesh and the B-S-mesh. The results on the B-S-mesh are much smaller than those on the S-mesh. The error in the ε-weighted H^1-norm on S-meshes shows the typical logarithmic factor which is not present on B-S-meshes. Concerning the error $\|u^N - I^N u\|_{LP}$, it is obvious that the predicted convergence order is achieved on B-S-meshes. On S-meshes, the situation is a little different since it seems than the predicted convergence order is not obtained. However, a careful look at the proof of Lemma 17 shows that error term $|(I^N u - u, b\cdot w^N)_{\Omega\setminus\Omega_{11}}|$ converges with the order $\mathcal{O}(N^{-(r+1)}\ln^{r+3/2} N)$ which dominates for the presented values of N. Moreover, the results on B-S-meshes are again much smaller than those on S-meshes.

Table 2 shows the error norms $\|u^N - I^N u\|_{LP}$ and $\|u - u^N\|_{1,\varepsilon}$ for different values of $\varepsilon \in \{10^{-4}, 10^{-6}, 10^{-8}, 10^{-10}, 10^{-12}\}$. It is clearly to see that the error estimates are robust with respect to the diffusion parameter ε since the errors are uniformly bounded in ε. Note again that the error on B-S-meshes are much smaller than on S-meshes. The difference in the ε-weighted H^1-norm are two orders of magnitude.

Table 1: $Q_3^+ - Q_3$, $\varepsilon = 10^{-8}$, S-mesh and B-S-mesh.

	$\|u^N - I^N u\|_{LP}$				$\|u - u^N\|_{1,\varepsilon}$				
	S-mesh		B-S-mesh		S-mesh			B-S-mesh	
N	error	ord	error	ord	error	ord	ln-ord	error	ord
2	2.026-01		2.026-01		6.122-02			6.122-02	
4	3.330-02	2.60	2.838-02	2.84	3.995-02	0.62	1.48	1.883-02	1.70
8	4.669-03	2.83	2.335-03	3.60	1.947-02	1.04	1.77	3.722-03	2.34
16	8.070-04	2.53	1.758-04	3.73	6.523-03	1.58	2.33	5.685-04	2.71
32	1.266-04	2.67	1.361-05	3.69	1.704-03	1.94	2.63	7.813-05	2.86
64	1.678-05	2.92	1.087-06	3.65	3.790-04	2.17	2.79	1.023-05	2.93
128	1.960-06	3.10	8.973-08	3.60	7.609-05	2.32	2.87	1.309-06	2.97
256	2.096-07	3.22	7.613-09	3.56	1.425-05	2.42	2.91	1.656-07	2.98
512	2.099-08	3.32	6.547-10	3.54	2.540-06	2.49	2.93	2.082-08	2.99

Table 2: $Q_3^+ - Q_3$, $N = 512$, S-mesh and B-S-mesh.

	$\|u^N - I^N u\|_{LP}$		$\|u - u^N\|_{1,\varepsilon}$	
ε	S-mesh	B-S-mesh	S-mesh	B-S-mesh
10^{-4}	2.092200-8	1.257126-10	2.539967-6	2.081494-8
10^{-6}	2.095331-8	4.429800-10	2.540087-6	2.081606-8
10^{-8}	2.099274-8	6.547342-10	2.540089-6	2.081608-8
10^{-10}	2.099442-8	6.588598-10	2.540089-6	2.081608-8
10^{-12}	2.099533-8	6.591398-10	2.540089-6	2.081608-8

6 Conclusions

We have applied the local projection method to higher order discretisations of convection-diffusion problems on two types of layer-adapted meshes, Shishkin meshes and Bakhvalov–Shishkin meshes. For both mesh types, an ε-uniform convergence order $\mathcal{O}(N^{-(r+1/2)})$ for the difference between the solution u^N of the stabilised discrete problem (15) and the interpolant $I^N u$ of the solution u of the continuous problem (10) was shown. Although the predicted order is the same on both types meshes, the errors on Bakhvalov–Shishkin meshes are much smaller. The convergence order of $\|u - u^N\|_{1,\varepsilon}$ shows on Shishkin meshes the typical logarithmic factor which is not present on Bakhvalov–Shishkin meshes. Furthermore, the errors on Bakhvalov–Shishkin meshes are again much smaller than on Shishkin meshes.

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