A SPLITTING OF $\mathrm{TMF}_0(7)$

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Abstract. We provide a splitting of $\mathrm{TMF}_0(7)$ at the prime 3 as TMF-module into two shifted copies of TMF and two shifted copies of $\mathrm{TMF}_1(2)$.

Contents

1. Introduction

The study of modules over the real K-theory spectrum KO has been central in Bousfield's work on the classification of K-local spectra [5]. If we localize further at a prime p, localization at K-theory becomes equivalent to localization at the first Johnson–Wilson theory $E(1)$. If we want to study $E(2)$-local spectra, topological modular forms are a natural substitute for KO.

Topological modular forms come in many variants. First, there is the periodic version TMF that is based on the moduli stack of elliptic curves $\mathcal{M}_{ell}$. It has the disadvantage that its homotopy groups are infinitely generated in most degrees, which is different in the refinement Tmf that is based on the compactified moduli stack $\overline{\mathcal{M}}_{ell}$. Its connective cover is called tmf. We refer to [15] as a basic reference for these spectra.

It has a long tradition in arithmetic geometry not only to consider the moduli stack of elliptic curves itself, but also to consider moduli of elliptic curves with level structures. A $\Gamma_0(n)$-level structure on an elliptic curve E/S is a sub-group scheme that is étale locally on S isomorphic to $(\mathbb{Z}/n)_S$. A $\Gamma_1(n)$-level structure on E is a sub-group scheme of E with a chosen isomorphism to $(\mathbb{Z}/n)_S$. This leads to moduli stacks $\mathcal{M}_0(n)$ and $\mathcal{M}_1(n)$ and to spectra $\mathrm{TMF}_0(n)$ and $\mathrm{TMF}_1(n)$. Hill and Lawson [21] were able to define spectra $\mathrm{Tmf}_0(n)$ and $\mathrm{Tmf}_1(n)$ based on the compactified moduli $\overline{\mathcal{M}}_0(n)$ and $\overline{\mathcal{M}}_1(n)$ as well. Note that n is here always inverted.

When studying Tmf-modules, $\mathrm{Tmf}_0(n)$ and $\mathrm{Tmf}_1(n)$ are among the first examples to consider. In [35], the first-named author has proven splittings for $\mathrm{Tmf}_1(n)$ and $\mathrm{Tmf}_0(n)$ in many cases if we localize at a prime p. If $p > 3$, there is an explicit criterion when these modules are free over Tmf. If $p = 3$, the splittings are into shifted copies of $\mathrm{Tmf}_1(2)_{(3)}$. As $\pi_* \mathrm{Tmf}_1(2)_{(3)}$ is torsionfree, splittings into shifted copies of $\mathrm{Tmf}_1(2)_{(3)}$ can only exist if $\pi_* \mathrm{Tmf}_0(n)_{(3)}$ is also torsionfree, which is not expected if 3 divides $|(\mathbb{Z}/n)^\times|$. The first case where this occurs is $\mathrm{Tmf}_0(7)$, where we can prove the following modified splitting result.

Theorem 1.1. *The* $\mathrm{Tmf}_{(3)}$*-module* $\mathrm{Tmf}_0(7)_{(3)}$ *decomposes as*

$$\mathrm{Tmf}_{(3)} \oplus \Sigma^4 \mathrm{Tmf}_1(2)_{(3)} \oplus \Sigma^8 \mathrm{Tmf}_1(2)_{(3)} \oplus L,$$

where $L \in \mathrm{Pic}(\mathrm{Tmf}_{(3)})$*, i.e.* L *is an invertible* $\mathrm{Tmf}_{(3)}$*-module.*

The $\mathrm{TMF}_{(3)}$*-module* $\mathrm{TMF}_0(7)_{(3)}$ *decomposes as*

$$\mathrm{TMF}_{(3)} \oplus \Sigma^4 \mathrm{TMF}_1(2)_{(3)} \oplus \mathrm{TMF}_1(2)_{(3)} \oplus \Sigma^{36} \mathrm{TMF}_{(3)}.$$

Using unpublished work of M. Olbermann, one can deduce that L is actually an exotic Picard element, i.e. is not of the form $\Sigma^k \mathrm{Tmf}_{(3)}$ for any k. The group $\mathrm{Pic}(\mathrm{Tmf}_{(3)})$ was determined in [34] and one can explictly identify L. This shows that exotic Picard group elements of Tmf actually occur quite naturally.

Our main theorem is based on the following algebraic theorem.

Theorem 1.2. *Let* $h\colon \overline{\mathcal{M}}_0(7)_{(3)} \to \overline{\mathcal{M}}_{ell,(3)}$ *and* $f\colon \overline{\mathcal{M}}_1(2)_{(3)} \to \overline{\mathcal{M}}_{ell,(3)}$ *be the maps induced by forgetting the level structure on an elliptic curve and let* $\mathcal{O}$ *be the structure sheaf of* $\overline{\mathcal{M}}_{ell,(3)}$*. Then the quasi-coherent sheaf* $h_*\mathcal{O}_{\overline{\mathcal{M}}_0(7)} \cong h_*h^*\mathcal{O}$ *on* $\overline{\mathcal{M}}_{ell,(3)}$ *is a vector bundle of rank* 8*, which can be decomposed as a sum*

$$\mathcal{O} \oplus \underline{\omega}^{-6} \oplus f_*f^*\mathcal{O} \otimes \underline{\omega}^{-2} \oplus f_*f^*\mathcal{O} \otimes \underline{\omega}^{-4}.$$

Here, $\underline{\omega}$ *is the generator of* $\mathrm{Pic}(\overline{\mathcal{M}}_{ell})$ *that can be constructed as the pushforward of the sheaf of differentials on the universal generalized elliptic curve.*

Let us simultaneously describe the proof strategy and give an overview of the different sections. We will always work (implicitly) 3-locally.

Let $\mathrm{mf}_1(7) = \mathrm{mf}(\Gamma_1(7), \mathbb{Z}_{(3)})$ be the subring of the ring of holomorphic $\Gamma_1(7)$-modular forms $\mathrm{mf}(\Gamma_1(7), \mathbb{C})$ with coefficients in $\mathbb{Z}_{(3)}$. This can be identified with the sections of $\underline{\omega}^{\otimes *}$ on $\overline{\mathcal{M}}_1(7)$. While abstractly this ring is easily identified, we will compute it together with its $(\mathbb{Z}/7)^\times$-action and explicit q-expansion of its generators. More precisely we exhibit an isomorphism $\mathrm{mf}_1(7) \cong \mathbb{Z}_{(3)}[z_1, z_2, z_3]/(z_1z_2+$

$z_2z_3 + z_3z_1$) and identify the $(\mathbb{Z}/7)^\times \cong \mathbb{Z}/2 \times \mathbb{Z}/3$-action to be given by the sign action of $\mathbb{Z}/2$ and a cyclic permutation action of $\mathbb{Z}/3$ on the z_i.

In the next step, we shows in Section 3 that the universal elliptic curve over $\mathcal{M}_1(n)$ always has a Weierstraß equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2,$$

where the a_i are *holomorphic* modular forms. Moreover, we provide an explicit method how to compute the q-expansions of the a_i. This allows us to identify them as concrete polynomials in the z_i.

Our strategy to show Theorem 1.2 is to show a statement about comodules. The precise relationship between quasi-coherent sheaves and graded comodules will be recalled in Section 4. It is both easier and yields stronger results to use this relationship not for $\overline{\mathcal{M}}_{ell}$ but for $\mathcal{M}_{cub}$ instead (as in [32]). The latter stack has a presentation by the Hopf algebroid (A, Γ) with $B = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$. To formulate a version of Theorem 1.2 on $\mathcal{M}_{cub}$ we define and explore cubical versions of $\mathcal{M}_1(n)$ and $\mathcal{M}_0(n)$ by a normalization procedure in Section 5. In particular, we provide a flatness criterion for the map $\mathcal{M}_1(n)_{cub} \to \mathcal{M}_{cub}$. We stress that $\mathcal{M}_0(n)_{cub}$ is not the stack quotient of $\mathcal{M}_1(7)_{cub}$ by the $(\mathbb{Z}/7)^\times$-action as this stack quotient is not representable over $\mathcal{M}_{cub}$.

The next step is to make the Hopf algebroids corresponding to $\mathcal{M}_1(7)_{cub}$ and $\mathcal{M}_0(7)_{cub}$ explicit. In Section 6, we produce explicit B-bases of B-algebras R_B and S_B, which are defined by

$$\operatorname{Spec} R_B \cong \mathcal{M}_1(7)_{cub} \times_{\mathcal{M}_{cub}} \operatorname{Spec} B \text{ and } \operatorname{Spec} S_B \cong \mathcal{M}_0(7)_{cub} \times_{\mathcal{M}_{cub}} \operatorname{Spec} B.$$

This allows us to prove in Section 7 a splitting of S_B as a comodule over (B, Γ), which implies Theorem 1.2. In Section 8 we apply standard techniques (the transfer and the descent spectral sequence) to deduce our topological main theorem.

We end with an appendix that gives an exposition of the theory of modular forms with level over general rings and their q-expansion. The reason for the length of this appendix is the subtle difference between so-called arithmetic and naive level structures, which only agree in the presence of an n-th root of unity. To achieve a q-expansion principle in the form we need, care is needed how to identify the sections of $\underline{\omega}^{\otimes *}$ on $\overline{\mathcal{M}}_1(n)_{\mathbb{C}}$ with holomorphic $\Gamma_1(n)$-modular forms in the classical sense.

After giving this overview, let us add two directions of further research. First, we can also ask for a splitting of the connective spectrum $\mathrm{tmf}_0(7) = \tau_{\geq 0} \mathrm{Tmf}_0(7)$. Indeed, our algebraic theorem suggests this as it works over not only over $\overline{\mathcal{M}}_{ell}$ but over $\mathcal{M}_{cub}$. We will come to this refined splitting and will show indeed that the stack associated with the ring spectrum $\mathrm{tmf}_0(7)$ is equivalent to $\mathcal{M}_0(7)_{cub}$. This opens the possibility that our definition of $\mathcal{M}_1(n)_{cub}$ and $\mathcal{M}_0(n)_{cub}$ is indeed the key to understand the stacks associated with $\mathrm{tmf}_1(n)$ and $\mathrm{tmf}_0(n)$ in general.

Secondly, our main topological theorem suggests the following conjecture.

Conjecture 1.3. *The spectrum* $\mathrm{TMF}_0(n)_{(3)}$ *decomposes for every* $n \geq 2$ *into shifted copies of* $\mathrm{TMF}_{(3)}$ *and of* $\mathrm{TMF}_1(2)_{(3)}$.

This is related to a question asked in [37], namely whether all vector bundles on $\mathcal{M}_{ell,(3)}$ decompose (up to tensoring with powers of $\underline{\omega}$) into the structure sheaf $\mathcal{O}$, the 3-dimensional indecomposable $f_*f^*\mathcal{O}$ and a certain vector bundle E_α of rank 2.

Furthermore, it is shown in [35] that $\mathrm{TMF}_1(n)_{(3)}$ always decomposes into shifted copies of $\mathrm{TMF}_1(2)_{(3)}$ after 3-completion.

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1.2. **Conventions.** All quotients of schemes by group schemes (like $\mathbb{G}_m$) are understood to be stack quotients. Unless clearly otherwise, all rings and algebras are assumed to be commutative and unital. Tensor products of quasi-coherent sheaves are always over the structure sheaf.

2. Modular forms of level 7

Our goal in this section is to understand the ring of modular forms $\mathrm{mf}(\Gamma_1(7);\mathbb{Z})$ with respect to the congruence group $\Gamma_1(7) \subset \mathrm{SL}_2(\mathbb{Z})$ together with the action of $(\mathbb{Z}/7)^\times \cong \mathbb{Z}/6$. We refer for the background about modular forms and in particular the q-expansion principle to the appendix.

We begin with some recollections about algebraic geometry. We denote by $\mathcal{M}_{ell}$ the moduli stack of elliptic curves and by $\overline{\mathcal{M}}_{ell}$ its compactification in the sense of $\mathfrak{M}_1$ of [12, Chapter III], i.e. the stack classifying generalized elliptic curves whose geometric fibers are elliptic curves or Néron 1-gons. By $\mathcal{M}_1(n)$ we denote the moduli stack of elliptic curves E with chosen *point* $P\colon S \to E$ *of exact order* n over schemes S with n invertible and by $\mathcal{M}_0(n)$ the moduli stack of elliptic curves with chosen *cyclic subgroup of order* n over such schemes. More precisely, we demand for $\mathcal{M}_1(n)$ that for every geometric point $s\colon \operatorname{Spec} K \to S$ the pullback s^*P spans a cyclic subgroup of order n in $E(K)$ or, equivalently, that P defines a closed immersion $(\mathbb{Z}/n)_S \to E$. Moreover, we call a group scheme over S *cyclic* if it is étale locally isomorphic to $(\mathbb{Z}/n)_S$.

We can define compactifications $\overline{\mathcal{M}}_1(n)$ and $\overline{\mathcal{M}}_0(n)$ as the normalizations of $\overline{\mathcal{M}}_{ell}$ in $\mathcal{M}_1(n)$ and $\mathcal{M}_0(n)$, respectively (for the definition of normalization see Section 5). The relevance for our purposes is that we have isomorphisms

$$\mathrm{mf}_k(\Gamma_1(n);\mathbb{Z}[\tfrac{1}{n}]) \cong H^0(\overline{\mathcal{M}}_1(n);\underline{\omega}^{\otimes k}) \text{ and } \mathrm{mf}_k(\Gamma_0(n);\mathbb{Z}[\frac{1}{n}]) \cong H^0(\overline{\mathcal{M}}_0(n);\underline{\omega}^{\otimes k}).$$

The relevant case for us is the first one and is explained in Appendix A.

Lemma 2.1. *The stack* $\overline{\mathcal{M}}_1(n)$ *is equivalent to* $\mathbb{P}^1_{\mathbb{Z}[\frac{1}{n}]}$ *for* $5 \leq n \leq 10$ *and* $n = 12$. *For* $n = 7$, *the line bundle* $\underline{\omega}$ *corresponds to* $\mathcal{O}(2)$.

Proof. The first statement is proven in Section 2 of [38]. The Picard group of $\mathbb{P}^1_{\mathbb{Z}[\frac{1}{n}]}$ is isomorphic to $\mathbb{Z}$. Indeed, by [19, Prop 6.5c], we have a short exact sequence

$$0 \to \mathbb{Z} \to \operatorname{Pic}\mathbb{P}^1_{\mathbb{Z}[\frac{1}{n}]} \to \operatorname{Pic}\mathbb{A}^1_{\mathbb{Z}[\frac{1}{n}]} \to 0,$$

where the first map is split by degree and $\operatorname{Pic}\mathbb{A}^1_{\mathbb{Z}[\frac{1}{n}]} \cong \operatorname{Pic}\mathbb{Z}[\frac{1}{n}] = 0$ by [19, Prop 6.6].

Thus, we have only to compute the degree of $\underline{\omega}$ on $\overline{\mathcal{M}}_1(7)$. In general, the degree of $\underline{\omega}$ on $\overline{\mathcal{M}}_1(n)$ is $\frac{1}{24}n^2\prod_{p|n}(1-\frac{1}{p^2})$ [38, Lemma 4.3]. So we conclude that for $n=7$, the degree is 2. $\square$

This directly implies that $\mathrm{mf}(\Gamma_1(7);\mathbb{Z}[\frac{1}{7}])$ is isomorphic to the subalgebra of $\mathbb{Z}[\frac{1}{7}][x,y]$ of polynomials of even degree. Next, we want to determine the action of $(\mathbb{Z}/7)^\times\cong\mathbb{Z}/6$ on $\mathrm{mf}(\Gamma_1(7);\mathbb{Z}[\frac{1}{7}])$. Observe that the action of $\Gamma_0(7)/\Gamma_1(7)\cong(\mathbb{Z}/7)^\times$ by precomposition is the same as the action induced by the inverse of the $(\mathbb{Z}/7)^\times\cong\mathbb{Z}/6$-action on the torsion points of precise order 7 in the modular interpretation as discussed in Remark A.9.

We will use Eisenstein series to identify $\mathrm{mf}_1(\Gamma_1(7);\mathbb{C})$ explicitly together with the $(\mathbb{Z}/7)^\times$-action on it. As explained in [14, Section 3.9], the genus of $X_1(7)\cong\overline{\mathcal{M}}_1(7)_{\mathbb{C}}$ is 0 (cf. Appendix A for discussion), and using [14, Theorem 3.6.1], we conclude that there are no cusp forms of weight 1 in $\mathrm{mf}(\Gamma_1(7);\mathbb{C})$. Thus there is a basis of Eisenstein series of weight 1 for $\mathrm{mf}_1(\Gamma_1(7);\mathbb{C})$, which we will describe.

Fix the generator $t=[3]$ in $(\mathbb{Z}/7)^\times$, inducing an isomorphism $(\mathbb{Z}/7)^\times\cong\mathbb{Z}/6$. Then the three odd characters $\varphi_1,\varphi_2,\varphi_3\colon(\mathbb{Z}/7)^\times\to\mathbb{C}^\times$ are described by

$$\begin{aligned}\varphi_1(t)&=\zeta_6,\\ \varphi_2(t)&=-1,\\ \varphi_3(t)&=-\zeta_6+1,\end{aligned}$$

where $\zeta_6=\exp(\frac{2\pi i}{6})$ is a sixth primitive root of unity.

By [14, Theorem 4.8.1], there are (modified) Eisenstein series $E(\varphi_1),E(\varphi_2),E(\varphi_3)$ which form the basis of $\mathrm{mf}_1(\Gamma_1(7);\mathbb{C})$ and on which the $\mathbb{Z}/6$-action is described exactly by the multiplication with the respective character. From [14, Section 4.8 and Formula (4.33)] (or [7]) we obtain

$$E(\varphi_j)(\tau)=-\frac{1}{14}\sum_{n=1}^{6}n\varphi_j(n)+\sum_{k=1}^{\infty}\left(\sum_{l|k,l>0}\varphi_j(l)\right)q^k,\ \text{with } q=\exp(2\pi i\tau).$$

`MAGMA`-calculations suggest to consider the following modular forms in $\mathrm{mf}_1(\Gamma_1(7);\mathbb{C})$:

$$\begin{aligned}z_1&=\frac{1}{3}(3\zeta_6-1)E(\varphi_1)+\frac{2}{3}E(\varphi_2)+\frac{1}{3}(-3\zeta_6+2)E(\varphi_3),\\ z_2&=\frac{1}{3}(-\zeta_6-2)E(\varphi_1)+\frac{2}{3}E(\varphi_2)+\frac{1}{3}(\zeta_6-3)E(\varphi_3),\\ z_3&=\frac{1}{3}(-2\zeta_6+3)E(\varphi_1)+\frac{2}{3}E(\varphi_2)+\frac{1}{3}(2\zeta_6+1)E(\varphi_3).\end{aligned}$$

Note that the base change matrix

$$\frac{1}{3}\begin{pmatrix}3\zeta_6-1 & -\zeta_6-2 & -2\zeta_6+3\\ 2 & 2 & 2\\ -3\zeta_6+2 & \zeta_6-3 & 2\zeta_6+1\end{pmatrix}$$

has determinant $\frac{2}{27}(84\zeta_6-42)$, which is invertible in $\mathbb{C}$. Thus, z_1,z_2,z_3 form a new $\mathbb{C}$-basis of $\mathrm{mf}_1(\Gamma_1(7);\mathbb{C})$.

Lemma 2.2. *The z_j have only $\mathbb{Z}$-coefficients in their q-expansion.*

Proof. Denote the coefficient of q^n in z_j by $c_n(z_j)$.

First, we compute $c_0(z_j)$. This calculation is somewhat different from the ones for higher coefficients:

$$\begin{aligned} c_0(z_1) =& \frac{1}{3}(3\zeta_6 - 1) \cdot \left(-\frac{1}{14} \sum_{n=1}^{6} n\varphi_1(n) \right) + \frac{2}{3} \cdot \left(-\frac{1}{14} \sum_{n=1}^{6} n\varphi_2(n) \right) \\ &+ \frac{1}{3}(-3\zeta_6 + 2) \cdot \left(-\frac{1}{14} \sum_{n=1}^{6} n\varphi_3(n) \right) \end{aligned}$$

Evaluating the sum for φ_1, we obtain

$$\sum_{n=1}^{6} n\varphi_1(n) = 1 + 2\zeta_6^2 + 3\zeta_6 + 4\zeta_6^4 + 5\zeta_6^5 + 6\zeta_6^3.$$

Using $\zeta_6^2 = \zeta_6 - 1$ and $\zeta_6^3 = -1$, we obtain

$$\begin{aligned} \sum_{n=1}^{6} n\varphi_1(n) = & \quad 1 + 2(\zeta_6 - 1) + 3\zeta_6 - 4\zeta_6 - 5(\zeta_6 - 1) - 6 \\ = & \quad -4\zeta_6 - 2. \end{aligned}$$

For φ_2, we obtain

$$\sum_{n=1}^{6} n\varphi_2(n) = 1 + 2 - 3 + 4 - 5 - 6 = -7.$$

For φ_3, recall that $1 - \zeta_6 = \zeta_6^5$, so we obtain

$$\sum_{n=1}^{6} n\varphi_3(n) = 1 + 2\zeta_6^4 + 3\zeta_6^5 + 4\zeta_6^2 + 5\zeta_6 + 6\zeta_6^3.$$

Using the properties of ζ_6 again, we obtain

$$\begin{aligned} \sum_{n=1}^{6} n\varphi_3(n) = & \quad 1 - 2\zeta_6 + 3(1 - \zeta_6) + 4(\zeta_6 - 1) + 5\zeta_6 - 6 \\ = & \quad 4\zeta_6 - 6. \end{aligned}$$

Inserting this values into the formula for $c_0(z_1)$, we obtain

$$\begin{aligned} c_0(z_1) =& \frac{1}{3}(3\zeta_6 - 1) \cdot \left(-\frac{1}{14}(-4\zeta_6 - 2) \right) + \frac{1}{3} \\ &+ \frac{1}{3}(-3\zeta_6 + 2) \cdot \left(-\frac{1}{14}(4\zeta_6 - 6) \right) = 0. \end{aligned}$$

Next, we use the values computed above to compute $c_0(z_2)$:

$$\begin{aligned} c_0(z_2) = \;& \frac{1}{3}(-\zeta_6 - 2) \cdot \left(-\frac{1}{14} \sum_{n=1}^{6} n\varphi_1(n) \right) + \frac{2}{3} \cdot \left(-\frac{1}{14} \sum_{n=1}^{6} n\varphi_2(n) \right) \\ & + \frac{1}{3}(\zeta_6 - 3) \cdot \left(-\frac{1}{14} \sum_{n=1}^{6} n\varphi_3(n) \right) \\ = \;& \frac{1}{3}(-\zeta_6 - 2) \cdot \left(-\frac{1}{14}(-4\zeta_6 - 2) \right) + \frac{1}{3} \\ & + \frac{1}{3}(\zeta_6 - 3) \cdot \left(-\frac{1}{14}(4\zeta_6 - 6) \right) = 0. \end{aligned}$$

Finally, we compute $c_0(z_3)$:

$$\begin{aligned} c_0(z_3) = \;& \frac{1}{3}(-2\zeta_6 + 3) \cdot \left(-\frac{1}{14} \sum_{n=1}^{6} n\varphi_1(n) \right) + \frac{2}{3} \left(-\frac{1}{14} \sum_{n=1}^{6} n\varphi_2(n) \right) \\ & + \frac{1}{3}(2\zeta_6 + 1) \cdot \left(-\frac{1}{14} \sum_{n=1}^{6} n\varphi_3(n) \right) \\ = \;& \frac{1}{3}(-2\zeta_6 + 3) \cdot \left(-\frac{1}{14}(-4\zeta_6 - 2) \right) + \frac{1}{3} \\ & + \frac{1}{3}(2\zeta_6 + 1) \cdot \left(-\frac{1}{14}(4\zeta_6 - 6) \right) = 1. \end{aligned}$$

Now we will show that $c_k(z_j)$ for $k > 0$ and $j \in \{1, 2, 3\}$ is always an integer. This is somewhat different from the previous argument. For z_1, we obtain

$$\begin{aligned} c_k(z_1) = \;& \frac{1}{3}(3\zeta_6 - 1) \cdot \left(\sum_{l|k, l>0} \varphi_1(l) \right) + \frac{2}{3} \cdot \left(\sum_{l|k, l>0} \varphi_2(l) \right) \\ & + \frac{1}{3}(-3\zeta_6 + 2) \cdot \left(\sum_{l|k, l>0} \varphi_3(l) \right) \\ = \;& \sum_{l|k, l>0} \frac{1}{3} \left((3\zeta_6 - 1)\varphi_1(l) + 2\varphi_2(l) + (-3\zeta_6 + 2)\varphi_3(l) \right). \end{aligned}$$

where l denotes also its congruence class in $\mathbb{Z}/7$.

We give the values of the summands depending on l: (Note we would only need to compute the values for one half because of the symmetry)

$l \mod 7$	0	1	2	3	4	5	6
Summand	0	1	-1	-2	2	1	-1

In particular, the sum we obtain has only integer summands, thus is itself an integer.

We now look at z_2:

$$\begin{aligned}
c_k(z_2) = {} & \frac{1}{3}(-\zeta_6 - 2) \cdot \left(\sum_{l|k, l>0} \varphi_1(l) \right) + \frac{2}{3} \cdot \left(\sum_{l|k, l>0} \varphi_2(l) \right) \\
& + \frac{1}{3}(\zeta_6 - 3) \cdot \left(\sum_{l|k, l>0} \varphi_3(l) \right) \\
= {} & \sum_{l|k, l>0} \frac{1}{3} \cdot \left((-\zeta_6 - 2)\varphi_1(l) + 2\varphi_2(l) + (\zeta_6 - 3)\varphi_3(l) \right)
\end{aligned}$$

We give again the values of the summands depending on l:

$l \mod 7$	0	1	2	3	4	5	6
Summand	0	-1	2	-1	1	-2	1

Finally, for z_3 we obtain:

$$\begin{aligned}
c_k(z_3) = {} & \frac{1}{3}(-2\zeta_6 + 3) \cdot \left(\sum_{l|k, l>0} \varphi_1(l) \right) + \frac{2}{3} \left(\sum_{l|k, l>0} \varphi_2(l) \right) \\
& + \frac{1}{3}(2\zeta_6 + 1) \cdot \left(\sum_{l|k, l>0} \varphi_3(l) \right) \\
= {} & \sum_{l|k, l>0} \frac{1}{3} \cdot \left((-2\zeta_6 + 3)\varphi_1(l) + 2\varphi_2(l) + (2\zeta_6 + 1)\varphi_3(l) \right).
\end{aligned}$$

Again, we put the values of the summands depending on l into a table:

$l \mod 7$	0	1	2	3	4	5	6
Summand	0	2	1	1	-1	-1	-2

Thus, we have seen that all coefficients of z_1, z_2, z_3 in the q-expansion are integers, so we have $z_1, z_2, z_3 \in \mathrm{mf}_1(\Gamma_1(7); \mathbb{Z})$. $\square$

We want to show that $z_1, z_2, z_3 \in \mathrm{mf}_1(\Gamma_1(7); \mathbb{Z})$ is a basis. For this, we consider the q-expansions of z_1, z_2, z_3 modulo q^3:

$$\begin{aligned}
z_1 &\equiv && q && \mod q^3 \\
z_2 &\equiv && -q + q^2 && \mod q^3 \\
z_3 &\equiv 1 && +2q + 3q^2 && \mod q^3
\end{aligned}$$

The right hand sides form obviously a $\mathbb{Z}$-basis of $\mathbb{Z}[\![q]\!]/(q^3)$. Thus, $\mathrm{mf}_1(\Gamma_1(7); \mathbb{Z}) \to \mathbb{Z}[\![q]\!]/(q^3)$ is surjective. Since the composite

$$\mathrm{mf}_1(\Gamma_1(7); \mathbb{Z}) \otimes \mathbb{C} \to \mathrm{mf}_1(\Gamma_1(7); \mathbb{C}) \to \mathbb{C}[\![q]\!]/(q^3)$$

is as a composition of an injection and a surjection between 3-dimensional $\mathbb{C}$-vector spaces an injection as well, we conclude that $\mathrm{mf}_1(\Gamma_1(7); \mathbb{Z}) \to \mathbb{Z}[\![q]\!]/(q^3)$ is actually an isomorphism. This implies that $z_1, z_2, z_3 \in \mathrm{mf}_1(\Gamma_1(7); \mathbb{Z})$ is indeed a basis. The same is thus true in $\mathrm{mf}_1(\Gamma_1(7); \mathbb{Z}[\frac{1}{7}])$.

Our next goal is to understand all of $\mathrm{mf}_1(7)$ in terms of z_i's. We will prove the following proposition:

Proposition 2.3. *There is an isomorphism of rings*

$$\mathbb{Z}\left[\tfrac{1}{7}\right][z_1, z_2, z_3]/(z_1z_2 + z_2z_3 + z_3z_1) \to \mathrm{mf}(\Gamma_1(7); \mathbb{Z}\left[\tfrac{1}{7}\right]).$$

Proof. We will first show that the relation $z_1z_2 + z_2z_3 + z_3z_1 = 0$ is satisfied in $\mathrm{mf}_2(\Gamma_1(7); \mathbb{Z})$. For this, we will use an analogous argument as for z_1, z_2, z_3 being a basis of weight 1 modular forms. More precisely, we will consider the q-expansion of the modular forms z_iz_j for $1 \le i \le j \le 3$ modulo q^5. This will be enough as we will see below.

First, the z_i themselves are given

$$\begin{aligned}
z_1 &\equiv & & q & & -q^3 + 2q^4 & & \mod q^5\\
z_2 &\equiv & & -q + q^2 & & -2q^3 + 2q^4 & & \mod q^5\\
z_3 &\equiv & 1 & +2q + 3q^2 & & +3q^3 + 2q^4 & & \mod q^5
\end{aligned}$$

One computes the following products of those:

$$\begin{aligned}
z_1^2 &\equiv & & & q^2 & & -2q^4 & & \mod q^5\\
z_1z_2 &\equiv & & & -q^2 + q^3 & & -q^4 & & \mod q^5\\
z_1z_3 &\equiv & q & & +2q^2 + 2q^3 & & +3q^4 & & \mod q^5\\
z_2^2 &\equiv & & & q^2 - 2q^3 & & +5q^4 & & \mod q^5\\
z_3^2 &\equiv & 1 + 4q & & +10q^2 + 18q^3 & & +25q^4 & & \mod q^5\\
z_2z_3 &\equiv & -q & & -q^2 - 3q^3 & & -2q^4 & & \mod q^5
\end{aligned}$$

First, observe that we immediately obtain that the q-expansion of $z_1z_2+z_2z_3+z_3z_1$ is $0 \mod q^5$. Next, we observe that the matrix mapping the basis $1, q, q^2, q^3, q^4$ to the truncated power series for $z_1^2, z_1z_2, z_1z_3, z_2^2, z_3^2$ is

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 1\\
0 & 0 & 1 & 0 & 4\\
1 & -1 & 2 & 1 & 10\\
0 & 1 & 2 & -2 & 18\\
-2 & -1 & 3 & 5 & 25
\end{pmatrix}$$

The determinant of this matrix is 1; thus it is invertible over $\mathbb{Z}$ and the first 5 truncated power series above are a basis of $\mathbb{Z}[\![q]\!]/(q^5)$. This implies that a $\Gamma_1(7)$-modular form of weight 2 is zero iff its q-expansion is zero modulo q^5 since using the formulae of [14, Section 3.9], the vector space $\mathrm{mf}_2(\Gamma_1(7); \mathbb{C})$ is 5-dimensional. This in turn implies the relation.

We have noted before that

$$\mathrm{mf}(\Gamma_1(7); \mathbb{Z}[\tfrac{1}{7}]) \cong H^0(\overline{\mathcal{M}}_1(7); \underline{\omega}^{\otimes *}) \cong H^0(\mathbb{P}^1_{\mathbb{Z}[\frac{1}{7}]}; \mathcal{O}(2*)).$$

Thus, this ring is abstractly isomorphic to polynomials of even degree in variables of degree 1 (and the degree of the modular form is half the degree of the polynomial). The ring of such polynomials is generated by the three monomials of degree 2 with one quadratic relation between those. Thus, the ring $\mathrm{mf}(\Gamma_1(7); \mathbb{Z}\left[\tfrac{1}{7}\right])$ is generated in degree 1 and so by the z_1, z_2, z_3. Thus, we get a surjective map

$$\mathbb{Z}[\tfrac{1}{7}][z_1, z_2, z_3]/(z_1z_2 + z_2z_3 + z_3z_1) \to \mathrm{mf}(\Gamma_1(7); \mathbb{Z}[\tfrac{1}{7}]),$$

which has to be an isomorphism by counting the ranks. □

Next, we want to identify the $(\mathbb{Z}/7)^\times$ action on the left-hand side in terms of the generator $t = [3] \in (\mathbb{Z}/7)^\times$, where we use the conventions from Remark A.9.

Lemma 2.4. *The action of $(\mathbb{Z}/7)^\times$ on $\mathrm{mf}(\Gamma_1(7); \mathbb{Z}[\frac{1}{7}])$ is given by $t.z_1 = -z_3$ and $t.z_2 = -z_1$ and $t.z_3 = -z_2$.*

Proof. Recall that we already know the action on the Eisenstein series by definition, so we can conclude as follows:

$$\begin{aligned} t.z_1 = &\ \frac{1}{3}(3\zeta_6 - 1)\varphi_1(t)E(\varphi_1) + \frac{2}{3}\varphi_2(t)E(\varphi_2) + \frac{1}{3}(-3\zeta_6 + 2)\varphi_3(t)E(\varphi_3) \\ = &\ \frac{1}{3}(3\zeta_6 - 1)\zeta_6 E(\varphi_1) - \frac{2}{3}E(\varphi_2) + \frac{1}{3}(-3\zeta_6 + 2)(-\zeta_6 + 1)E(\varphi_3) \\ = &\ \frac{1}{3}(2\zeta_6 - 3)E(\varphi_1) - \frac{2}{3}E(\varphi_2) - \frac{1}{3}(2\zeta_6 + 1)E(\varphi_3) \\ = &\ -z_3. \end{aligned}$$

Similarly, one obtains $t.z_2 = -z_1$ and $t.z_3 = -z_2$. □

Note that the resulting action on $\mathbb{Z}[z_1, z_2, z_3]$ makes it isomorphic as a $\mathbb{Z}/6 \cong \mathbb{Z}/2 \times \mathbb{Z}/3$-representation to $\mathbb{Z}^{\mathrm{sign}} \otimes \mathbb{Z}[z_1, z_2, z_3]$, where $\mathbb{Z}^{\mathrm{sign}}$ is the permutation representation of $\mathbb{Z}/2$ and $\mathbb{Z}/3$ acts on $\mathbb{Z}[z_1, z_2, z_3]$ now permuting the variables as indicated above. We record without proof the following consequence.

Proposition 2.5. *If we denote by $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{Z}[z_1, z_2, z_3]$ the elementary symmetric polynomials, we obtain that the invariants $\mathrm{mf}(\Gamma_0(7); \mathbb{Z}[\frac{1}{7}]) \cong H^0(\mathbb{Z}/6, \mathbb{Z}[z_1, z_2, z_3]/\sigma_2)$ are the even degrees of the free $\mathbb{Z}[\sigma_1, \sigma_3]$-module on 1 and $z_1^2 z_2 + z_2^2 z_3 + z_3^2 z_1$.*

3. q-expansion of α_i as $\Gamma_1(7)$-modular forms using Tate curve

The aim of this section is to obtain q-expansions for the coefficients of the Weierstraß equations of elliptic curve with a $\Gamma_1(7)$-level structure. It is known that for such curves, the coefficients of the Weierstraß equations yield at least meromorphic $\Gamma_1(7)$-modular forms. Our computations show that they are indeed holomorphic, and thus we can identify them under the isomorphism of Proposition 2.3 with polynomials in z_1, z_2, z_3.

3.1. **Coordinates for generalized elliptic curves.** In this section, we will need some results about Weierstraß equations for generalized elliptic curves in the sense of [12, Definition II.1.12]. In the following theorem we will summarize the necessary input from [11, §1].

Theorem 3.1. *Let S be a scheme and let $p\colon \mathcal{E} \to S$ be a generalized elliptic curve whose geometric fibers are either smooth or Néron 1-gons. Let furthermore (e) be the relative Cartier divisor defined by the unit section $e\colon S \to \mathcal{E}$.*

(a) The sheaves $p_*\mathcal{O}_\mathcal{E}(ke)$ are locally free of rank k for $k > 0$ and are related by short exact sequences

$$0 \to p_*\mathcal{O}_\mathcal{E}((k-1)e) \to p_*\mathcal{O}_\mathcal{E}(ke) \to \omega_\mathcal{E}^{\otimes(-k)} \to 0 \tag{3.2}$$

for $k > 1$. The morphisms $\mathcal{O}_S \to p_*\mathcal{O}_\mathcal{E} \to p_*\mathcal{O}_\mathcal{E}(e)$ are isomorphisms. Moreover, $R^1p_*\mathcal{O}(ke) = 0$ for $k > 0$.

(b) *Zariski locally one can choose a trivialization of $\omega_{\mathcal{E}}$ and splittings of* (3.2) *for $k = 2$ and 3 and these choices define a trivialization*

$$(1, x, y) \colon \mathcal{O}_S^3 \xrightarrow{\cong} p_*\mathcal{O}(3e).$$

This gives rise to a closed embedding $\mathcal{E} \to \mathbb{P}^2_S$ with image cut out by a cubic equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with $a_1, a_2, a_3, a_4, a_6 \in \Gamma(\mathcal{O}_S)$ and the a_i are completely determined by the previous choices.

Remark 3.3. We want to discuss the role of gradings in the preceding theorem. Let $p \colon \mathcal{E} \to S$ be a generalized elliptic curve as in the theorem with chosen splittings of (3.2) for $k = 2$ and $k = 3$. This yields an isomorphism

$$p_*\mathcal{O}_{\mathcal{E}} \cong \mathcal{O}_S \oplus \omega_{\mathcal{E}}^{\otimes(-2)} \oplus \omega_{\mathcal{E}}^{\otimes(-3)}.$$

Let $q \colon T \to S$ be a morphism with a trivialization $\mathcal{O}_T \xrightarrow{\omega} \omega_{\mathcal{E}_T}$, where $p^T : \mathcal{E}_T \to T$ is the pullback of p along q. By [50, Proposition 4.37], the natural map $q^*p_*\mathcal{O}_{\mathcal{E}} \to p^T_*\mathcal{O}_{\mathcal{E}_T}$ is an isomorphism. The resulting isomorphism $\mathcal{O}_T^3 \to p^T_*\mathcal{O}_{\mathcal{E}_T}$ sends the basis $(1, 1, 1)$ to $(1, x, y)$ and we get associated quantities $a_i \in \Gamma(\mathcal{O}_T)$. Changing the trivialization to $\lambda\omega$ for some $\lambda \in \mathbb{G}_m(T)$ produces new coordinates $(1, x', y') = (1, \lambda^{-2}x, \lambda^{-3}y)$ with associated quantities $a_i' = \lambda^{-i}a_i$.

In particular, we can consider the $\mathbb{G}_m$-torsor $q \colon T \to S$ given by $T = \underline{\mathrm{Spec}}\left(\bigoplus_{i\in\mathbb{Z}} \omega_{\mathcal{E}}^{\otimes i}\right)$. This comes with a canonical trivialization of $q^*\omega_{\mathcal{E}} \cong \omega_{\mathcal{E}_T}$ from (A.4), and this produces elements $a_i \in \Gamma(\mathcal{O}_T)$. The computation above shows that the degree of a_i is i, where the grading on $\Gamma(\mathcal{O}_T)$ comes from the $\mathbb{G}_m$-action on T or equivalently from the identification with $\left(\bigoplus_{i\in\mathbb{Z}} \omega_{\mathcal{E}}^{\otimes i}\right)(S)$.

The following standard fact will be needed for the proof of Proposition 3.5.

Lemma 3.4. *Let R be a nonnegatively graded ring, Z the vanishing locus of the ideal generated by the positive degree homogeneous elements and U its complement in* $\operatorname{Spec} R$. *Then $U/\mathbb{G}_m$ is separated.*

Proof. By the valuative criterion it suffices to show that for every valuation ring V with field of fractions K, the map $p_V \colon (U/\mathbb{G}_m)(V) \to (U/\mathbb{G}_m)(K)$ of groupoids is fully faithful [29, Proposition 7.8]. As every $\mathbb{G}_m$-torsor over V is trivial, the groupoid $(U/\mathbb{G}_m)(V)$ is equivalent to one with objects ring morphisms $f \colon R \to V$ such that $f(r)$ is invertible for some r homogeneous of positive degree. A morphism $g \to f$ is given by an element $\lambda \in \mathbb{G}_m(V)$ such that $g(r) = \lambda^i f(r)$ for all homogeneous r of degree i. The description of $(U/\mathbb{G}_m)(K)$ is analogous. As $\mathbb{G}_m(V)$ includes into $\mathbb{G}_m(K)$, we see that p_V is faithful. Now suppose that $f, g \colon R \to V$ are two objects in $(U/\mathbb{G}_m)(V)$ and $\lambda \in \mathbb{G}_m(K)$ satisfies $g(r) = \lambda^i f(r)$ for all homogeneous r of degree i. We can choose an $r \in R$ such that $f(r)$ is invertible in V and thus $\lambda^i \in V$ and hence $\lambda \in V$ as V is normal. Repeating this argument for an $r' \in R$ such that $g(r')$ is invertible yields that $\lambda^{-1} \in V$ and hence $\lambda \in \mathbb{G}_m(V)$. Thus p_V is full. □

Proposition 3.5. *Let*

$$\pi \colon \overline{\mathcal{M}}_1^1(n) = \underline{\mathrm{Spec}}\left(\bigoplus_{i\in\mathbb{Z}} \underline{\omega}^{\otimes i}\right) \to \overline{\mathcal{M}}_1(n)$$

be the $\mathbb{G}_m$-torsor trivializing $\underline{\omega}$. Let $p\colon \mathcal{E} \to \overline{\mathcal{M}}_1(n)$ be the generalized elliptic curve classified by the natural map $\overline{\mathcal{M}}_1(n) \to \overline{\mathcal{M}}_{ell}$.

Then there are

$$a_1, \ldots, a_6 \in \mathrm{mf}_1(n) = H^0(\overline{\mathcal{M}}_1^1(n), \mathcal{O}_{\overline{\mathcal{M}}_1^1(n)})$$

with $|a_i| = i$ such that the generalized elliptic curve $\pi^*\mathcal{E}$ is defined by the cubic equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Consider moreover the canonical morphism $\overline{\mathcal{M}}_1^1(n) \to \operatorname{Spec} \mathrm{mf}_1(n)$. It is an open immersion onto the complement of the common vanishing locus of c_4 and Δ for the usual quantities c_4 and Δ associated with $a_1, \ldots, a_6$. Moreover, c_4 and Δ coincide with the images of the classes with the same name along $\mathrm{mf}_1(1) \to \mathrm{mf}_1(n)$.

Proof. To apply Theorem 3.1 and Remark 3.3, it suffices to show that the inclusions

$$p_*\mathcal{O}_{\mathcal{E}}((k-1)e) \to p_*\mathcal{O}_{\mathcal{E}}(ke)$$

split for $k > 1$. By induction, we can assume $p_*\mathcal{O}_{\mathcal{E}}((k-1)e)$ to be isomorphic to $\mathcal{O}_{\overline{\mathcal{M}}_1(n)} \oplus \underline{\omega}^{\otimes(-2)} \oplus \cdots \oplus \underline{\omega}^{\otimes(-k+1)}$. The vector bundle $p_*\mathcal{O}_{\mathcal{E}}(ke)$ is an extension of $p_*\mathcal{O}_{\mathcal{E}}((k-1)e)$ and $\underline{\omega}^{\otimes(-k)}$ and thus we have to show the vanishing of a class χ in

$$\mathrm{Ext}_{\mathcal{O}_{\overline{\mathcal{M}}_1(n)}}(\underline{\omega}^{\otimes(-k)}, p_*\mathcal{O}_{\mathcal{E}}((k-1)e)) \cong H^1(\overline{\mathcal{M}}_1(n); \underline{\omega}^{\otimes k} \oplus \underline{\omega}^{\otimes(k-2)} \oplus \cdots \oplus \underline{\omega})$$

The vanishing result [38, Proposition 2.14] implies that the projection to

$$\mathrm{Ext}_{\mathcal{O}_{\overline{\mathcal{M}}_1(n)}}(\underline{\omega}^{\otimes(-k)}, p_*\mathcal{O}_{\mathcal{E}}((k-1)e)/p_*\mathcal{O}_{\mathcal{E}}((k-2)e)) \cong H^1(\overline{\mathcal{M}}_1(n); \underline{\omega})$$

is an isomorphism and thus it suffices to show the vanishing of the projection χ' of χ. As $\mathcal{E}$ is the pullback of the universal generalized elliptic curve $\mathcal{E}^{uni} \to \overline{\mathcal{M}}_{ell}$ along $\overline{\mathcal{M}}_1(n) \to \overline{\mathcal{M}}_{ell}$, the class χ' actually lies in the image from

$$\mathrm{Ext}_{\mathcal{O}_{\overline{\mathcal{M}}_{ell}}}(\underline{\omega}^{\otimes(-k)}, p_*\mathcal{O}_{\mathcal{E}^{uni}}((k-1)e)/p_*\mathcal{O}_{\mathcal{E}^{uni}}((k-2)e)) \cong H^1(\overline{\mathcal{M}}_{ell}; \underline{\omega}) \cong \mathbb{Z}/2,$$

whose non-trivial element we call η. As shown in [38, Proposition 2.16], the image of η in $H^1(\overline{\mathcal{M}}_1(n); \underline{\omega})$ is trivial and thus the inclusions $p_*\mathcal{O}_{\mathcal{E}}((k-1)e) \to p_*\mathcal{O}_{\mathcal{E}}(ke)$ are split for $k > 1$.

As $\underline{\omega}$ is ample on $\overline{\mathcal{M}}_1(n)$ [39, Lemma 5.11], the pullback $\pi^*\underline{\omega}$ on $\overline{\mathcal{M}}_1^1(n)$ is both trivial and ample and thus $\overline{\mathcal{M}}_1^1(n)$ is quasi-affine, i.e. the canonical morphism $\overline{\mathcal{M}}_1^1(n) \to \operatorname{Spec} H^0(\overline{\mathcal{M}}_1^1(n), \mathcal{O}_{\overline{\mathcal{M}}_1^1(n)}) = \operatorname{Spec} \mathrm{mf}_1(n)$ is an open immersion [18, Propositions 13.83 and 13.80]. As a cubic curve defines a generalized elliptic curve only if c_4 and Δ vanish nowhere simultaneously [46, Proposition III.1.4], we see that the immersion $\overline{\mathcal{M}}_1^1(n) \to \operatorname{Spec} \mathrm{mf}_1(n)$ has image in the complement U of the common vanishing locus of c_4 and Δ. Note that the inclusion $\overline{\mathcal{M}}_1^1(n) \to U$ is $\mathbb{G}_m$-equivariant.

By Lemma 3.4, $U/\mathbb{G}_m$ is separated. As $\overline{\mathcal{M}}_1(n)$ is proper over $\operatorname{Spec} \mathbb{Z}[\frac{1}{n}]$ and $U/\mathbb{G}_m$ is separated, we obtain analogously to [19, Corollary II.4.6] that the open immersion $\overline{\mathcal{M}}_1(n) \hookrightarrow U/\mathbb{G}_m$ is proper. Thus, the image is closed and $\overline{\mathcal{M}}_1(n) \hookrightarrow U/\mathbb{G}_m$ is an isomorphism.

The last point is an instance of the following more general observation. If $S \to \overline{\mathcal{M}}_{ell}$ classifies a generalized elliptic curve $E \to S$ given by a Weierstraß equation, then the images of $c_4, \Delta \in H^0(\overline{\mathcal{M}}_{ell}; \underline{\omega}^{\otimes *})$ in $H^0(S; \omega_E^{\otimes *})$ coincide with the corresponding polynomials in the coefficients a_i of the Weierstraß equation. $\square$

Lemma 3.6. *Let $\mathcal{E} \to S := \operatorname{Spec} R$ be an elliptic curve given by a Weierstraß equation. Assume that the degrees of the associated quantities $a_i \in \Gamma(\mathcal{O}_S) = R$ have degree i, where the grading on $\Gamma(\mathcal{O}_S)$ comes from a $\mathbb{G}_m$-action on S (equivalently, a grading on R). Let there furthermore be a section $P\colon S \to \mathcal{E}$ of exact order $n \geq 3$. Then there are coordinates for $\mathcal{E}$ such that the associated Weierstraß equation is of the form*

$$y^2 + \alpha_1 xy + \alpha_3 y = x^3 + \alpha_2 x^2$$

and P corresponds to the point $(0,0)$ and $|\alpha_i| = i$.

This special form of the Weierstraß equation is called *Tate normal form* or also sometimes *Kubert–Tate normal form.*

Proof. We perform a similar transformation as in the proof of [4, Theorem 1.1.1]. First, observe from formulae in [46, Section III.1] that any Weierstraß equation of the form

$$y^2 + a_1 xy + a_3 x = x^3 + a_2 x^2 + a_4 x + a_6$$

can be transformed so that the chosen torsion point (x_0, y_0) on this curve is moving to $(0,0)$. This transformation has the transformation parameter

$$r = x_0, \quad t = y_0, \quad s = 0.$$

Thus we may assume $a_6 = 0$ and the torsion point to be $(0,0)$. Over any field K, it follows from [23, Remark 4.2.1] that if $a_3 = 0$ over this field, then $(0,0)$ would be either singular or a 2-torsion point, contradicting the assumption that it is a torsion point of strict order $n \geq 4$. Thus, a_3 is invertible in R since it maps to a non-zero element in every field for any ring map $R \to K$.

This allows to define a transformation with transformation parameters

$$r = 0, \quad t = 0, \quad s = \frac{a_4}{a_3},$$

and the resulting coefficients are

$$y^2 + \alpha_1 xy + \alpha_3 y = x^3 + \alpha_2 x^2.$$

$\square$

Remark 3.7. If we start with the Weierstraß equation $y^2 + a_1 xy + a_3 x = x^3 + a_2 x^2 + a_4 x + a_6$ as above, we want to record for later use the values of α_i obtained by the procedure in the proof of the lemma above. Denote by s' the auxiliary quantity (the invertibility of the denominator follows from the lemma above)

$$s' = \frac{a_4 + 2a_2 x_0 - a_1 y_0 + 3x_0^2}{a_3 + a_1 x_0 + 2y_0}.$$

Then we obtain

$$\begin{aligned}
\alpha_1 =& \frac{a_1 a_3 + 2a_4 + (a_1^2 + 4a_2)x_0 + 6x_0^2}{a_3 + a_1 x_0 + 2y_0} \\
\alpha_2 =& a_2 + 3x_0 - a_1 s' - (s')^2 \\
\alpha_3 =& a_3 + a_1 x_0 + 2y_0.
\end{aligned}$$

3.2. **The q-expansions of the α_i.** In the last subsection, we showed that the universal elliptic curve over $\mathcal{M}_1^1(n)$ has a Weierstraß equation in Tate normal form and that the corresponding quantities α_i are meromorphic modular forms (i.e. elements in $H^0(\mathcal{M}_1^1(n); \mathcal{O}_{\mathcal{M}_1^1(n)}))$ of degree i. In this subsection, we will show that the α_i are actually *holomorphic* modular forms and provide a general formula for the q-expansion. This will allow us to identify them with explicit polynomials in the z_i in the case $n = 7$. Our first goal will be to prove a criterion how to check that a meromorphic $\Gamma_1(n)$-modular form is actually holomorphic.

Given a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ with positive determinant, an integer k and a meromorphic function $g\colon \mathbb{H} \to \mathbb{C}$, one defines a new meromorphic function $g[\gamma]_k$ as follows:

$$\begin{aligned} g[\gamma]_k \colon \mathbb{H} &\to \mathbb{C} \\ z &\mapsto (cz+d)^{-k} g\left(\frac{az+b}{cz+d}\right). \end{aligned}$$

Moreover, if g was holomorphic, so is $g[\gamma]_k$.

Whether a $\Gamma_1(n)$-modular form g of weight k(in the analytic sense) is holomorphic, can by definition be checked as follows: For representatives $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of cosets $\mathrm{SL}_2(\mathbb{Z})/\Gamma_1(n)$, we need the map $g[\gamma]_k$, given by

$$z \mapsto (cz+d)^{-k} g\left(\frac{az+b}{cz+d}\right)$$

to be holomorphic at ∞.

Let $g \in \mathrm{Nat}_k(\mathrm{Ell}_{\Gamma_1(n)}(-), \Gamma(-))$ be a Katz modular form over $\mathbb{C}$. Let $\beta_1(g)$ denote the corresponding complex modular form as in Lemma A.15. We want to analyze the effect of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ on this modular form.

$$\tau \mapsto \beta_1(g)\left(\frac{a\tau+b}{c\tau+d}\right) = g\left(\mathbb{C}/\mathbb{Z} + \mathbb{Z}n\frac{a\tau+b}{c\tau+d}, dz, \frac{a\tau+b}{c\tau+d}\right)$$

Similarly to e.g. [13, Section 4], we use certain operators to rephrase holomorphy of a $\Gamma_1(n)$-modular forms.

Lemma 3.8. *Let $g\colon \mathbb{H} \to \mathbb{C}$ be a meromorphic $\Gamma_1(n)$-modular form of weight k. Then g is holomorphic if and only if the map*

$$\begin{aligned} W_m g := g[W_m]_k \colon \mathbb{H} &\to \mathbb{C} \\ \tau &\mapsto (m\tau)^{-k} g\left(-\frac{1}{m\tau}\right) \end{aligned}$$

is holomorphic at ∞ for some fixed divisor $m \geq 1$ of n, where W_m denotes the matrix $\begin{pmatrix} 0 & -1 \\ m & 0 \end{pmatrix}$.

Proof. Given any matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(n)$, observe that the matrix $W_m \gamma W_m^{-1} = \begin{pmatrix} d & -\frac{c}{m} \\ -bm & a \end{pmatrix}$ also lies in $\Gamma_1(n)$. By [14, Lemma 1.2.2], the operation $\gamma \to g[\gamma]_k$ is compatible with matrix multiplication in $\mathrm{SL}_2(\mathbb{Z})$ and the same computation shows that it is also true for matrices in $\mathrm{GL}_2(\mathbb{R})^+$. We obtain the identity

$$((g[W_m]_k)[\gamma]_k)(\tau) = g[W_m\gamma W_m^{-1}]_k[W_m]_k(\tau).$$

Thus the map $g[W_m]_k$ is itself a $\Gamma_1(n)$-weak modular form and asking for holomorphy at ∞ makes sense.

As $W_1 \in SL_2(\mathbb{Z})$, we see that $W_1\,\mathrm{SL}_2(\mathbb{Z})W_1^{-1} = \mathrm{SL}_2(\mathbb{Z})$ and thus g is holomorphic if and only if $g[W_1]_k$ is holomorphic. This proves the statement for $m = 1$. Otherwise, observe that

$$g[W_m]_k(\tau) = g[W_1]_k(m\tau).$$

Thus, the claim follows from the fact that a function satisfying the $\Gamma_1(n)$-transformation formula is holomorphic at ∞ if and only if it is bounded as $\mathrm{Im}(\tau) \to \infty$. □

For the next lemma recall the Tate curve $\mathrm{Tate}(q^n)$ from the discussion after Theorem A.18, as well as the description of torsion points on this curve.

Lemma 3.9. *Let $g \in \mathrm{Nat}_k(\mathrm{Ell}_{\Gamma_1(n)}(-), \Gamma(-))$ be a Katz modular form over $\mathbb{C}$. Assume that the evaluation at the Tate curve $\mathrm{Tate}(q^n)$ with its invariant differential η^{can} over Conv_{q^n} yields a power series (as opposed to a general Laurent series) for any choice of torsion point $(X(\zeta^d q^c, q^n), Y(\zeta^d q^c, q^n), 1)$. Then the associated modular form $\beta_1(g) \in \mathrm{MF}(\Gamma_1(n); \mathbb{C})$ is actually holomorphic.*

Proof. Throughout this proof, let τ be an arbitrary point in the upper half plane. By definition, $\beta_1(g)(\tau) = g(\mathbb{C}/\mathbb{Z} + n\tau\mathbb{Z}, dz, \tau)$. We claim that $(W_n\beta_1(g))(\tau) = (-n)^{-k}g(\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}, dz, \frac{1}{n})$. Indeed, multiplication by $-\tau$ induces an isomorphism from the elliptic curve $\mathbb{C}/\mathbb{Z} + n\frac{-1}{n\tau}\mathbb{Z}$ to $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$. Thus, $(\mathbb{C}/\mathbb{Z} + n\frac{-1}{n\tau}\mathbb{Z}, dz, \frac{-1}{n\tau})$ is isomorphic to $(\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}, (-\tau)^{-1}dz, \frac{1}{n})$. We obtain

$$(W_n\beta_1(g))(\tau) = (n\tau)^{-k}\beta_1(g)(-\frac{1}{n\tau}) = (n\tau)^{-k}(-\tau)^k g(\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}, dz, \frac{1}{n})$$

as was to be shown.

The assignment $g' \colon \mathbb{H} \to \mathbb{C}$, given by $\tau \mapsto g(\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}, dz, \frac{1}{n})$, can be checked to define a meromorphic $\Gamma_1(n)$-modular form. In particular, it suffices by Lemma 3.8 and our previous computation to show that g' is holomorphic. By definition, this means that all $g'[\gamma]_k$ are holomorphic at ∞, where γ runs over the cosets $\Gamma_1(n) \setminus \mathrm{SL}_2(\mathbb{Z})$. If we write $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$\begin{aligned} g'[\gamma]_k(\tau) &= (c\tau + d)^{-k} g\left(\mathbb{C}/\mathbb{Z} + \frac{a\tau + b}{c\tau + d}\mathbb{Z}, dz, \frac{1}{n}\right) \\ &= g\left(\mathbb{C}/\mathbb{Z} + \frac{a\tau + b}{c\tau + d}\mathbb{Z}, (c\tau + d)dz, \frac{1}{n}\right) \\ &= g\left(\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}, dz, \frac{c\tau + d}{n}\right) \\ &= g\left(\mathbb{C}^\times/q_0^{n\mathbb{Z}}, \frac{du}{u}, \zeta^d q_0^c\right) \end{aligned}$$

with $q_0 = e^{2\pi i\tau}$. Recall that over Conv_{q^n}, we have the Tate curve with chosen torsion point $(E_{q^n}, \eta^{can}, (X(\zeta^d q^c, q^n), Y(\zeta^d q^c, q^n), 1))$. If we pull it back the evaluation map $\mathrm{ev}_{q_0}\colon \mathrm{Conv}_{q^n} \to \mathbb{C}$, the resulting elliptic curve with chosen torsion point is isomorphic to $\left(\mathbb{C}^\times/q_0^{n\mathbb{Z}}, \frac{du}{u}, \zeta^d q_0^c\right)$. By naturality, the values of g are also related via ev_{q_0}. Recall that $g(E_{q^n}, \eta^{can}, (X(\zeta^d q^c, q^n), Y(\zeta^d q^c, q^n), 1))$ is a power series by assumption. Thus, $g'[\gamma]$ is holomorphic at ∞ for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

□

From now on we will assume $n \geq 3$. Our aim now is to compute the q-expansions of the coefficients α_i of the Tate normal form

$$y^2 + \alpha_1 xy + \alpha_3 y = x^3 + \alpha_2 x^2$$

for the Tate curve $\mathrm{Tate}(q^n)$ given by $y^2 + xy = x^3 + a_4(q^n)x + a_6(q^n)$ over Conv_{q^n} with a chosen n-torsion point (x_0, y_0). The values from Remark 3.7 specialize to

$$\begin{aligned} s' &= \frac{a_4(q^n) - y_0 + 3x_0^2}{x_0 + 2y_0}, \\ \alpha_1 &= \frac{x_0 + 6x_0^2 + 2a_4(q^n)}{x_0 + 2y_0}, \\ \alpha_2 &= 3x_0 - s' - (s')^2, \\ \alpha_3 &= x_0 + 2y_0. \end{aligned}$$

Next, we want to specify the torsion point (x_0, y_0) on the Tate curve $\mathrm{Tate}(q^n)$. We use methods from [47, Section V.3], to simplify the expressions for $X(vq^k, q^n)$ and $Y(vq^k, q^n)$ in our case, where v and q are complex numbers with $|v| = 1$ and $|q| < 1$ and $0 \leq k < n$. First, we reindex the sum over positive natural numbers:

$$\begin{aligned} X(vq^k, q^n) &= \sum_{m\in\mathbb{Z}} \frac{vq^{mn+k}}{(1 - vq^{mn+k})^2} - 2s_1(q^n), \\ &= \frac{vq^k}{(1 - vq^k)^2} + \sum_{m\geq 1} \left(\frac{vq^{mn+k}}{(1 - vq^{mn+k})^2} + \frac{v^{-1}q^{mn-k}}{(1 - v^{-1}q^{mn-k})^2} \right. \\ &\qquad \left. -2\frac{q^{mn}}{(1 - q^{mn})^2} \right). \end{aligned}$$

Recall the following formulae for $|x| < 1$, obtained e.g. by differentiating the geometric series:

$$\frac{x}{(1-x)^2} = \sum_{l\geq 1} l x^l \text{ and } \frac{x^2}{(1-x)^3} = \sum_{l\geq 1} \frac{l(l-1)}{2} x^l \text{ and } \frac{x}{(1-x)^3} = \sum_{l\geq 0} \frac{l(l+1)}{2} x^l.$$

Inserting this into the expression for $X(vq^k, q^n)$, we obtain for $k > 0$

$$X(vq^k, q^n) = \sum_{l\geq 1} l v^l q^{kl} + \sum_{m\geq 1}\sum_{l\geq 1} \left(l v^l q^{(mn+k)l} + l v^{-l} q^{(mn-k)l} - 2l q^{mnl} \right)$$

For $k = 0$ and $v \neq 1$ we obtain similarly

$$X(v, q^n) = \frac{v}{(1-v)^2} + \sum_{m>0} \left(\sum_{l|m} l(v^l + v^{-l} - 2) \right) q^{mn}$$

For $Y(vq^k, q^n)$ we get

$$\begin{aligned} Y(vq^k, q^n) = & \sum_{m\in\mathbb{Z}} \frac{v^2 q^{2(mn+k)}}{(1-vq^{mn+k})^3} + s_1(q^n), \\ = & \frac{v^2q^{2k}}{(1-vq^k)^3} + \sum_{m\geq 1}\left(\frac{v^2q^{2(mn+k)}}{(1-vq^{mn+k})^3} - \frac{v^{-1}q^{mn-k}}{(1-v^{-1}q^{mn-k})^3}\right. \\ & \left. + \frac{q^{mn}}{(1-q^{mn})^2}\right). \end{aligned}$$

Using again the formulae derived from geometric series, we obtain

$$\begin{aligned} Y(vq^k, q^n) = & \sum_{l\geq 2} \frac{(l-1)l}{2} v^l q^{kl} + \sum_{m\geq 1}\sum_{l\geq 1}\left(\frac{(l-1)l}{2} v^l q^{(mn+k)l}\right. \\ & \left. - \frac{l(l+1)}{2} v^{-l} q^{(mn-k)l} + l q^{mnl}\right). \end{aligned}$$

For $k = 0$ and $v \neq 1$, we obtain instead

$$Y(v, q^n) = \frac{v^2}{(1-v)^3} + \sum_{m>0}\left(\sum_{l|m}\left(\frac{l(l-1)}{2}v^l - \frac{l(l+1)}{2}v^{-l} + 1\right)\right) q^{mn}$$

Lemma 3.10. *As before let $|q| < 1$, $|v| = 1$ and $0 \leq k < n$. In terms of the $X(vq^k, q^n)$ and $Y(vq^k, q^n)$ the Laurent series α_1, α_2 and α_3 are actually power series as well if $v \neq \pm 1$ if $k = 0$ or $k = \frac{n}{2}$.*

Proof. Note that in each of the cases above both $X(vq^k, q^n)$ and $Y(vq^k, q^n)$ are not just Laurent series in q, but actually power series. In particular, so is $\alpha_3 = X + 2Y$. Given the expressions for α_1 and α_2, we only need to check that

$$s' = \frac{a_4(q^n) - X(vq^k, q^n) + 3X(vq^k, q^n)^2}{X(vq^k, q^n) + 2Y(vq^k, q^n)}$$

is a power series to obtain that α_1 and α_2 are power series as well. In our Tate curve, we have

$$a_4(q^n) = -5s_3(q^n) = -5\sum_{m\geq 1}\sigma_3(m)q^{mn}$$

so this power series has $n > k$ as lowest exponent of q. Thus, the lowest power of q occuring in the numerator is the same as for X (unless the numerator is 0 and thus $s' = 0$). It thus suffices to show that the lowest term of the power series for X has at least the order of the lowest term of the power series defining $X + 2Y$. In the table below we will compute the lowest term in the power series defining X, Y and $X + 2Y$ in the different cases.

	X	Y	$X+2Y$
$k=0$	$\frac{v}{(1-v)^2}$	$\frac{v^2}{(1-v)^3}$	$\frac{v+v^2}{(1-v)^3}$
$0<k<\frac{n}{2}$	vq^k	higher term	vq^k
$k=\frac{n}{2}$	$(v+v^{-1})q^{\frac{n}{2}}$	$-v^{-1}q^{\frac{n}{2}}$	$(v-v^{-1})q^{\frac{n}{2}}$
$\frac{n}{2}<k<n$	$v^{-1}q^{n-k}$	$-v^{-1}q^{n-k}$	$-v^{-1}q^{n-k}$

Note that $\frac{v+v^2}{(1-v)^3}=0$ only if $v=-1$ and $v-v^{-1}=0$ only if $v=\pm 1$. □

Proposition 3.11. *If $n\geq 3$, the universal elliptic curve over $\mathcal{M}_1^1(n)$ has a Weierstraß equation of the form*

$$y^2+\alpha_1xy+\alpha_3y=x^3+\alpha_2x^2,$$

where the α_i are holomorphic *modular forms in* $\mathrm{mf}(\Gamma_1(n);\mathbb{Z}[\frac{1}{n}])$ *of degree i.*

Proof. Lemma 3.10 checks exactly the holomorphicity criterion Lemma 3.9 once we observe that $P=(X(\zeta^dq^k,q^n),Y(\zeta^dq^k,q^n),1)$ cannot be a point of exact order n if $k=0$ or $k=\frac{n}{2}$ and $\zeta^d=\pm1$ because this would imply that P is of order 2. □

According to the conventions from Appendix A.4 we obtain the q-expansions of the α_i by specializing to the torsion point $(X(q,q^n),Y(q,q^n),1)$ above and use our explicit expressions of the α_i in terms of X and Y.

In our case of $n=7$ we obtain the following q-expansions for X and Y:

$$X = q+2q^2+3q^3+4q^4+5q^5+7q^6+5q^7+9q^8+\cdots$$
$$Y = q^2+3q^3+6q^4+10q^5+14q^6+22q^7+28q^8+\cdots$$

The form of the q-expansions of z_1,z_2 and z_3 implies that there are elements of $\mathrm{mf}_k(\Gamma_1(7);\mathbb{Z}[\frac{1}{7}])$ with q-expansions of the form q^i+higher terms where i runs over all integers in $[0,2k]$. As $\mathrm{mf}_k(\Gamma_1(7);\mathbb{Z}[\frac{1}{7}])$ is free of rank $2k+1$, these elements form automatically a basis and thus every element in $\mathrm{mf}_k(\Gamma_1(7);\mathbb{Z}[\frac{1}{7}])$ is determined by its q-expansion modulo q^{2k+1}. Comparing the q-expansions of the z_i with those of the α_i using `MAGMA` implies the following theorem.

Theorem 3.12. *The elliptic curve classified by the composition* $\operatorname{Spec}\mathrm{MF}(\Gamma_1(7),\mathbb{Z}[\frac{1}{7}])\to\mathcal{M}_1(7)\to\mathcal{M}_{ell}$ *is given by the equation*

$$y^2+\alpha_1xy+\alpha_3y=x^3+\alpha_2x^2$$

with

$$\begin{aligned}\alpha_1 &= z_1-z_2+z_3,\\ \alpha_2 &= z_1z_2+z_1z_3,\\ \alpha_3 &= z_1z_3^2.\end{aligned}$$

4. Graded Hopf algebroids and stacks

In this section, all gradings can be taken to be either over $\mathbb{Z}$ (as convenient in the algebraic setting) or over $2\mathbb{Z}$ (as convenient in the topological setting) if this choice is done consistently. In either case, we assume our graded rings to be commutative and not just graded commutative. All comodules over Hopf algebroids, a notion which we will recall in this section, are chosen to be left comodules.

4.1. **General theory.** Let (B, Σ) be a graded Hopf algebroid, i.e. a cogroupoid object in the category of graded rings. To such a graded Hopf algebroid, we can associate an ungraded Hopf algebroid $(B, \Sigma[u^{\pm 1}])$ as follows. The structure maps η_L and ε are essentially unchanged. The right unit $\eta_R^{(B,\Sigma[u^{\pm 1}])}\colon B \to \Sigma[u^{\pm 1}]$ is given via

$$\eta_R^{(B,\Sigma[u^{\pm 1}])}(x) = u^i \eta_R^{(B,\Sigma)}(x)$$

if $x \in B$ is a homogeneous element of degree i. The comultiplication $\psi^{(B,\Sigma[u^{\pm 1}])}\colon \Sigma[u^{\pm 1}] \to \Sigma[u^{\pm 1}] \otimes_B \Sigma[u^{\pm 1}]$ is given by

$$\psi^{(B,\Sigma[u^{\pm 1}])}(s) = u^i \psi^{(B,\Sigma)}(s)$$

for homogeneous elements $s \in \Sigma$ of degree i and $\psi(u) = 1 \otimes u + u \otimes 1$. One can show that the category of (evenly) graded comodules over (B, Σ) is equivalent to that of comodules over $(B, \Sigma[u^{\pm 1}])$.

In the following, we will assume that (B, Σ) is *flat*, i.e. that Σ is flat as a B-module. We observe that Σ is flat as a B-module with respect to left module structure given by η_L if and only if it is flat as a B-module with respect to the right module structure given by η_R.

Definition 4.1. The *associated stack* for a graded Hopf algebroid (B, Σ) is the stack $\mathcal{X}(B, \Sigma)$ associated to the (ungraded) Hopf algebroid $(B, \Sigma[u^{\pm 1}])$ defined above, i.e. the stackification of the represented presheaf of groupoids.

As explained in [41, Section 3] the stack $\mathcal{X} = \mathcal{X}(B, \Sigma)$ is automatically algebraic in the sense of op. cit. and actually an Artin stack if Σ is a finitely presented B-algebra (see [29, Théorème 10.1]). Moreover, it comes with a map $\operatorname{Spec} B \to \mathcal{X}$ and the pullback $\operatorname{Spec} B \times_{\mathcal{X}} \operatorname{Spec} B$ can be identified with $\operatorname{Spec} \Sigma[u^{\pm 1}]$. The pullback

$$\mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\operatorname{Spec} B) \simeq B\text{-mod}$$

refines to an equivalences from $\mathrm{QCoh}(\mathcal{X})$ to $(B, \Sigma[u^{\pm 1}])$-comodules by [41, Section 3.4].

Example 4.2. Let B be a graded ring viewed as a graded Hopf algebroid (B, B). Its associated stack is $\operatorname{Spec} B/\mathbb{G}_m$ with the $\mathbb{G}_m$ action corresponding to the grading. Graded B-modules are the same as graded comodules over (B, B) and are thus equivalent to $\mathrm{QCoh}(\operatorname{Spec} B/\mathbb{G}_m)$.

Composing the equivalence betweeen $\mathrm{QCoh}(\mathcal{X})$ and $(B, \Sigma[u^{\pm 1}])$-comodules and between the latter and graded (B, Σ)-comodules, we obtain the following.

Proposition 4.3. *The map* $(\mathrm{id}_B, \varepsilon)\colon (B, \Sigma) \to (B, B)$ *of graded Hopf algebroids induces a map* $f^B\colon \operatorname{Spec} B/\mathbb{G}_m \to \mathcal{X}$ *and the pullback functor* $(f^B)^*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\operatorname{Spec} B/\mathbb{G}_m)$ *refines to an equivalence of quasi-coherent sheaves on* $\mathcal{X}$ *to graded comodules over* (B, Σ).

As just described, the pullback functor $\mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\operatorname{Spec} B/\mathbb{G}_m)$ translates into the forgetful functor from graded (B,Σ)-comodules to graded B-modules. This forgetful functor has a right adjoint, sending a graded B-module M to the *extended* graded (B,Σ)-comodule $\Sigma \otimes_B M$ with the comodule structure $\psi \otimes \mathrm{id}_M \colon \Sigma \otimes_B M \to \Sigma \otimes_B \Sigma \otimes_B M$ [44, Definition A1.2.1]. Under the equivalence above this translates into the right adjoint $f^B_* \colon \mathrm{QCoh}(\operatorname{Spec} B/\mathbb{G}_m) \to \mathrm{QCoh}(\mathcal{X})$ to $(f^B)^*$.

The pullback $\operatorname{Spec} B \times_{\mathcal{X}} \operatorname{Spec} B/\mathbb{G}_m$ can be identified with $\operatorname{Spec} \Sigma$ and the $\mathbb{G}_m$-action on $\operatorname{Spec} B$ induces a $\mathbb{G}_m$-action on $\operatorname{Spec} \Sigma$ that corresponds to the grading on Σ. By [41, Section 3.3], the map $\operatorname{Spec} B \to \mathcal{X}$ is fpqc and as $\operatorname{Spec} B \to \operatorname{Spec} B/\mathbb{G}_m$ is also fpqc, $\operatorname{Spec} B/\mathbb{G}_m \to \mathcal{X}$ is fpqc as well.

In the next lemma we investigate what happens after base change along a morphism $B \to C$.

Lemma 4.4. *Let $B \to C$ and $B \to D$ be grading preserving ring morphisms. This induces a morphism $f^C \colon \operatorname{Spec} C/\mathbb{G}_m \to \operatorname{Spec} B/\mathbb{G}_m \to \mathcal{X}$ and similarly for D.*

1. *The pullback $\operatorname{Spec} B \times_{\mathcal{X}} \operatorname{Spec} C/\mathbb{G}_m$ is equivalent to $\operatorname{Spec} \Sigma \otimes_B C$, where Σ is a B-module via the right unit η_R.*
2. *Under the equivalence from Proposition 4.3, the quasi-coherent sheaf $f^C_* \mathcal{O}_{\operatorname{Spec} C/\mathbb{G}_m}$ corresponds to the extended (B,Σ)-comodule structure on $\Sigma \otimes_B C$.*
3. *The pullbacks $\operatorname{Spec} C \times_{\mathcal{X}} \operatorname{Spec} D/\mathbb{G}_m$ and $\operatorname{Spec} D/\mathbb{G}_m \times_{\mathcal{X}} \operatorname{Spec} C$ are equivalent to $\operatorname{Spec} \Omega$ with $\Omega = C \otimes_B \Sigma \otimes_B D$ or $\Omega = D \otimes_B \Sigma \otimes_B C$, respectively. If $C = D$ and the maps $B \to C$ coincide, then (C,Ω) obtains the structure of a graded Hopf algebroid.*
4. *The stack associated with (C,Ω) is equivalent to $\mathcal{X}$.*

Proof. Recall that $\operatorname{Spec} B \times_{\mathcal{X}} \operatorname{Spec} B/\mathbb{G}_m \simeq \operatorname{Spec} \Sigma$. Pulling back along $\operatorname{Spec} C/\mathbb{G}_m \to \operatorname{Spec} B/\mathbb{G}_m$ gives the equivalence in the first item. The second item follows by the remarks above as we can factor f^C into the map $\operatorname{Spec} C/\mathbb{G}_m \to \operatorname{Spec} B/\mathbb{G}_m$ and f^B. Pulling back the equivalence from the first item along $\operatorname{Spec} D \to \operatorname{Spec} B$ gives the equivalences in the third item. The pullback $\operatorname{Spec} C \times_{\mathcal{X}} \operatorname{Spec} C$ is of the form $\Omega[u^{\pm 1}]$ for analogous reasons. By [41, Section 3.3], $\mathcal{X}$ is the stack associated with the (ungraded) Hopf algebroid $(C, \Omega[u^{\pm 1}])$ and thus also with the graded Hopf algebroid (C,Ω). □

4.2. **Stacks related to elliptic curves.** Recall that we are working with the moduli stack of elliptic curves $\mathcal{M}_{ell}$. Let

$$A := \mathbb{Z}[a_1, a_2, a_3, a_4, a_6] \text{ and } \Gamma := A[r,s,t].$$

There is an element $\Delta \in A$ corresponding to the discriminant for cubical curves, see e.g. [46, Section III.1] for a precise formula. The stack $\mathcal{M}_{ell}$ is equivalent to the stack associated with the graded Weierstraß Hopf algebroid $(A[\Delta^{-1}], \Gamma[\Delta^{-1}])$ in the sense recalled in Section 4.1. For the precise structure maps, see e.g. [2]; note the name comes from the fact that this Hopf algebroid is related to Weierstraß equation for elliptic curves and the right unit η_R comes from change-of-coordinates formulas for these. Our grading convention is that $|a_i| = i$.

One observes that the structure maps do not use the fact that Δ was involved, so one can consider the graded Weierstraß Hopf algebroid (A,Γ).

Definition 4.5. Let *the moduli stack of cubical curves* $\mathcal{M}_{cub}$ be the Artin stack associated with the graded Weierstraß Hopf algebroid (A, Γ).

The name is justified, as there is a modular interpretation for this stack, see [32, Section 3.1], and in particular the morphism $\operatorname{Spec} A \to \mathcal{M}_{cub}$ classifies the Weierstraß cubical curve over A given by the usual equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

We record the relationship with the moduli stack of elliptic curves.

Lemma 4.6. *There is a pullback square*

$$\begin{tikzcd} \operatorname{Spec} A[\Delta^{-1}] \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ \mathcal{M}_{ell} \arrow[r] & \mathcal{M}_{cub} \end{tikzcd}$$

In particular, $\mathcal{M}_{cub}$ contains $\mathcal{M}_{ell}$ as an open substack and the inclusion is an affine morphism.

Proof. The existence of the pullback square corresponds to the fact that a cubical curve given by a Weierstraß equation is an elliptic curve if and only if its discriminant Δ is invertible.

As noted in the last subsection, $\operatorname{Spec} A \to \mathcal{M}_{cub}$ is fpqc. Since both being open immersion and being affine can be checked after faithfully flat base change, the remaining claims follow. $\square$

Definition 4.7. We define the line bundle $\underline{\omega}$ on $\mathcal{M}_{cub}$ to be the one corresponding to the shift $A[1]$ under the equivalence between quasi-coherent sheaves on $\mathcal{M}_{cub}$ and graded (A, Γ)-comodules.

By [11, (1.2)], our definition of $\underline{\omega}$ agrees with the more geometric definition of Deligne; in particular, our definition restricts to the corresponding definition on $\overline{\mathcal{M}}_{ell}$ we give in the appendix. Next, we will need the following easy lemma.

Lemma 4.8. *The morphism* $\operatorname{Spec} A/\mathbb{G}_m \to \mathcal{M}_{cub}$ *is smooth.*

Proof. As noted in the last subsection, $\operatorname{Spec} A \to \mathcal{M}_{cub}$ is fpqc. Moreover, $\Gamma = A[r, s, t]$ is smooth over A and $\operatorname{Spec}\Gamma \simeq \operatorname{Spec} A \times_{\mathcal{M}_{cub}} \operatorname{Spec} A/\mathbb{G}_m$. As smoothness can be tested after base change along an fpqc morphism [48, Tag 02VL], we obtain the claim. $\square$

When working at the prime 3, it turns out to be more convenient to work with a different smooth cover of $\mathcal{M}_{cub} = \mathcal{M}_{cub,\mathbb{Z}_{(3)}}$, namely with $\operatorname{Spec} \widetilde{A} \to \mathcal{M}_{cub}$, where $\widetilde{A} := \mathbb{Z}_{(3)}[\overline{a}_2, \overline{a}_4, \overline{a}_6]$ and the morphism is given by composition of the canonical morphism $\operatorname{Spec} A \to \mathcal{M}_{cub}$ with the one induced by the ring map $A \to \widetilde{A}$, given by

$$a_1, a_3 \mapsto 0 \text{ and } a_i \mapsto \overline{a}_i \text{ for } i \in \{2, 4, 6\}.$$

The cubical curve corresponding to the morphism $\operatorname{Spec} \widetilde{A} \to \mathcal{M}_{cub}$ is

$$y^2 = x^3 + \overline{a}_2x^2 + \overline{a}_4x + \overline{a}_6.$$

Note that there are different conventions for simplifying the elliptic or cubical curves when 2 is inverted. In particular, the convention used in [46] differs from ours.

We want to show that this smooth cover induces a different Hopf algebroid $(\widetilde{A}, \widetilde{\Gamma})$ with $\widetilde{\Gamma} := \widetilde{A} \otimes_A \Gamma \otimes_A \widetilde{A}$ representing $\mathcal{M}_{cub}$. For more details on the explicit description of $(\widetilde{A}, \widetilde{\Gamma})$, see [2, Section 3]. We will first prove that it is indeed a presentation for $\mathcal{M}_{cub}$, and then recall some of the structure maps we will be using later.

Lemma 4.9. *At the prime* 3*, the stack associated to the graded Hopf algebroid* $(\widetilde{A}, \widetilde{\Gamma})$ *is equivalent to* $\mathcal{M}_{cub}$*. In particular, there is an equivalence between quasi-coherent sheaves on* $\mathcal{M}_{cub}$ *and graded* $(\widetilde{A}, \widetilde{\Gamma})$*-comodules. Moreover, the morphism* $\operatorname{Spec} \widetilde{A}/\mathbb{G}_m \to \mathcal{M}_{cub}$ *is a smooth cover.*

Proof. We would like to apply Lemma 4.4. Thus, we only have to check that the composition $\operatorname{Spec} \widetilde{A}/\mathbb{G}_m \to \operatorname{Spec} A/\mathbb{G}_m \to \mathcal{M}_{cub}$ is fpqc. As explained in Section 4.1, the map $\operatorname{Spec} A \to \mathcal{M}_{cub}$ is fpqc, so by faithfully flat descent, it is enough to check that the pullback map $\operatorname{Spec} A \times_{\mathcal{M}_{cub}} \operatorname{Spec} \widetilde{A}/\mathbb{G}_m \to \operatorname{Spec} A$ is smooth. By Lemma 4.4, the source can be identified with $\operatorname{Spec} \Gamma \otimes_A \widetilde{A}$. By inspection, we arrive at the isomorphism of A-modules

$$\Gamma \otimes_A \widetilde{A} \cong A[r, s, t]/(\eta_R(a_1), \eta_R(a_3)).$$

Using the right unit formulae

$$\begin{aligned} \eta_R(a_1) &= a_1 + 2s, \\ \eta_R(a_3) &= a_3 + a_1 r + 2t, \end{aligned}$$

and the fact that we inverted 2, we get $\Gamma \otimes_A \widetilde{A} \cong A[r]$. In particular, this is a smooth A-module, thus we conclude the claim. □

By the proof of the previous lemma we obtain

$$\widetilde{\Gamma} \cong \tilde{A} \otimes_A A[r] \cong \widetilde{A}[r].$$

The structure formulae for the Hopf algebroid (A, Γ) determine under this identification the formulae

$$\begin{aligned} \eta_R(\overline{a}_2) &= \overline{a}_2 + 3r, \\ \eta_R(\overline{a}_4) &= \overline{a}_4 + 2r\overline{a}_2 + 3r^2, \\ \eta_R(\overline{a}_6) &= \overline{a}_6 + r\overline{a}_4 + r^2\overline{a}_2 + r^3, \end{aligned}$$

whereas η_L is the canonical inclusion of $\widetilde{A}$.

Remark 4.10. Note that there is an even easier smooth cover of $\mathcal{M}_{cub}$ coming from $\mathcal{M}_1(2)_{cub}$ and a corresponding graded Hopf algebroid yielding $\mathcal{M}_{cub}$ again. For our approach, it has the following disadvantage: there is no meaningful module structure on $\mathrm{mf}_1(7)$ over the corresponding ring. On the contrary, $\mathrm{mf}_1(7)$ is a $\widetilde{A}$-module corresponding to a cubical curve discussed later.

5. The definition and properties of $\mathcal{M}_1(7)_{cub}$ and $\mathcal{M}_0(7)_{cub}$

Fix throughout the section a number $n \geq 2$ and the notation

$$A = \mathbb{Z}\left[\tfrac{1}{n}\right][a_1, \dots, a_4, a_6].$$

We want to extend the moduli stacks $\mathcal{M}_0(n) = \mathcal{M}_0(n)_{\mathbb{Z}[\frac{1}{n}]}$ and $\mathcal{M}_1(n) = \mathcal{M}_1(n)_{\mathbb{Z}[\frac{1}{n}]}$ to algebraic stacks that are finite over $\mathcal{M}_{cub}$ via a normalization construction.

5.1. **Definition and basic properties of $\mathcal{M}_1(n)_{cub}$ and $\mathcal{M}_0(n)_{cub}$.** Let us recall the notion of normalization. Let $\mathcal{X}$ be an Artin stack and $\mathcal{A}$ a quasi-coherent sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras. Let $\mathcal{A}' \subset \mathcal{A}$ be the presheaf that evaluated on any smooth $\operatorname{Spec} C \to \mathcal{X}$ consists of those elements in $\mathcal{A}(\operatorname{Spec} C)$ that are integral over C. This is an fpqc (and in particular étale) sheaf because being integral for an element can be tested fpqc-locally (as generating a finite module can be checked fpqc-locally). Thus, we obtain a sheaf on the subsite of the lisse-étale site of $\mathcal{X}$ consisting of affine schemes (see [29, Section 12] or [48, Tag 0786] for the definition).

Lemma 5.1. *This construction has the following properties.*

1. *If $\mathcal{X} = \operatorname{Spec} D$ is affine, then $\mathcal{A}'$ is the quasi-coherent sheaf associated with the D-algebra that is the normalization of D in $\mathcal{A}(\operatorname{Spec} D)$.*
2. *If $p\colon \mathcal{Y} \to \mathcal{X}$ is a smooth morphism of Artin stacks, the map $p^*(\mathcal{A}') \to (p^*\mathcal{A})'$ is an isomorphism.*
3. *The sheaf $\mathcal{A}'$ is quasi-coherent for general Artin stacks $\mathcal{X}$.*

Proof. Let $\mathcal{X} = \operatorname{Spec} D$ affine and let $D \to C$ be a smooth map of rings. By [48, Tag 03GG] and using that $\mathcal{A}$ is quasi-coherent we obtain that the canonical map $\mathcal{A}'(\operatorname{Spec} D) \otimes_D C \to \mathcal{A}'(\operatorname{Spec} C)$ is an isomorphism. This implies that $\mathcal{A}'$ is quasi-coherent in this case.

Let now $p\colon \mathcal{Y} \to \mathcal{X}$ be a smooth morphism of Artin stacks with $\mathcal{X}$ general again and let $\operatorname{Spec} C \xrightarrow{q} \mathcal{Y}$ be smooth as well. Then both $p^*(\mathcal{A}')(\operatorname{Spec} C)$ and $(p^*\mathcal{A})'(\operatorname{Spec} C)$ are computed as the normalization of C in $(q^*p^*\mathcal{A})(C) = \mathcal{A}(C)$.

Finally, let

$$\begin{tikzcd} \operatorname{Spec} C \arrow[rr, "r"] \arrow[dr, "p"'] & & \operatorname{Spec} D \arrow[dl, "q"] \\ & \mathcal{X} & \end{tikzcd}$$

be a 2-commutative diagram where the vertical maps are smooth. To show the quasi-coherence of $\mathcal{A}'$, we need to show that the natural map

$$\mathcal{A}'(\operatorname{Spec} D) \otimes_D C \to \mathcal{A}'(\operatorname{Spec} C) \tag{5.2}$$

is an isomorphism. From the above, $q^*\mathcal{A}'$ is quasi-coherent on $\operatorname{Spec} D$. Hence $(r^*q^*\mathcal{A}')(\operatorname{Spec} C)$ can be computed as $(q^*\mathcal{A}')(\operatorname{Spec} D) \otimes_D C = \mathcal{A}'(\operatorname{Spec} D) \otimes_D C$. Thus we can identify the map (5.2) with the evaluation of the isomorphism $r^*q^*\mathcal{A}'_* \cong p^*\mathcal{A}'$ on $\operatorname{Spec} C$. ☐

Definition 5.3. We define the *normalization of $\mathcal{X}$ in $\mathcal{A}$* to be the relative Spec of $\mathcal{A}'$ over $\mathcal{X}$. For a quasi-compact and quasi-separated morphism $p\colon \mathcal{Y} \to \mathcal{X}$, we define the *normalization* of $\mathcal{X}$ in $\mathcal{Y}$ to be the normalization of $\mathcal{X}$ in $p_*\mathcal{O}_{\mathcal{Y}}$, where p_* denotes the pushforward of quasi-coherent sheaves as in [48, Tag 070A].

Directly from Lemma 5.1 and [29, Proposition 13.1.9] we obtain:

Lemma 5.4. *Relative normalization commutes with smooth base change.*

Recall that the compactifications $\overline{\mathcal{M}}_0(n)$ and $\overline{\mathcal{M}}_1(n)$ are defined as the normalizations of $\overline{\mathcal{M}}_{ell}$ in $\mathcal{M}_0(n)$ and $\mathcal{M}_1(n)$, respectively. This motivates the following definition.

Definition 5.5. We define $\mathcal{M}_0(n)_{cub}$ and $\mathcal{M}_1(n)_{cub}$ as the normalizations of $\mathcal{M}_{cub}$ in $\mathcal{M}_0(n)$ and $\mathcal{M}_1(n)$.

We warn that there is no reason to expect the map $\mathcal{M}_1(n)_{cub} \to \mathcal{M}_0(n)_{cub}$ to be the stack quotient by the natural $(\mathbb{Z}/n)^\times$-action on the source.

Note that the normalization maps are by definition affine and in the next lemma we will even show finiteness.

Lemma 5.6. *The maps $\mathcal{M}_1(n)_{cub} \to \mathcal{M}_{cub}$ and $\mathcal{M}_0(n)_{cub} \to \mathcal{M}_{cub}$ are finite.*

Proof. Let $\mathcal{X} \to \mathcal{M}_{ell}$ be an affine map of finite type from a reduced Artin stack. Note that reducedness is local in the smooth topology [48, Tag 034E]. We want to show that the normalization $\mathcal{X}'$ of $\mathcal{M}_{cub}$ in $\mathcal{X}$ is finite over $\mathcal{M}_{cub}$. The relevant cases for us are $\mathcal{X} = \mathcal{M}_1(n)$ and $\mathcal{X} = \mathcal{M}_0(n)$.

Let $\operatorname{Spec} A \to \mathcal{M}_{cub}$ be the usual smooth cover. Denote by T the global sections of the pullback $\mathcal{X} \times_{\mathcal{M}_{cub}} \operatorname{Spec} A$, which is an affine scheme. By Lemma 5.4, the pullback $\mathcal{X}' \times_{\mathcal{M}_{cub}} \operatorname{Spec} A$ is equivalent to the spectrum of the normalization A' of A in T. As finiteness can be checked after faithfully flat base change, it thus suffices to show that A' is a finite A-module.

By [48, Tag 03GR], we just have to check that A is a Nagata ring, $\operatorname{Spec} T \to \operatorname{Spec} A$ is of finite type and T is reduced. As A is a polynomial ring over a quasi-excellent ring, it is quasi-excellent again and hence Nagata [48, Tag 07QS]. The second point is clear by base change. For the last one note that $\operatorname{Spec} T$ is equivalent to $\operatorname{Spec} A[\Delta^{-1}] \times_{\mathcal{M}_{ell}} \mathcal{X}$ by Lemma 4.6, and also that this pullback is affine. Moreover, $\operatorname{Spec} A[\Delta^{-1}] \to \mathcal{M}_{ell}$ is smooth by Lemma 4.8. Since being reduced is local in the smooth topology, we conclude that T is reduced. Hence, we obtain finiteness of A' over A and thus of $\mathcal{X}'$ over $\mathcal{M}_{cub}$. □

5.2. **Commutative algebra of rings of modular forms.** For structural results about $\mathcal{M}_1(n)_{cub}$ we need some information about the commutative algebra of rings of modular forms. Throughout this subsection, we will use the abbreviation $\mathrm{mf}_1(n)$ for $\mathrm{mf}(\Gamma_1(n); \mathbb{Z}[\frac{1}{n}])$. Recall from A.5 that $\mathrm{mf}_k(\Gamma_1(n); R) \cong H^0(\overline{\mathcal{M}}_1(n)_R; \underline{\omega}^{\otimes k})$ for every subring $R \subset \mathbb{C}$ and thus we set in general $\mathrm{mf}_k(\Gamma_1(n); R) = H^0(\overline{\mathcal{M}}_1(n)_R; \underline{\omega}^{\otimes k})$ for every $\mathbb{Z}[\frac{1}{n}]$-algebra R. In general, $\mathrm{mf}(\Gamma_1(n); R)$ differs from $\mathrm{mf}_1(n) \otimes R$, but we have the following useful lemma.

Lemma 5.7. *Let $R \to S$ be a flat ring extension of $\mathbb{Z}[\frac{1}{n}]$-algebras. Then the canonical map $\mathrm{mf}(\Gamma_1(n); R) \otimes_R S \to \mathrm{mf}(\Gamma_1(n); S)$ is an isomorphism.*

Proof. This is a variant of flat base change applied to the sheaf $\underline{\omega}^{\otimes i}$ on $\mathcal{M}_1(n)$. If $\mathcal{M}_1(n)$ is a scheme, we can apply [30, Lemma 5.2.26] directly. For Deligne–Mumford stacks, the proof is the same using étale instead of Zariski coverings. □

Our next goal is to investigate when $\mathrm{mf}_1(n)$ is Cohen–Macaulay. Recall that a commutative ring R is called *Cohen–Macaulay* if the depth of every maximal ideal $\mathfrak{m}R_\mathfrak{m}$ equals the Krull dimension of $R_\mathfrak{m}$. On the other hand $\mathrm{mf}_1(n)$ is a graded ring and it might appear more natural to consider the following graded analogue: A graded ring R is *graded Cohen–Macaulay* if for every *homogeneous* ideal $\mathfrak{m} \subset R$ the graded depth of $\mathfrak{m}R_\mathfrak{m}$ equals the Krull dimension of $R_\mathfrak{m}$, where the graded depth is at least d if there is a regular sequence of homogeneous elements $x_0, \ldots, x_d$. Note that the Krull dimension of $R_\mathfrak{m}$ agrees with the length of the maximal chain of homogeneous ideals by [6, Theorem 1.5.8]. It turns out that under usual circumstances there is no difference between the two notions of Cohen–Macaulay for graded rings.

Lemma 5.8. *Let R be a nonnegatively graded noetherian commutative ring. Then R is Cohen–Macaulay if and only if it is graded Cohen–Macaulay.*

Proof. The ring R is Cohen–Macaulay if and only if for every maximal ideal $\mathfrak{m} \subset R_0$ the ring $R_\mathfrak{m}$ is Cohen–Macaulay. Likewise, the ring R is graded Cohen–Macaulay if and only if every maximal ideal $\mathfrak{m} \subset R_0$ the ring $R_\mathfrak{m}$ is graded Cohen–Macaulay. Thus we can assume that R_0 is local and thus R is local in the graded sense, i.e. has a unique maximal homogeneous ideal, namely the ideal I generated by the elements of positive degree.

By [6, Exercise 2.1.27], R is Cohen–Macaulay if and only if R_I is Cohen–Macaulay. The Krull dimension d of R equals that of R_I. By definition, R_I is Cohen–Macaulay if there is a regular sequence of length d in IR_I. This is equivalent to there being a regular sequence of length d in I itself by (the proof of) [24, Theorem 135]. By [6, Proposition 1.5.11], this happens if and only if there is a regular sequence of homogeneous elements of length d in I, i.e. that R is graded Cohen–Macaulay. □

Remark 5.9. By [6, Corollary 2.2.6] every regular ring is Cohen–Macaulay, but the ring $\mathrm{mf}_1(n)$ is not regular in general, even over the complex numbers. Indeed, the maximal ideal of $\mathrm{mf}_1(n) \otimes \mathbb{C}$ is generated by all elements of positive degree and thus needs at least $\dim_\mathbb{C} \mathrm{mf}_1(n) \otimes \mathbb{C}$ many generators. On the other hand, $\mathrm{mf}_1(n)_1 \otimes \mathbb{C}$ has Krull dimension 2 as in the proof of [38, Theorem 5.14] and thus $\mathrm{mf}_1(n) \otimes \mathbb{C}$ can only be regular if $\mathrm{mf}_1(n) \otimes \mathbb{C}$ is of dimension at most 2. This does not happen for $n \geq 7$ as the dimension of $\mathrm{mf}_1(n)_1 \otimes \mathbb{C}$ is at least half the number of regular cusps, i.e. at least $\frac{1}{4}\sum_{d|n} \varphi(d)\varphi(\frac{n}{d})$ by [14, Theorem 3.6.1 and Figure 3.3], where φ denotes Euler's totient function.

Proposition 5.10. *The ring $\mathrm{mf}_1(n)$ is a Cohen–Macaulay ring if and only if $\mathrm{mf}_1(n)_1 \to \mathrm{mf}_1(\Gamma_1(n); \mathbb{F}_l)$ is surjective for all primes l not dividing n. This happens if and only if $H^1(\overline{\mathcal{M}}_1(n); \underline{\omega})$ is torsionfree.*

Proof. This follows from [38, Theorem 5.14] and the fact that a ring is Cohen–Macaulay if all its localizations at maximal ideals are Cohen–Macaulay. □

Example 5.11. As noted in [38, Remark 3.14], the condition of Proposition 5.10 is equivalent to the existence of a *cusp form* in $\mathrm{mf}_1(\Gamma_1(n); \mathbb{F}_l)$ that is not liftable to a cusp form in $\mathrm{mf}_1(n)_1$. For $n \leq 28$, Buzzard [8] shows that only for $n = 23$ there is a nonvanishing cusp form in $\mathrm{mf}(\Gamma_1(n); \mathbb{F}_l)$. But [39, Corollary 5.8] shows that on $\overline{\mathcal{M}}_1(23)$ there is an isomorphism $\Omega^1_{\overline{\mathcal{M}}_1(23)/\mathbb{Z}[\frac{1}{n}]} \cong \underline{\omega}$. By [38, Proposition 2.11], this implies that we can identify the reduction map $\mathrm{mf}_1(n)_1 \to \mathrm{mf}_1(\Gamma_1(n); \mathbb{F}_l)$ with the surjection $\mathbb{Z}[\frac{1}{n}] \to \mathbb{F}_l$. (A similar argument also appears in [8].)

Thus, $\mathrm{mf}_1(n)$ is Cohen–Macaulay for all $2 \leq n \leq 28$. It is not Cohen–Macaulay for example for $n = 74$ or $n = 82$ (see [38, Remark 3.14]).

In the context of normalizations it is furthermore an important question whether the rings $\mathrm{mf}_1(n)$ are normal.

Conjecture 5.12. *The ring $\mathrm{mf}_1(n)$ is normal for every $n \geq 2$.*

We plan to come back to this conjecture in a forthcoming article. For the present article, the following example suffices.

Example 5.13. The rings $\mathrm{mf}_1(n)$ for $2 \leq n \leq 6$ are polynomial rings over $\mathbb{Z}[\frac{1}{n}]$ in two variables and thus regular. The ring $\mathrm{mf}_1(7) \cong \mathbb{Z}[\frac{1}{7}][z_1, z_2, z_3]/(z_1z_2 + z_2z_3 + z_3z_1)$ is no longer regular by Remark 5.9, but still Cohen–Macaulay and normal. The former follows either directly or by Example 5.11. Serre's criterion states that a ring is normal if it satisfies (R1) and (S2). The latter is automatic for Cohen–Macaulay rings and the former states that the localization at every prime ideal of height 1 is regular. The singular locus of $\operatorname{Spec} \mathrm{mf}_1(7)$ is the common vanishing locus of the derivatives of $z_1z_2 + z_2z_3 + z_3z_1$, i.e. the ideal $I = (z_2 + z_3, z_1 + z_3, z_1 + z_2)$. The quotient $\mathrm{mf}_1(7)/I$ is isomorphic to $\mathbb{Z}[\frac{1}{7}][x]/(2x, x^2)$ and thus the vanishing locus $V(I)$ has codimension 2. In particular, the localization at every prime ideal of height 1 is regular and $\mathrm{mf}_1(7)$ is normal.

5.3. **The flatness of $\mathcal{M}_1(n)_{cub} \to \mathcal{M}_{cub}$.** Throughout this section let n be chosen so that $\mathrm{mf}_1(n)$ is normal. Conjecturally, this is true for all $n \geq 2$ and we have shown it in Example 5.13 for $2 \leq n \leq 7$. Our first aim is to show that $\mathcal{M}_1(n)_{cub}$ agrees with $\operatorname{Spec} \mathrm{mf}_1(n)/\mathbb{G}_m$, but before this we state the following simple observation.

Lemma 5.14. *Let R be a graded normal domain with a graded ring map $A \to R$. Consider the induced map*

$$\operatorname{Spec} R[\Delta^{-1}]/\mathbb{G}_m \to \operatorname{Spec} A[\Delta^{-1}]/\mathbb{G}_m \to \mathcal{M}_{ell}.$$

Then the normalization of $\mathcal{M}_{cub}$ in $\operatorname{Spec} R[\Delta^{-1}]/\mathbb{G}_m$ is equivalent to $\operatorname{Spec} R/\mathbb{G}_m$ if $R_A = \Gamma \otimes_A R$ is finite over A.

Proof. According to Lemma 4.4, the pullback $\operatorname{Spec} A \times_{\mathcal{M}_{cub}} \operatorname{Spec} R/\mathbb{G}_m$ is equivalent to $\operatorname{Spec} R_A$. Thus, $\operatorname{Spec} R/\mathbb{G}_m$ is finite over $\mathcal{M}_{cub}$.

Let now $\operatorname{Spec} C \to \mathcal{M}_{cub}$ be any smooth map and denote by $\operatorname{Spec} R_C$ the fiber product $\operatorname{Spec} C \times_{\mathcal{M}_{cub}} \operatorname{Spec} R/\mathbb{G}_m$. As R_C is finite over C, every element of R_C is integral over C. As R is normal and R_C is smooth over R, also R_C is normal [48, Tag 033C]. Thus, every element that is integral over C (and hence R_C) in $R_C[\Delta^{-1}]$ is already in R_C. Thus, R_C is the normalization of C in $R_C[\Delta^{-1}]$. As $\operatorname{Spec} R_C[\Delta^{-1}]$ is equivalent to the fiber product $\operatorname{Spec} C \times_{\mathcal{M}_{cub}} \operatorname{Spec} R[\Delta^{-1}]/\mathbb{G}_m$, this shows the result. ☐

For the rest of the section, we reinstate the convention that we work implicitly over $\mathbb{Z}[\frac{1}{n}]$, i.e. that $\mathcal{M}_{cub}$ means $\mathcal{M}_{cub,\mathbb{Z}[\frac{1}{n}]}$ etc.

Proposition 5.15. *The maps $\overline{\mathcal{M}}_1(n) \to \overline{\mathcal{M}}_{ell}$ and $\overline{\mathcal{M}}_0(n) \to \overline{\mathcal{M}}_{ell}$ are finite and flat. In particular, also $\mathcal{M}_1(n) \to \mathcal{M}_{ell}$ and $\mathcal{M}_0(n) \to \mathcal{M}_{ell}$ are finite and flat. The degree d_n of $\overline{\mathcal{M}}_1(n) \to \overline{\mathcal{M}}_{ell}$ satisfies $d_n = n^2 \prod_{p|n}(1 - \frac{1}{p^2})$ and $\overline{\mathcal{M}}_0(n) \to \overline{\mathcal{M}}_{ell}$ is of degree $\frac{d_n}{\varphi(n)}$ for φ Euler's totient function.*

Proof. The first part is contained in Theorem 4.1.1 of [9]. For the formula for d_n note that the degree of $\overline{\mathcal{M}}_1(n) \to \overline{\mathcal{M}}_{ell}$ agrees with that of $\overline{\mathcal{M}}_1(n)_{\mathbb{C}} \to \overline{\mathcal{M}}_{ell,\mathbb{C}}$ as $\overline{\mathcal{M}}_{ell}$ is connected. As recalled in Appendix A.2.3, for $n \geq 5$ the analytification of $\overline{\mathcal{M}}_1(n)_{\mathbb{C}}$ agrees with $X_1(n)$ and as the generic point of $\overline{\mathcal{M}}_{ell,\mathbb{C}}$ has automorphism group of order 2, the degree d_n is twice the degree of $X_1(n) \to X_1(1)$, which is computed in [14, Sections 3.8+3.9]. The cases $n = 2, 3$ and 4 are easily computed by hand.

The degree of $\overline{\mathcal{M}}_0(n) \to \overline{\mathcal{M}}_{ell}$ agrees with that of $\mathcal{M}_0(n) \to \mathcal{M}_{ell}$. As $\mathcal{M}_1(n) \to \mathcal{M}_0(n)$ is a $(\mathbb{Z}/n)^{\times}$-Galois cover, it has degree $\varphi(n)$. The formula for the degree of $\mathcal{M}_0(n) \to \mathcal{M}_{ell}$ follows. ☐

Before we come to the next proposition, consider again the $\mathbb{G}_m$-torsor $\overline{\mathcal{M}}_1^1(n) \to \overline{\mathcal{M}}_1(n)$ that trivializes $\underline{\omega}$. As $H^0(\overline{\mathcal{M}}_1^1(n), \mathcal{O}_{\overline{\mathcal{M}}_1^1(n)}) = \mathrm{mf}_1(n)$, we obtain a $\mathbb{G}_m$-equivariant map $\overline{\mathcal{M}}_1^1(n) \to \operatorname{Spec} \mathrm{mf}_1(n)$ and thus $\overline{\mathcal{M}}_1(n) \to \operatorname{Spec} \mathrm{mf}_1(n)/\mathbb{G}_m$, where the $\mathbb{G}_m$ action on $\operatorname{Spec} \mathrm{mf}_1(n)$ corresponds to the standard grading on the ring of modular forms.

Lemma 5.16. *The map $\overline{\mathcal{M}}_1(n) \to \overline{\mathcal{M}}_{ell} \to \mathcal{M}_{cub}$ factors over $\operatorname{Spec} \mathrm{mf}_1(n)/\mathbb{G}_m$, resulting in the following commutative square:*

$$\begin{tikzcd}
\overline{\mathcal{M}}_1(n) \arrow[r, "j"] \arrow[d, "h"'] & \operatorname{Spec} \mathrm{mf}_1(n)/\mathbb{G}_m \arrow[d, "\widetilde{h}"] \\
\overline{\mathcal{M}}_{ell} \arrow[r, "i"'] & \mathcal{M}_{cub}
\end{tikzcd}$$

Proof. Proposition 3.5 yields a commutative square

$$\begin{tikzcd}
\overline{\mathcal{M}}_1^1(n) \arrow[r] \arrow[dd] & \operatorname{Spec} \mathrm{mf}_1(n) \arrow[d] \\
 & \operatorname{Spec} A \arrow[d] \\
\overline{\mathcal{M}}_{ell} \arrow[r] & \mathcal{M}_{cub}
\end{tikzcd}$$

Quotiening by $\mathbb{G}_m$ gives the result since the map $A \to \mathrm{mf}_1(n)$ is grading preserving (i.e. $|a_i| = i$). $\square$

Proposition 5.17. *The stack $\mathcal{M}_1(n)_{cub}$ is equivalent to $\operatorname{Spec} \mathrm{mf}_1(n)/\mathbb{G}_m$ and the A-module $R_A = \Gamma \otimes_A \mathrm{mf}_1(n)$ is finite.*

Proof. The stack $\mathcal{M}_1^1(n)$ is representable by an affine scheme for $n \geq 2$ (see e.g [38, Proposition 2.4, Example 2.5]). As $\mathrm{MF}_1(n)$ coincides with the global sections of $\mathcal{O}_{\mathcal{M}_1^1(n)}$, we obtain $\mathcal{M}_1^1(n) \simeq \operatorname{Spec} \mathrm{MF}_1(n)$ and thus $\mathcal{M}_1(n) = \operatorname{Spec} \mathrm{MF}_1(n)/\mathbb{G}_m$ for all $n \geq 2$.

Moreover by assumption $\mathrm{mf}_1(n)$ is normal. Thus, Lemma 5.14 reduces the proof to showing that R_A is finite over A for $R = \mathrm{mf}_1(n)$, where we use the map $\operatorname{Spec} R/\mathbb{G}_m \to \mathcal{M}_{cub}$ from the commutative square introduced in Lemma 5.16:

$$\begin{tikzcd}
\overline{\mathcal{M}}_1(n) \arrow[r, "j"] \arrow[d, "f"'] & \operatorname{Spec} R/\mathbb{G}_m \arrow[d, "\widetilde{f}"] \\
\overline{\mathcal{M}}_{ell} \arrow[r, "i"'] & \mathcal{M}_{cub}
\end{tikzcd}$$

Consider the cartesian square

$$\begin{tikzcd}
U \arrow[r, "k"] \arrow[d, "q"'] & \operatorname{Spec} A \arrow[d, "p"] \\
\overline{\mathcal{M}}_{ell} \arrow[r, "i"'] & \mathcal{M}_{cub},
\end{tikzcd}$$

where k is an open immersion onto the complement of the common vanishing locus $V(c_4, \Delta)$ of c_4 and Δ by [46, Proposition III.1.4]. As a quasi-coherent sheaf on

$\operatorname{Spec} R/\mathbb{G}_m$ is determined by its graded global sections as explained in Example 4.2, we see that $j_*\mathcal{O}_{\overline{\mathcal{M}}_1(n)}$ is exactly $\mathcal{O}_{\operatorname{Spec} R/\mathbb{G}_m}$. We see that R_A are the global sections of

$$p^*\widetilde{f}_*j_*\mathcal{O}_{\overline{\mathcal{M}}_1(n)} \cong p^*i_*f_*\mathcal{O}_{\overline{\mathcal{M}}_1(n)}.$$

As p is flat, we have an isomorphism $p^*i_*f_*\mathcal{O}_{\overline{\mathcal{M}}_1(n)} \cong k_*q^*f_*\mathcal{O}_{\overline{\mathcal{M}}_1(n)}$. As f is finite flat by the last proposition, $q^*f_*\mathcal{O}_{\overline{\mathcal{M}}_1(n)}$ is a vector bundle. We claim that for every reflexive sheaf $\mathcal{F}$ on U the pushforward $k_*\mathcal{F}$ is reflexive and hence coherent. In particular, this would imply that $\Gamma(k_*q^*f_*\mathcal{O}_{\overline{\mathcal{M}}_1(n)}) = R_A$ is a finitely generated A-module if we apply the claim to $\mathcal{F} = q^*f_*\mathcal{O}_{\overline{\mathcal{M}}_1(n)}$.

To finish the proof, let $\mathcal{F}$ be a reflexive sheaf on U. It is possible to extend $\mathcal{F}$ to a reflexive sheaf $\mathcal{E}$ on $\operatorname{Spec} A$ (see e.g. [37, Lemma 3.2]). By [20, Proposition 1.6], we see that $k_*\mathcal{F} \cong k_*k^*\mathcal{E} \cong \mathcal{E}$ as A is normal and the complement $V(c_4, \Delta)$ of U has codimension 2. Thus, $k_*\mathcal{F}$ is reflexive. □

Lemma 5.18. *Let R be a graded A-algebra that is Cohen–Macaulay. Assume furthermore that R is concentrated in nonnegative degrees and satisfies $R_0 = \mathbb{Z}\left[\frac{1}{n}\right]$. Then $R_A = \Gamma \otimes_A R$ is flat over A if it is finite over A.*

Proof. As $R_A \cong R[r,s,t]$, we see that R_A is Cohen–Macaulay as well. It suffices to show that $(R_A)_{(p)}$ is flat over $A_{(p)}$ for every prime p. Note that both $A_{(p)}$ and $(R_A)_{(p)}$ are graded local rings. As R_A is finite over A, we see that $\dim R_{(p)} = \dim (R_A)_{(p)}$. We obtain by a graded version of Hironaka's flatness criterion (see e.g. [17, Theorem 18.16]) that $(R_A)_{(p)}$ is flat over $A_{(p)}$. □

Recall for the next proposition our standing assumption that we only consider n such that $\mathrm{mf}_1(n)$ is normal.

Proposition 5.19. *If $\mathrm{mf}_1(n)_1 \to \mathrm{mf}_1(\Gamma_1(n);\mathbb{F}_l)$ is surjective for all primes l not dividing n, the map $\mathcal{M}_1(n)_{cub} \to \mathcal{M}_{cub}$ is flat. This is in particular true for all those $n \leq 28$ for which $\mathrm{mf}_1(n)$ is normal.*

Proof. This follows from Lemma 5.18, Proposition 5.10 and Example 5.11. □

We will fix the notation of previous lemma in the following, specializing the earlier choices.

Notation 5.20. For a given n and an A-algebra C, we define R_C and S_C by the equivalences

$$\operatorname{Spec} R_C \simeq \operatorname{Spec} C \times_{\mathcal{M}_{cub}} \mathcal{M}_1(n)_{cub}$$
$$\operatorname{Spec} S_C \simeq \operatorname{Spec} C \times_{\mathcal{M}_{cub}} \mathcal{M}_0(n)_{cub}.$$

Here we use that by construction $\mathcal{M}_1(n)_{cub} \to \mathcal{M}_{cub}$ and $\mathcal{M}_0(n)_{cub} \to \mathcal{M}_{cub}$ are affine. We will show in the next lemma that $S_C = (R_C)^{(\mathbb{Z}/n)^\times}$.

As normalization commutes with smooth base change by Lemma 5.4, $\mathcal{M}_{ell} \times_{\mathcal{M}_{cub}} \mathcal{M}_1(n)_{cub}$ is the normalization of $\mathcal{M}_{ell}$ in $\mathcal{M}_1(n)$, which is $\mathcal{M}_1(n)$ itself as $\mathcal{M}_1(n) \to \mathcal{M}_{ell}$ is finite by Proposition 5.15. Thus

$$\operatorname{Spec} C \times_{\mathcal{M}_{cub}} \mathcal{M}_1(n) \simeq \operatorname{Spec} R_C \times_{\mathcal{M}_{cub}} \mathcal{M}_{ell}.$$

As R_C is an A-algebra and $\operatorname{Spec} A \times_{\mathcal{M}_{cub}} \mathcal{M}_{ell} \simeq \operatorname{Spec} A[\Delta^{-1}]$ by Lemma 4.6, we obtain an equivalence $\operatorname{Spec} C \times_{\mathcal{M}_{cub}} \mathcal{M}_1(n) \simeq \operatorname{Spec} R_C[\Delta^{-1}]$ that forms a commutative square with defining equivalence of R_C and the obvious maps. Similarly, we obtain $\operatorname{Spec} C \times_{\mathcal{M}_{cub}} \mathcal{M}_0(n) \simeq \operatorname{Spec} S_C[\Delta^{-1}]$ with the analogous property.

Lemma 5.21. *Let C be an A-algebra such that the composite* $\operatorname{Spec} C \to \operatorname{Spec} A \to \mathcal{M}_{cub}$ *is smooth.*

1. *The map $S_C \to R_C^{(\mathbb{Z}/n)^\times}$ is an isomorphism.*
2. *The ring of invariants $R_{C[\Delta^{-1}]}^{(\mathbb{Z}/n)^\times}$ is projective over $C[\Delta^{-1}]$. Its rank is precisely the degree of the map $\overline{\mathcal{M}}_0(n) \to \overline{\mathcal{M}}_{ell}$.*

A formula for the degree of $\overline{\mathcal{M}}_0(n) \to \overline{\mathcal{M}}_{ell}$ was recalled in Proposition 5.15.

Proof. We start by analyzing the situation after inverting Δ. As the map $\mathcal{M}_1(n) \to \mathcal{M}_0(n)$ is a $(\mathbb{Z}/n)^\times$-torsor, the pullback

$$\operatorname{Spec} C \times_{\mathcal{M}_{cub}} \mathcal{M}_1(n) \to \operatorname{Spec} C \times_{\mathcal{M}_{cub}} \mathcal{M}_0(n)$$

is a $(\mathbb{Z}/n)^\times$-torsor as well. This map can be identified with $\operatorname{Spec} R_C[\Delta^{-1}] \to \operatorname{Spec} S_C[\Delta^{-1}]$ and thus the map $S_C[\Delta^{-1}] \to R_C[\Delta^{-1}]^{(\mathbb{Z}/n)^\times}$ is an isomorphism. As $\mathcal{M}_0(n) \to \mathcal{M}_{ell}$ is finite and flat, we deduce that $S_C[\Delta^{-1}]$ is projective over $C[\Delta^{-1}]$. By base change, the rank of this module is the degree of $\mathcal{M}_1(n) \to \mathcal{M}_0(n)$ (or equivalently of $\overline{\mathcal{M}}_0(n) \to \overline{\mathcal{M}}_{ell}$).

Next we want to show that the map $S_C \to (R_C)^{(\mathbb{Z}/n)^\times}$ is an isomorphism. By Lemma 5.4, S_C consists of those elements in $S_C[\Delta^{-1}]$ that are integral over C and in particular $S_C \to S_C[\Delta^{-1}]$ is an injection. For analogous reasons $R_C \to R_C[\Delta^{-1}]$ is injective as well. Thus, the two maps in the composition

$$S_C \to R_C^{(\mathbb{Z}/n)^\times} \to R_C^{(\mathbb{Z}/n)^\times}[\Delta^{-1}] \cong S_C[\Delta^{-1}]$$

are injections. Hence, it remains to show that every element in $R_C^{(\mathbb{Z}/n)^\times}$ is integral over C to obtain that $S_C \to R_C^{(\mathbb{Z}/n)^\times}$ is surjective as well.

As R_A is a finite A-module by Proposition 5.17, $R_C \cong C \otimes_A R_A$ is a finite C-module. Moreover, C is noetherian as it is smooth and hence finitely presented over the stack $\mathcal{M}_{cub}$ and the latter is noetherian because A is. Hence, $(R_C)^{(\mathbb{Z}/n)^\times}$ is finite over C and thus every element of it is indeed integral over C. ☐

6. Computation of invariants

Throughout this section, we localize implicitly at the prime 3.

Recall that at the prime 3, there is a smooth cover $\operatorname{Spec} \widetilde{A} \to \mathcal{M}_{cub}$, where $\widetilde{A} := \mathbb{Z}_{(3)}[\overline{a}_2, \overline{a}_4, \overline{a}_6]$, discussed in Section 4.2. This defines rings $R_{\widetilde{A}}$ and $S_{\widetilde{A}}$ as in Notation 5.20. Lemma 5.21 identifies $S_{\widetilde{A}}$ with the invariants $R_{\widetilde{A}}^{(\mathbb{Z}/7)^\times}$. Here and in the following we consider the case $n = 7$ so that $\operatorname{Spec} R_{\widetilde{A}} \simeq \operatorname{Spec} \widetilde{A} \times_{\mathcal{M}_{cub}} \mathcal{M}_1(7)_{cub}$. Our main goal in this section is to compute explicitly $R_{\widetilde{A}}$ together with its $(\mathbb{Z}/7)^\times$-action and also the invariants $S_{\widetilde{A}}$. Later we will see that the pushforward of $\mathcal{O}_{\mathcal{M}_0(n)_{cub}}$ to $\mathcal{M}_{cub}$ corresponds under the equivalence from Lemma 4.9 to a $(\widetilde{A}, \widetilde{\Gamma})$-comodule structure on $S_{\widetilde{A}}$. Thus, the computation of $S_{\widetilde{A}}$ will be key to our splitting results.

We will need to recall from Section 4.2 the graded Hopf algebroid $(\widetilde{A}, \widetilde{\Gamma})$, in particular, that $\widetilde{\Gamma} \cong \widetilde{A}[r]$, and that η_R is determined under this identification by

$$\begin{aligned}
\eta_R(\overline{a}_2) &= \overline{a}_2 + 3r, \\
\eta_R(\overline{a}_4) &= \overline{a}_4 + 2r\overline{a}_2 + 3r^2, \\
\eta_R(\overline{a}_6) &= \overline{a}_6 + r\overline{a}_4 + r^2\overline{a}_2 + r^3,
\end{aligned}$$

whereas η_L is the canonical inclusion of $\widetilde{A}$.

We transform the Tate normal form of the universal cubical curve over

$$\mathrm{mf}_1(7) \cong \mathbb{Z}_{(3)}[z_1, z_2, z_3]/(z_1z_2 + z_2z_3 + z_3z_1)$$

into the Weierstraß form

$$y^2 = x^3 + \kappa(\overline{a}_2)x^2 + \kappa(\overline{a}_4) + \kappa(\overline{a}_6),$$

determining a map $\kappa\colon \widetilde{A} \to \mathrm{mf}_1(7)$ which makes the diagram

$$\begin{tikzcd}
\operatorname{Spec} \mathrm{mf}_1(7) \arrow[r, "\kappa"] \arrow[d] & \operatorname{Spec} \widetilde{A} \arrow[d] \\
\mathcal{M}_1(7)_{cub} \arrow[r] & \mathcal{M}_{cub}
\end{tikzcd}$$

commutative. Note that from Theorem 3.12, we can deduce that the module structure on $\mathrm{mf}_1(7)$ is given by the map $\kappa\colon \widetilde{A} \to \mathrm{mf}_1(7)$ determined via

$$\begin{aligned}
\overline{a}_2 &\mapsto \frac{1}{4}\alpha_1^2 + \alpha_2 = \frac{1}{4}(z_1 - z_2 + z_3)^2 - z_2z_3, \\
\overline{a}_4 &\mapsto \frac{1}{2}\alpha_1\alpha_3 = \frac{1}{2}z_1z_3^2(z_1 - z_2 + z_3), \\
\overline{a}_6 &\mapsto \frac{1}{4}\alpha_3^2 = \frac{1}{4}z_1^2z_3^4.
\end{aligned}$$

Using Lemma 4.4 and Lemma 4.9, the map $\widetilde{A} \to \mathrm{mf}_1(7)$ allows us to rewrite $R_{\widetilde{A}}$ as follows:

$$R_{\widetilde{A}} \cong \widetilde{\Gamma}_{\eta_R} \underset{\widetilde{A}}{\otimes} \mathrm{mf}_1(7).$$

Proposition 6.1. *$R_{\widetilde{A}}$ is a free $\widetilde{A}$-module of rank* 48.

Proof. Recall that $\operatorname{Spec} R_{\widetilde{A}}$ is the pullback $\operatorname{Spec} \widetilde{A} \times_{\mathcal{M}_{cub}} \mathcal{M}_1(n)_{cub}$. The map $\mathcal{M}_1(n)_{cub} \to \mathcal{M}_{cub}$ is finite and flat by Lemma 5.6, Proposition 5.19 and Example 5.13. Thus, $R_{\widetilde{A}}$ is a finite projective module over $\widetilde{A}$. As $\widetilde{A}$ is a polynomial ring over a discrete valuation ring, the Quillen-Suslin-Theorem [43], [49] implies that $R_{\widetilde{A}}$ is already free. Its rank coincides with the degree of the map $\mathcal{M}_1(7)_{cub} \to \mathcal{M}_{cub}$ that coincides with that of the restriction $\overline{\mathcal{M}}_1(7) \to \overline{\mathcal{M}}_{ell}$. By Proposition 5.15, this is $7^2 - 1 = 48$. $\square$

We want to identify $R_{\widetilde{A}}$ with $\mathrm{mf}_1(7)[r]$. The tensor product $R_{\widetilde{A}} \cong \widetilde{\Gamma}_{\eta_R} \underset{\widetilde{A}}{\otimes} \mathrm{mf}_1(7)$ can be described as

$$R_{\widetilde{A}} \cong \mathrm{mf}_1(7)[\overline{a}_2, \overline{a}_4, \overline{a}_6, r]/(\eta_R(\overline{a}_2) = \kappa(\overline{a}_2), \eta_R(\overline{a}_4) = \kappa(\overline{a}_4), \eta_R(\overline{a}_6) = \kappa(\overline{a}_6)).$$

Looking closely at the formulae, we can eliminate $\overline{a}_2, \overline{a}_4, \overline{a}_6$ and this yields a ring isomorphism to $\mathrm{mf}_1(7)[r]$. The resulting composite

$$\lambda\colon \widetilde{A} \xrightarrow{\eta_L \otimes 1} \widetilde{\Gamma} \otimes_{\widetilde{A}} \mathrm{mf}_1(7) \cong \mathrm{mf}_1(7)[r]$$

defines a rather complicated $\widetilde{A}$-module structure. Concretely it is given by:

$$\begin{aligned}
\overline{a}_2 \mapsto \quad & \frac{1}{4}(z_1 - z_2 + z_3)^2 - z_2z_3 - 3r, \\
\overline{a}_4 \mapsto \quad & \frac{1}{2}z_1z_3^2(z_1 - z_2 + z_3) - 2r(\frac{1}{4}(z_1 - z_2 + z_3)^2 - z_2z_3 - 3r) - 3r^2, \\
\overline{a}_6 \mapsto \quad & \frac{1}{4}z_1^2z_3^4 - r(\frac{1}{2}z_1z_3^2(z_1 - z_2 + z_3) - 2r(\frac{1}{4}(z_1 - z_2 + z_3)^2 - z_2z_3 - 3r) - 3r^2) \\
& - r^2(\frac{1}{4}(z_1 - z_2 + z_3)^2 - z_2z_3 - 3r) - r^3.
\end{aligned}$$

The map λ corresponds to the projection $\operatorname{Spec}\widetilde{A} \times_{\mathcal{M}_{cub}} \mathcal{M}_1(7)_{cub} \to \operatorname{Spec}\widetilde{A}$ under the identification of the source with $\operatorname{Spec}\mathrm{mf}_1(7)[r]$.

Our next aim is to make Proposition 6.1 explicit. More precisely, we claim that there is an $\widetilde{A}$-basis of $R_{\widetilde{A}}$ of the form $X \sqcup Xr \sqcup Xr^2$, where X is a 16-element subset of the image of $\mathrm{mf}_1(7)$ in $R_{\widetilde{A}}$. To prove this, we will use the following graded version of the Nakayama lemma.

Lemma 6.2. *Let R be a nonnegatively graded commutative ring such that R_0 is local with maximal ideal $\mathfrak{m}_0$. Let $\mathfrak{m}$ be the homogeneous ideal generated by $\mathfrak{m}_0$ and the ideal R_+ of all homogeneous elements of positive degree. Let furthermore M and N be graded R-modules that are finitely generated over R_0 in every degree. Then a map $M \to N$ is surjective if $M/\mathfrak{m} \to N/\mathfrak{m}$ is surjective.*

Proof. It suffices to show that $N = 0$ if $N/\mathfrak{m} = 0$. By the usual Nakayama lemma it suffices to show that $N/(\mathfrak{m}_0)$ is zero. Assume otherwise and let i be the minimal non-vanishing degree of $N/(\mathfrak{m}_0)$. As $(N/(\mathfrak{m}_0))_i \cong (N/\mathfrak{m})_i$, we see that $(N/(\mathfrak{m}_0))_i$ vanishes as well. $\square$

Recall the notation $\sigma_1 = z_1 + z_2 + z_3$ and $\sigma_3 = z_1z_2z_3$ for elementary symmetric polynomials in z_i.

Lemma 6.3. *The subset*

$$\begin{aligned}
X = & \{1\} \cup \{\sigma_1, z_2, z_3\} \cup \{\sigma_1^2, \sigma_1z_2, \sigma_1z_3, z_2z_3\} \cup \{\sigma_1^3, \sigma_1^2z_2, \sigma_1^2z_3, \sigma_3\} \\
& \cup \{\sigma_1^4, \sigma_1^3z_2, \sigma_1^3z_3\} \cup \{\sigma_1^4z_2\}.
\end{aligned}$$

of $R_{\widetilde{A}}$ gives an $\widetilde{A}$-basis of $R_{\widetilde{A}}$ of the form $X \sqcup Xr \sqcup Xr^2$.

Proof. The verification is heavily based on a `MAGMA`-computation.

As we already know by Proposition 6.1 that $R_{\widetilde{A}}$ is a free $\widetilde{A}$-module of rank 48 and $\widetilde{A}$ is noetherian, it is enough to show that $X \sqcup Xr \sqcup Xr^2$ is a generating system (since it has precisely 48 elements).

We want to apply the graded Nakayama Lemma 6.2 to the ideal $I = (3, \overline{a}_2, \overline{a}_4, \overline{a}_6)$ in the ring $\widetilde{A}$.

It is enough to show that the images of $X \sqcup Xr \sqcup Xr^2$ form a basis of $R_{\widetilde{A}}/I$. This is done by the following `MAGMA` code.

```
F3:=FiniteField(3);
M<z1, z2, z3, r>:=PolynomialRing(F3,4);
ka2:=(z1-z2+z3)^2/4-z2*z3;
ka4:=z1*z3^2*(z1-z2+z3)/2;
ka6:=z1^2*z3^4/4;
```

```
la2:=ka2-3*r;
la4:=ka4-2*r*la2-3*r^2;
la6:=ka6-r*la4-r^2*la2-r^3;
RAtildeModI:=quo<M|z1*z2+z2*z3+z3*z1,la2,la4,la6>;
RAtildeModIasVSp, pr:=VectorSpace(RAtildeModI);
Dimension(RAtildeModIasVSp);
sigma1:=z1+z2+z3;
sigma3:=z1*z2*z3;
X:={1, sigma1, z2, z3, sigma1^2, sigma1*z2, sigma1*z3, z2*z3,
sigma1^3, sigma1^2*z2, sigma1^2*z3, sigma3,
sigma1^4, sigma1^3*z2, sigma1^3*z3, sigma1^4*z2};
Xr:={x*r: x in X};
Xr2:={x*r^2: x in X};
IsIndependent(pr(X) join pr(Xr) join pr(Xr2));
```

Here `ka2` denotes $\kappa(\overline{a}_2)$ and `la2` denotes $\lambda(\overline{a}_2)$ etc. We first check the quotient $R_{\widetilde{A}}/I$ to be 48-dimensional $\mathbb{F}_3$ vector space and then show that $X \sqcup Xr \sqcup Xr^2$ is linearly independent. $\square$

Now that we have some understanding of $R_{\widetilde{A}}$ as $\widetilde{A}$-module, we can look at the $(\mathbb{Z}/7)^\times$-action on it and the invariants under this action. Recall we have chosen the generator $\tau = [3] \in (\mathbb{Z}/7)^\times$ to act on the $z_i \in \mathrm{mf}_1(7)$ via

$$\begin{aligned}\tau(z_1) &= -z_3,\\ \tau(z_2) &= -z_1,\\ \tau(z_3) &= -z_2.\end{aligned}$$

This grading-preserving action induces an action on $R_{\widetilde{A}}$ by the identification of its spectrum with $\operatorname{Spec}\widetilde{A} \times_{\mathcal{M}_{cub}} (\operatorname{Spec}\mathrm{mf}_1(7)/\mathbb{G}_m)$ via Proposition 5.17 and thus on $\mathrm{mf}_1(7)[r]$ as well. By definition, the projections onto both factors are $(\mathbb{Z}/7)^\times$-equivariant with the trivial action on $\widetilde{A}$ and the action above on $\mathrm{mf}_1(7)$. Thus, the obvious inclusion $\mathrm{mf}_1(7) \to \mathrm{mf}_1(7)[r]$ is $(\mathbb{Z}/7)^\times$-equivariant and so is $\lambda\colon \widetilde{A} \to \mathrm{mf}_1(7)[r]$. In particular, $\tau(\lambda(\overline{a}_2)) = \lambda(\overline{a}_2)$ enforces $\tau(r) = r + z_2z_3$.

The computation of invariants relies on heavy `MAGMA` computations. We will list now some elements which can be checked to be invariant, and the remainder of the section is devoted to the proof that these elements actually form a basis of $S_{\widetilde{A}}$ as an $\widetilde{A}$-module, in particular proving that this module is free.

We consider the elements $1, \sigma_1^2, \sigma_1^4, \sigma_3^2$ as well as

$$\begin{aligned}n_4 &:= \sigma_1^2 r - z_1^3 z_3 - z_1 z_2^3 - z_1^2 z_3^2,\\ \sigma_1^2 n_4 &= \sigma_1^4 r - \sigma_1^2 \cdot (z_1^3 z_3 + z_1 z_2^3 + z_1^2 z_3^2),\\ n_6 &:= \sigma_1^2 r^2 - 2z_1^3 z_3 r - 2z_1 z_2^3 r - 2z_1^2 z_3^2 r + 2z_1^3 z_3^3 - z_1^2 z_3^4\\ &= 2n_4 r - \sigma_1^2 r^2 + 2z_1^3 z_3^3 - z_1^2 z_3^4,\\ \sigma_1^2 n_6 &= \sigma_1^2 \cdot (\sigma_1^2 r^2 - 2z_1^3 z_3 r - 2z_1 z_2^3 r - 2z_1^2 z_3^2 r + 2z_1^3 z_3^3 - z_1^2 z_3^4),\end{aligned}$$

which we claim to elements in $S_{\widetilde{A}}$.

Indeed, to check that the non-obvious elements n_4 and n_6 are invariant, we use `MAGMA`. For the example of n_4, we have used the following code:

```
QQ:=RationalField();
M<z1, z2, z3, r>:=PolynomialRing(QQ,4);
RAtildeQ:=quo<M|z1*z2+z2*z3+z3*z1>;
tau:=hom<M ->RAtildeQ| -z3, -z1, -z2, r+z2*z3>;
proj:=hom<M ->RAtildeQ| z1, z2, z3, r>;
sigma1:=z1+z2+z3;
n4:=sigma1^2*r -z1^3*z3-z1*z2^3-z1^2*z3^2;
tau(n4)-proj(n4);
```

Our aim is to prove the following proposition.

Proposition 6.4. *The elements*

$$1, \sigma_1^2, \sigma_1^4, n_4, \sigma_1^2 n_4, n_6, \sigma_1^2 n_6, \sigma_3^2$$

form a $\widetilde{A}$-basis of $S_{\widetilde{A}}$; in particular, $S_{\widetilde{A}}$ is a free $\widetilde{A}$-module of rank 8.

Proof. The proof will proceed in several steps.

Step 1: As a first step, we compute using MAGMA the following expressions for the listed invariants in terms of the basis $X \sqcup Xr \sqcup Xr^2$:

$$\begin{aligned}
1 =& 1\\
\sigma_1^2 =& \sigma_1^2\\
\sigma_1^4 =& \sigma_1^4\\
n_4 =& \frac{1}{2}\sigma_1^3 z_3 + 4\sigma_1^2 r - 6\sigma_1 z_3 r - 2\overline{a}_2\sigma_1 z_3 + \overline{a}_2\sigma_1^2 - 4\overline{a}_2^2 + 12\overline{a}_4\\
n_6 =& -\frac{33}{32}\sigma_1^4 r + \frac{3}{8}\sigma_1^3 z_2 r + \frac{13}{4}\sigma_1^3 z_3 r + \frac{233}{8}\sigma_1^2 r^2 - \frac{21}{4}\sigma_1 z_2 r^2 - 42\sigma_1 z_3 r^2\\
&+ \frac{3}{2} z_2 z_3 r^2 - 18\overline{a}_6 - \frac{7}{8}\overline{a}_4\sigma_1^2 - \frac{1}{4}\overline{a}_4\sigma_1 z_2 - \overline{a}_4\sigma_1 z_3 + \frac{13}{2}\overline{a}_4 z_2 z_3 + \frac{123}{2}\overline{a}_4 r\\
&- \frac{11}{2}\overline{a}_2^3 + \frac{11}{4}\overline{a}_2^2\sigma_1^2 - \frac{1}{2}\overline{a}_2^2\sigma_1 z_2 - 3\overline{a}_2^2\sigma_1 z_3 - 2\overline{a}_2^2 z_2 z_3 - \frac{41}{2}\overline{a}_2^2 r\\
&+ \frac{37}{2}\overline{a}_2\overline{a}_4 - \frac{11}{32}\overline{a}_2\sigma_1^4 + \frac{1}{8}\overline{a}_2\sigma_1^3 z_2 + \frac{3}{4}\overline{a}_2\sigma_1^3 z_3 + \frac{67}{4}\overline{a}_2\sigma_1^2 r\\
&- \frac{7}{2}\overline{a}_2\sigma_1 z_2 r - 24\overline{a}_2\sigma_1 z_3 r + \overline{a}_2 z_2 z_3 r\\
\sigma_1^2 n_4 =& -8\sigma_1^4 r + 6\sigma_1^3 z_2 r + 24\sigma_1^3 z_3 r + 252\sigma_1^2 r^2 - 336\sigma_1 z_3 r^2\\
&+ 24\overline{a}_4\sigma_1 z_2 - 16\overline{a}_4\sigma_1 z_3 + 96\overline{a}_4 z_2 z_3 + 576\overline{a}_4 r\\
&- 64\overline{a}_2^3 + 28\overline{a}_2^2\sigma_1^2 - 8\overline{a}_2^2\sigma_1 z_2 - 32\overline{a}_2^2\sigma_1 z_3 - 32\overline{a}_2^2 z_2 z_3 - 192\overline{a}_2^2 r\\
&+ 192\overline{a}_2\overline{a}_4 - 3\overline{a}_2\sigma_1^4 + 2\overline{a}_2\sigma_1^3 z_2 + 8\overline{a}_2\sigma_1^3 z_3 + 168\overline{a}_2\sigma_1^2 r - 224\overline{a}_2\sigma_1 z_3 r\\
\sigma_3^2 =& \frac{81}{64}\sigma_1^4 r - \frac{3}{16}\sigma_1^3 z_2 r - \frac{33}{8}\sigma_1^3 z_3 r - \frac{537}{16}\sigma_1^2 r^2 + \frac{69}{8}\sigma_1 z_2 r^2 + 51\sigma_1 z_3 r^2 - \frac{3}{4} z_2 z_3 r^2\\
&- 9\overline{a}_6 - \frac{17}{16}\overline{a}_4\sigma_1^2 + \frac{17}{8}\overline{a}_4\sigma_1 z_2 + \frac{1}{2}\overline{a}_4\sigma_1 z_3 - \frac{13}{4}\overline{a}_4 z_2 z_3 - \frac{267}{4}\overline{a}_4 r\\
&+ \frac{27}{4}\overline{a}_2^3 - \frac{27}{8}\overline{a}_2^2\sigma_1^2 + \frac{1}{4}\overline{a}_2^2\sigma_1 z_2 + \frac{11}{2}\overline{a}_2^2\sigma_1 z_3 + \overline{a}_2^2 z_2 z_3 + \frac{89}{4}\overline{a}_2^2 r\\
&- \frac{77}{4}\overline{a}_2\overline{a}_4 + \frac{27}{64}\overline{a}_2\sigma_1^4 - \frac{1}{16}\overline{a}_2\sigma_1^3 z_2 - \frac{11}{8}\overline{a}_2\sigma_1^3 z_3\\
&- \frac{179}{8}\overline{a}_2\sigma_1^2 r + \frac{23}{4}\overline{a}_2\sigma_1 z_2 r + 34\overline{a}_2\sigma_1 z_3 r - \frac{1}{2}\overline{a}_2 z_2 z_3 r
\end{aligned}$$

$$\begin{aligned}
\sigma_1^2 n_6 =&\frac{5933}{3488}\sigma_1^4 r^2 + \frac{7599}{872}\sigma_1^3 z_2 r^2 - \frac{255}{872}\sigma_1^3 z_3 r^2 \\
&+ \frac{2997}{218}\overline{a}_6\sigma_1^2 - \frac{11475}{109}\overline{a}_6\sigma_1 z_2 + \frac{816}{109}\overline{a}_6\sigma_1 z_3 \\
&- \frac{2339}{436}\overline{a}_4\sigma_1^4 + \frac{4267}{1744}\overline{a}_4\sigma_1^3 z_2 + \frac{21951}{1744}\overline{a}_4\sigma_1^3 z_3 + \frac{52113}{436}\overline{a}_4\sigma_1^2 r \\
&- \frac{28203}{436}\overline{a}_4\sigma_1 z_2 r - \frac{64187}{436}\overline{a}_4\sigma_1 z_3 r + \frac{2397}{109}\overline{a}_4 z_2 z_3 r - \frac{13005}{218}\overline{a}_4 r^2 \\
&+ \frac{16659}{109}\overline{a}_4^2 + \frac{11279}{1744}\overline{a}_2\sigma_1^4 r + \frac{789}{436}\overline{a}_2\sigma_1^3 z_2 r - \frac{7061}{436}\overline{a}_2\sigma_1^3 z_3 r - 168\overline{a}_2\sigma_1^2 r^2 \\
&+ 224\overline{a}_2\sigma_1 z_3 r^2 + \frac{15373}{436}\overline{a}_2\overline{a}_4\sigma_1^2 - \frac{1077}{436}\overline{a}_2\overline{a}_4\sigma_1 z_2 - \frac{17833}{436}\overline{a}_2\overline{a}_4\sigma_1 z_3 \\
&- \frac{6177}{109}\overline{a}_2\overline{a}_4 z_2 z_3 - \frac{46191}{109}\overline{a}_2\overline{a}_4 r + \frac{13485}{3488}\overline{a}_2^2\sigma_1^4 - \frac{2059}{1744}\overline{a}_2^2\sigma_1^3 z_2 \\
&- \frac{16675}{1744}\overline{a}_2^2\sigma_1^3 z_3 - \frac{66203}{436}\overline{a}_2^2\sigma_1^2 r + \frac{9401}{436}\overline{a}_2^2\sigma_1 z_2 r + \frac{86505}{436}\overline{a}_2^2\sigma_1 z_3 r \\
&- \frac{799}{109}\overline{a}_2^2 z_2 z_3 r + \frac{4335}{218}\overline{a}_2^2 r^2 - \frac{51561}{218}\overline{a}_2^2\overline{a}_4 + \frac{13485}{218}\overline{a}_2^4 - \frac{13485}{436}\overline{a}_2^3\sigma_1^2 \\
&+ \frac{2059}{436}\overline{a}_2^3\sigma_1 z_2 + \frac{16675}{436}\overline{a}_2^3\sigma_1 z_3 + \frac{2059}{109}\overline{a}_2^3 z_2 z_3 + \frac{15397}{109}\overline{a}_2^3 r
\end{aligned}$$

Step 2: We want to show that the 8 invariants listed in the statement of the proposition are $\widetilde{A}$-linearly independent elements of $R_{\widetilde{A}}$. Since $\widetilde{A}$ is torsion-free, it is enough to check linearly independency over $\widetilde{A}\otimes_{\mathbb{Z}}\mathbb{Q}$. We observe that there is a non-vanishing 8×8-minor in the 48×8-matrix corresponding to the map $\widetilde{A}^8 \to R_{\widetilde{A}} \cong \widetilde{A}^{48}$ given by the invariants above. More precisely, after tensoring with $\mathbb{Q}$, the determinant of the following matrix

$$\begin{array}{c|c}
 & \begin{matrix} 1 & \sigma_1^2 & \sigma_1^4 & n_4 & n_6 & \sigma_1^2 n_4 & \sigma_3^2 & \sigma_1^2 n_6 \end{matrix} \\
\begin{matrix} 1 \\ \sigma_1^2 \\ \sigma_1^4 \\ \sigma_1^2 r \\ \sigma_1^4 r \\ \sigma_1 z_2 r^2 \\ z_2 z_3 r^2 \\ \sigma_1^4 r^2 \end{matrix} &
\begin{pmatrix}
1 & * & * & * & * & * & * & * \\
0 & 1 & * & * & * & * & * & * \\
0 & 0 & 1 & * & * & * & * & * \\
0 & 0 & 0 & 4 & * & * & * & * \\
0 & 0 & 0 & 0 & -\frac{33}{32} & -8 & \frac{81}{64} & * \\
0 & 0 & 0 & 0 & -\frac{21}{4} & 0 & \frac{69}{8} & * \\
0 & 0 & 0 & 0 & \frac{3}{2} & 0 & -\frac{3}{4} & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5933}{3488}
\end{pmatrix}
\end{array}$$

is invertible in $\widetilde{A}\otimes_{\mathbb{Z}}\mathbb{Q}$. On the left, we recorded the elements of our chosen basis to which the selected 8 out of 48 rows correspond. This shows that the map $(\widetilde{A}\otimes_{\mathbb{Z}}\mathbb{Q})^8 \to R_{\widetilde{A}}\otimes_{\mathbb{Z}}\mathbb{Q}$ given by the invariants above is an inclusion of a direct $\widetilde{A}\otimes_{\mathbb{Z}}\mathbb{Q}$-summand.

Thus, we have shown that the 8 invariants listed above are $\widetilde{A}$-linearly independent elements of $R_{\widetilde{A}}$, so they generate a free sub-$\widetilde{A}$-module of $R_{\widetilde{A}}$ of rank 8, which we denote by V.

Step 3: Our next goal is to show that this module V is already all of $S_{\widetilde{A}}$ when tensored with $\mathbb{Q}$. Recall that we identify $S_{\widetilde{A}}$ with $R_{\widetilde{A}}^{(\mathbb{Z}/7)^\times}$ as in Lemma 5.21 and likewise $S_{\widetilde{A}\otimes\mathbb{Q}}$ with $R_{\widetilde{A}\otimes\mathbb{Q}}^{(\mathbb{Z}/7)^\times}$.

Moreover, there is an isomorphism $S_{\widetilde{A}\otimes\mathbb{Q}} \cong S_{\widetilde{A}} \otimes_{\mathbb{Z}} \mathbb{Q}$ since $R_{\widetilde{A}} \otimes_{\mathbb{Z}} \mathbb{Q}$ can be written as directed colimit of the form

$$R_{\widetilde{A}} \xrightarrow{\cdot 2} R_{\widetilde{A}} \xrightarrow{\cdot 3} \dots$$

and directed colimits commute with finite limits in the finitely presentable category of abelian groups (see e.g. [1, Proposition 1.59]).

As the order of $(\mathbb{Z}/7)^\times$ is invertible in $\widetilde{A}\otimes\mathbb{Q}$, the invariants $S_{\widetilde{A}\otimes\mathbb{Q}}$ are a direct summand of the free $\widetilde{A}\otimes\mathbb{Q}$-module $R_{\widetilde{A}\otimes\mathbb{Q}}$ and thus projective. By the Quillen-Suslin Theorem, it implies that $S_{\widetilde{A}\otimes\mathbb{Q}}$ is also free, automatically of rank 8 as this is true after inverting Δ by the second part of Lemma 5.21.

Since the map $V\otimes\mathbb{Q} \to R_{\widetilde{A}}\otimes\mathbb{Q}$ is split injective, so is the map $V\otimes\mathbb{Q} \to S_{\widetilde{A}}\otimes\mathbb{Q}$. Since we have shown now both sides to be free $\widetilde{A}\otimes_{\mathbb{Z}}\mathbb{Q}$-modules of rank 8, this map is also surjective and thus an isomorphism.

Step 4: In this step, we reduce the proof of the proposition to showing that the map $V \to S_{\widetilde{A}}$ (or to $R_{\widetilde{A}}$) is injective when we tensor it with $\mathbb{F}_3$. This will imply surjectivity of $V \to S_{\widetilde{A}}$. Indeed, let $x \in S_{\widetilde{A}}$ be some element. By the rational statement, we know that there is an element y in V and $k \in \mathbb{N}$ such that $3^k x = y$. If $k = 0$, we are done; otherwise we can conclude that y is mapped to 0 in $R_{\widetilde{A}}$ after tensoring with $\mathbb{F}_3$, so by injectivity of the map $V\otimes\mathbb{F}_3 \to R_{\widetilde{A}}\otimes\mathbb{F}_3$ it can be divided by 3 in V. Inductively, this implies the claim.

Step 5: Finally, we show that the map $V \to R_{\widetilde{A}}$ is still injective after tensoring $\mathbb{F}_3$. We do a similar computation for $\mathbb{F}_3$ as we did above rationally. This time, we consider the following 8×8-minor of the 48×8-matrix describing the inclusion

$V \to R_{\widetilde{A}}$, again with the corresponding basis elements displayed on the left:

$$\begin{array}{c} \\ 1 \\ \sigma_1^2 \\ \sigma_1^4 \\ \sigma_1^2 r \\ \sigma_1^3 z_3 \\ \sigma_1^4 r \\ \sigma_1^3 z_3 r \\ \sigma_1^4 r^2 \end{array}
\begin{array}{c} \begin{array}{cccccccc} 1 & \sigma_1^2 & \sigma_1^4 & n_4 & n_6 & \sigma_1^2 n_4 & \sigma_3^2 & \sigma_1^2 n_6 \end{array} \\
\begin{pmatrix}
1 & * & * & * & * & * & * & * \\
0 & 1 & * & * & * & * & * & * \\
0 & 0 & 1 & * & * & * & * & * \\
0 & 0 & 0 & 4 & \frac{67}{4}\overline{a}_2 & 168\overline{a}_2 & -\frac{179}{8}\overline{a}_2 & * \\
0 & 0 & 0 & \frac{1}{2} & \frac{3}{4}\overline{a}_2 & 8\overline{a}_2 & -\frac{11}{8}\overline{a}_2 & * \\
0 & 0 & 0 & 0 & -\frac{33}{32} & -8 & \frac{81}{64} & * \\
0 & 0 & 0 & 0 & \frac{13}{4} & 24 & -\frac{33}{8} & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5933}{3488}
\end{pmatrix} \end{array}$$

Its determinant is a rational multiple of $\overline{a}_2$ not divisible by 3. It shows that the map $V \to R_{\widetilde{A}}$ is still injective after tensoring with $\mathbb{F}_3$, as desired. This completes the proof of $V = S_{\widetilde{A}}$.

□

7. Comodule Structures

Recall from Section 4.2 that we denote by $\widetilde{A}$ the ring $\widetilde{A} = \mathbb{Z}_{(3)}[\overline{a}_2, \overline{a}_4, \overline{a}_6]$. Recall moreover that we obtain a graded Hopf algebroid $(\widetilde{A}, \widetilde{\Gamma})$ representing $\mathcal{M}_{cub,\mathbb{Z}_{(3)}}$, and that quasi-coherent sheaves on $\mathcal{M}_{cub,\mathbb{Z}_{(3)}}$ are equivalent to graded $(\widetilde{A}, \widetilde{\Gamma})$-comodules. Thus it suffices for our main algebraic theorem to provide an isomorphism of certain comodules, which will describe explicitly.

Throughout this section we will again (implicitly) localize everything at the prime 3. Moreover, we will denote by f_n the natural map $\mathcal{M}_1(n) \to \mathcal{M}_{ell}$ and by f'_n the resulting map $\mathcal{M}_1(n)_{cub} \to \mathcal{M}_{cub}$ from the normalization. In the case $n = 2$, we will use the abbreviations $f = f_2$ and $f' = f'_2$. Lastly, we use h_n for the natural map $\mathcal{M}_0(n) \to \mathcal{M}_{ell}$ and h'_n for the resulting map $\mathcal{M}_0(n)_{cub} \to \mathcal{M}_{cub}$

7.1. The comodule corresponding to $f_*f^*\mathcal{O}$.

We will use $\mathcal{O}$ as a shorthand notation for the structure sheaf on $\mathcal{M}_{ell}$ or $\mathcal{M}_{cub}$. Recall that the map $f' \colon \mathcal{M}_1(2)_{cub} \to \mathcal{M}_{cub}$ is affine by construction. This implies that the pushforward sheaf $(f')_*(f')^*\mathcal{O}$ is quasi-coherent. By the discussion above, it is equivalent to a certain $(\widetilde{A}, \widetilde{\Gamma})$-comodule.

At the prime 3, the universal elliptic curve with a $\Gamma_1(2)$-structure has an equation of the form

$$y^2 = x^3 + b_2x^2 + b_4x$$

with $(0, 0)$ being the chosen point of order 2, resulting in an identification $\mathcal{M}_1(2) \simeq \operatorname{Spec} \mathbb{Z}_{(3)}[b_2, b_4, \Delta^{-1}]/\mathbb{G}_m$ (see e.g. [3, Section 1.3]). As in [38, Example 2.1] one can deduce $\mathrm{mf}_1(2) \cong \mathbb{Z}_{(3)}[b_2, b_4]$. The resulting $\widetilde{A}$-module structure is given by

$$\overline{a}_2 \mapsto b_2 \text{ and } \overline{a}_4 \mapsto b_4 \text{ and } \overline{a}_6 \mapsto 0.$$

The corresponding $(\widetilde{A}, \widetilde{\Gamma})$-comodule is given by $\widetilde{\Gamma} \otimes_{\widetilde{A}} \mathrm{mf}_1(2)$ with extended comodule structure by Lemma 4.4. In this tensor product, we use the right $\widetilde{A}$-module structure of $\widetilde{\Gamma}$. A similar computation to the following appears in [2].

Lemma 7.1. *There is a ring isomorphism* $\widetilde{\Gamma} \otimes_{\widetilde{A}} \mathrm{mf}_1(2) \cong \widetilde{A}[r]/(\overline{a}_6 + \overline{a}_4 r + \overline{a}_2 r^2 + r^3)$, *and the comodule structure is an* $\widetilde{A}$*-module map determined by* $r \mapsto 1 \otimes r + r \otimes 1$ *(and* $\overline{a}_i \mapsto 1 \otimes \overline{a}_i$*).*

Forgetting the ring structure, we can identify this comodule with the free $\widetilde{A}$*-module* $\widetilde{A}w_1 \oplus \widetilde{A}w_2 \oplus \widetilde{A}w_3$ *with* $(\widetilde{A}, \widetilde{\Gamma})$*-comodule structure given by*

$$\begin{aligned} w_1 &\mapsto \quad 1 \otimes w_1, \\ w_2 &\mapsto \quad 1 \otimes w_2 + r \otimes w_1, \\ w_3 &\mapsto \quad 1 \otimes w_3 + 2r \otimes w_2 + r^2 \otimes w_1. \end{aligned}$$

Proof. Using the formulae for η_R, we obtain a ring isomorphism

$$\widetilde{\Gamma} \otimes_{\widetilde{A}} \mathrm{mf}_1(2) \cong \widetilde{A}[r, b_2, b_4]/(R),$$

where the relations R are generated by

$$\begin{aligned} &\overline{a}_2 + 3r = b_2, \\ &\overline{a}_4 + 2\overline{a}_2 r + 3r^2 = b_4, \\ &\overline{a}_6 + \overline{a}_4 r + \overline{a}_2 r^2 + r^3 = 0. \end{aligned}$$

This immediately implies the first claim.

The first statement about the comodule structure is straightforward since $\widetilde{\Gamma} \otimes_{\widetilde{A}} \mathrm{mf}_1(2)$ carries the extended comodule structure.

For the second description of the comodule structure, observe that there is an isomorphism of $\widetilde{A}$-modules

$$\widetilde{A}w_1 \oplus \widetilde{A}w_2 \oplus \widetilde{A}w_3 \to \widetilde{A}[r]/(\overline{a}_6 + \overline{a}_4 r + \overline{a}_2 r^2 + r^3)$$

given by $w_i \mapsto r^{i-1}$. Thus, to identify the comodule structure, we only need to compute it on $1, r, r^2$ on the left-hand side, and transfer it via this isomorphism, using the compatibility of comodule structure with the ring structure of $\widetilde{A}[r]/(\overline{a}_6 + \overline{a}_4 r + \overline{a}_2 r^2 + r^3)$. This yields the claim. □

We will now identify the dual of the vector bundle $f_* f^* \mathcal{O}$ on $\overline{\mathcal{M}}_1(2)$, which will be useful in the next section. For the identification, we use the translation into comodules. Given a graded left $(\widetilde{A}, \widetilde{\Gamma})$-comodule M that is finitely generated free as a $\widetilde{A}$-module, we can define a right comodule structure on $\mathrm{Hom}_{\widetilde{A}}(M, \widetilde{A})$ whose coaction is given as in [44, Definition A1.1.6] by the composite of

$$\mathrm{Hom}_{\widetilde{A}}(M, \widetilde{A}) \xrightarrow{\widetilde{\Gamma} \otimes \mathrm{id}} \mathrm{Hom}_{\widetilde{A}}(\widetilde{\Gamma} \otimes_{\widetilde{A}} M, \widetilde{\Gamma}) \xrightarrow{\Psi_M^*} \mathrm{Hom}_{\widetilde{A}}(M, \widetilde{\Gamma})$$

with the inverse of the isomorphism

$$\mathrm{Hom}_{\widetilde{A}}(M, \widetilde{A}) \otimes_{\widetilde{A}} \widetilde{\Gamma} \xrightarrow{\cong} \mathrm{Hom}_{\widetilde{A}}(M, \widetilde{\Gamma}).$$

Using the conjugation c on $\widetilde{\Gamma}$ we can transform this into a left comodule. Recall from Lemma 4.9 the equivalence of quasi-coherent sheaves on $\mathcal{M}_{cub}$ to graded left

$(\widetilde{A},\widetilde{\Gamma})$-comodules and denote by $\mathcal{F}$ the sheaf corresponding to M. Under the same equivalence the graded left $(\widetilde{A},\widetilde{\Gamma})$-comodule $\operatorname{Hom}_{\widetilde{A}}(M,\widetilde{A})$ corresponds to the sheaf $\mathcal{H}om_{\mathcal{O}_{\mathcal{M}_{cub}}}(\mathcal{F},\mathcal{O}_{\mathcal{M}_{cub}})$.

Lemma 7.2. *The dual of $f_*f^*\mathcal{O}$ is isomorphic to $f_*f^*\underline{\omega}^{\otimes(-4)}$.*

Proof. Observe that the conjugation on $\widetilde{\Gamma}$ is given by $\overline{a}_i \mapsto \eta_R(\overline{a}_i)$ and $r \mapsto -r$ because it corresponds on the level of represented functors to inverting an isomorphism between Weierstraß curves. Inserting this into the above description of the internal hom and using Lemma 7.1, we arrive at the following left comodule structure on $\widetilde{A}w_1^*\oplus\widetilde{A}w_2^*\oplus\widetilde{A}w_3^*$:

$$\begin{aligned}
\widetilde{A}w_1^*\oplus\widetilde{A}w_2^*\oplus\widetilde{A}w_3^* \longrightarrow & \widetilde{\Gamma}\otimes_{\widetilde{A}}\left(\widetilde{A}w_1^*\oplus\widetilde{A}w_2^*\oplus\widetilde{A}w_3^*\right)\\
w_1^* \mapsto \quad & 1\otimes w_1^* - r\otimes w_2^* + r^2\otimes w_3^*\\
w_2^* \mapsto \quad & \qquad\qquad 1\otimes w_2^* - 2r\otimes w_3^*\\
w_3^* \mapsto \quad & \qquad\qquad\qquad\qquad 1\otimes w_3^*.
\end{aligned}$$

Looking more closely shows that this comodule is actually isomorphic to $\widetilde{A}w_1\oplus\widetilde{A}w_2\oplus\widetilde{A}w_3$ as an ungraded comodule from Lemma 7.1 via the following isomorphism:

$$w_1\mapsto w_3^*,\quad w_2\mapsto -\frac{1}{2}w_2^*,\quad w_3\mapsto w_1^*.$$

This map shifts grading by 4 and thus $\mathcal{H}om_{\mathcal{O}}(f'_*(f')^*\mathcal{O},\mathcal{O})\cong f'_*(f')^*\mathcal{O}\otimes\underline{\omega}^{\otimes(-4)}$ and by the projection formula this yields the result. □

7.2. **The comodule corresponding to $(f_7)_*(f_7)^*\mathcal{O}$.** Recall from Proposition 2.3 that 3-locally, $\mathrm{mf}_1(7)\cong\mathbb{Z}_{(3)}[z_1,z_2,z_3]/(\sigma_2)$. Again by Lemma 4.4 we obtain that $R_{\widetilde{A}}\cong\widetilde{\Gamma}\otimes_{\widetilde{A}}\mathrm{mf}_1(7)$ is equipped with the extended comodule structure. Recall further from the beginning of Section 6 the induced $\widetilde{A}$-module structure on $\mathrm{mf}_1(7)[r]$ from the identification $R_{\widetilde{A}}\cong\mathrm{mf}_1(7)[r]$.

Lemma 7.3. *Consider the natural morphism $f_7\colon\mathcal{M}_1(7)_{cub}\to\mathcal{M}_{cub}$. Under the ring isomorphism $R_{\widetilde{A}}\cong\mathbb{Z}_{(3)}[z_1,z_2,z_3,r]/(\sigma_2)$ the $(\widetilde{A},\widetilde{\Gamma})$-comodule structure corresponding to $(f_7)_*(f_7)^*\mathcal{O}$ is completely determined by*

$$\begin{aligned}
z_i &\mapsto 1\otimes z_i,\ \textit{for } i\in\{1,2,3\},\\
r &\mapsto 1\otimes r + r\otimes 1.
\end{aligned}$$

7.3. **The comodule corresponding to $(h_7)_*(h_7)^*\mathcal{O}$.** Recall that we identified $S_{\widetilde{A}}\cong(R_{\widetilde{A}})^{(\mathbb{Z}/7)^\times}$ (cf. Lemma 5.21) in Proposition 6.4 as $\widetilde{A}$-module with a free 8-dimensional $\widetilde{A}$-module with basis

$$1,\sigma_1^2,\sigma_1^4,n_4,\sigma_1^2n_4,n_6,\sigma_1^2n_6,\sigma_3^2.$$

We will now describe the comodule structure on this $\widetilde{A}$-module.

Lemma 7.4. *The graded $(\widetilde{A}, \widetilde{\Gamma})$-comodule structure on $S_{\widetilde{A}}$ is given by*

$$\begin{aligned}
1 &\mapsto 1 \otimes 1, \\
\sigma_1^2 &\mapsto 1 \otimes \sigma_1^2, \\
\sigma_1^4 &\mapsto 1 \otimes \sigma_1^4, \\
n_4 &\mapsto 1 \otimes n_4 + r \otimes \sigma_1^2, \\
\sigma_1^2 n_4 &\mapsto 1 \otimes \sigma_1^2 n_4 + r \otimes \sigma_1^4, \\
n_6 &\mapsto 1 \otimes n_6 + 2r \otimes n_4 + r^2 \otimes \sigma_1^2, \\
\sigma_1^2 n_6 &\mapsto 1 \otimes \sigma_1^2 n_6 + 2r \otimes \sigma_1^2 n_4 + r^2 \otimes \sigma_1^4, \\
\sigma_3^2 &\mapsto 1 \otimes \sigma_3^2.
\end{aligned}$$

Proof. This follows from Lemma 7.3 and Proposition 6.4 by a straightforward computation. □

7.4. **The conclusion.** We continue to work 3-locally.

Proposition 7.5. *There is an isomorphism of $(\widetilde{A}, \widetilde{\Gamma})$-comodules*

$$\widetilde{A} \oplus (\widetilde{\Gamma} \otimes_{\widetilde{A}} \mathrm{mf}_1(2))[2] \oplus (\widetilde{\Gamma} \otimes_{\widetilde{A}} \mathrm{mf}_1(2))[4] \oplus \widetilde{A}[6] \to S_{\widetilde{A}},$$

given by

$$\begin{aligned}
1_B &\mapsto 1, \\
w_1[2] &\mapsto \sigma_1^2, \\
w_2[2] &\mapsto n_4, \\
w_3[2] &\mapsto n_6, \\
w_1[4] &\mapsto \sigma_1^4, \\
w_2[4] &\mapsto \sigma_1^2 n_4, \\
w_3[4] &\mapsto \sigma_1^2 n_6, \\
1_B[6] &\mapsto \sigma_3^2.
\end{aligned}$$

Proof. This follows by inspection from Lemma 7.1 and Lemma 7.4. □

This implies our main algebraic theorem by the equivalence of $(\widetilde{A}, \widetilde{\Gamma})$-comodules and quasi-coherent sheaves on $\mathcal{M}_{cub,\mathbb{Z}_{(3)}}$ from Lemma 4.9:

Theorem 7.6. *There is 3-locally an isomorphism*

$$\begin{aligned}
(h_7')_* \mathcal{O}_{\mathcal{M}_0(7)_{cub}} \cong & \mathcal{O}_{\mathcal{M}_{cub}} \oplus \underline{\omega}^{\otimes(-6)} \oplus \Big((f')_* \mathcal{O}_{\mathcal{M}_1(2)_{cub}} \otimes \underline{\omega}^{\otimes(-2)} \Big) \\
& \oplus \Big((f')_* \mathcal{O}_{\mathcal{M}_1(2)_{cub}} \otimes \underline{\omega}^{\otimes(-4)} \Big)
\end{aligned}$$

of vector bundles on $\mathcal{M}_{cub}$.

By restricting to the open substack $\mathcal{M}_{ell,(3)}$, this implies Theorem 1.2. Similarly, we obtain a splitting result on $\overline{\mathcal{M}}_{ell,(3)}$ by restriction as well.

8. Topological conclusions

Recall from [15] that one obtains the spectrum Tmf as the global sections of a sheaf of E_∞-ring spectra $\mathcal{O}^{top}$ on the étale site of $\overline{\mathcal{M}}_{ell}$. Given any sheaf of spectra $\mathcal{F}$ on the étale site of any Deligne–Mumford stack $\mathcal{X}$, there is a *descent spectral sequence*

$$H^q(\mathcal{X}; \pi_p\mathcal{F}) \Rightarrow \pi_{p-q}(\mathcal{F}(\mathcal{X})),$$

where $\pi_*\mathcal{F}$ denotes the *sheafification* of the naive presheaf of homotopy groups [15, Chapter 5]. We have $\pi_{2p-1}\mathcal{O}^{top} = 0$ and $\pi_{2p}\mathcal{O}^{top} \cong \underline{\omega}^{\otimes p}$ and in particular $\pi_0\mathcal{O}^{top} \cong \mathcal{O}_{\overline{\mathcal{M}}_{ell}}$. Thus the descent spectral sequence takes the form

$$H^q(\overline{\mathcal{M}}_{ell}; \underline{\omega}^{\otimes p}) \Rightarrow \pi_{2p-q}\operatorname{Tmf}.$$

In general, the edge homomorphism takes the form $\pi_n(\mathcal{F}(\mathcal{X})) \to (\pi_n\mathcal{F})(\mathcal{X})$. In the case of $\mathcal{O}^{top}$, this produces a morphism $\pi_{2n}\operatorname{Tmf} \to \operatorname{mf}_n(\mathrm{SL}_2(\mathbb{Z}); \mathbb{Z})$ that is not an isomorphism integrally even for $n \geq 0$.

Actually, the approach of [15, Chapter 12] defines sheaves of E_∞-ring spectra $\mathcal{O}^{top}_R$ on $\overline{\mathcal{M}}_{ell,R}$ for every localization R of the integers by varying the set of primes in the arithmetic square following Remark 1.6 in op. cit. By construction, $\pi_*\mathcal{O}^{top}_R$ is again concentrated in even degrees with $\pi_{2k}\mathcal{O}^{top}_R$ being the pullback of $\underline{\omega}^{\otimes k}$ to $\overline{\mathcal{M}}_{ell,R}$. As R is a filtered colimit over the integers, we can form the analogous filtered homotopy colimit over Tmf to obtain a spectrum Tmf_R with $\pi_*\operatorname{Tmf}_R \cong (\pi_*\operatorname{Tmf}) \otimes R$. As homotopy colimits do not commute with global sections in general, we have to prove the following lemma about the global section $\Gamma(\mathcal{O}^{top}_R) = \mathcal{O}^{top}_R(\overline{\mathcal{M}}_{ell,R})$.

Lemma 8.1. *The map* $\operatorname{Tmf} \to \Gamma(\mathcal{O}^{top}_R)$ *factors over an equivalence* $\operatorname{Tmf}_R \to \Gamma(\mathcal{O}^{top}_R)$.

Proof. The map $(\overline{\mathcal{M}}_{ell,R}, \mathcal{O}^{top}_R) \to (\overline{\mathcal{M}}_{ell}, \mathcal{O}^{top})$ induces a map of descent spectral sequences in the opposite direction. As R is flat over $\mathbb{Z}$ and cohomology commutes with flat base change, this map of spectral sequences is just tensoring with R. The map converges moreover to a map $\pi_*\operatorname{Tmf} \to \pi_*(\mathcal{O}^{top}_R(\overline{\mathcal{M}}_{ell,R}))$ and we claim that the induced map

$$\pi_*\operatorname{Tmf} \otimes R \cong \pi_*\operatorname{Tmf}_R \to \pi_*\Gamma(\mathcal{O}^{top}_R)$$

is an isomorphism. This is true because the E_∞-pages of these descent spectral sequences are concentrated in finitely many lines, either by computation [28] or conceptually as in [31, Theorem 3.14]. As $\Gamma(\mathcal{O}^{top}_R)$ is R-local, the map $\operatorname{Tmf} \to \Gamma(\mathcal{O}^{top}_R)$ factors over a map $\operatorname{Tmf}_R \to \Gamma(\mathcal{O}^{top}_R)$ that is an equivalence by the argument above. □

To avoid cluttering the notation, we will set $\overline{\mathcal{M}}_{ell} = \overline{\mathcal{M}}_{ell,R}$ and $\operatorname{Tmf} = \operatorname{Tmf}_R$ etc. in the following.

We will work in the homotopy category of $\mathcal{O}^{top}$-modules. We denote the derived smash product over $\mathcal{O}^{top}$ by $\otimes_{\mathcal{O}^{top}}$ and the internal Hom in this category by $\mathcal{H}om_{\mathcal{O}^{top}}$ (see [39, Section 2.2] for details on the latter). Given two $\mathcal{O}^{top}$-modules $\mathcal{F}$ and $\mathcal{G}$, we denote by $[\mathcal{F}, \mathcal{G}]^{\mathcal{O}^{top}}$ the morphism set in the homotopy category and this coincides with π_0 of the global sections $\operatorname{Hom}_{\mathcal{O}^{top}}(\mathcal{F}, \mathcal{G})$ of the sheaf of spectra $\mathcal{H}om_{\mathcal{O}^{top}}(\mathcal{F}, \mathcal{G})$.

Definition 8.2. An $\mathcal{O}^{top}$-module $\mathcal{F}$ is *locally free of rank* n if there is an étale covering $\{U_i \to \overline{\mathcal{M}}_{ell}\}$ such that $\mathcal{F}$ restricted to U_i is equivalent to $\bigoplus_n \mathcal{O}^{top}|_{U_i}$.

Lemma 8.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}^{top}$-module and assume $\mathcal{F}$ to be locally free.*

(1) The homotopy groups $\pi_p\mathcal{F}$ are zero for p odd and isomorphic to $\pi_0\mathcal{F} \otimes \underline{\omega}^{\otimes \frac{p}{2}}$ for p even.

(2) The map $\pi_p \mathcal{H}om_{\mathcal{O}^{top}}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_{\overline{\mathcal{M}}_{ell}}}(\pi_0\mathcal{F}, \pi_p\mathcal{G})$ is an isomorphism for every $p \in \mathbb{Z}$.

Proof. The sheaf $\pi_p\mathcal{F}$ vanishes for p odd as it vanishes locally. For p even, we can write $\pi_p\mathcal{F} \cong \pi_0\Sigma^{-p}\mathcal{F} \cong \pi_0\left(\Sigma^{-p}\mathcal{O}^{top} \wedge_{\mathcal{O}^{top}} \mathcal{F}\right)$. The map

$$\underline{\omega}^{\otimes \frac{-p}{2}} \otimes \pi_0\mathcal{F} \cong \pi_0\Sigma^{-p}\mathcal{O}^{top} \otimes_{\pi_0\mathcal{O}^{top}} \pi_0\mathcal{F} \to \pi_0\Sigma^{-p}\mathcal{F} \cong \pi_0\left(\Sigma^{-p}\mathcal{O}^{top} \wedge_{\mathcal{O}^{top}} \mathcal{F}\right)$$

is an isomorphism as it is an isomorphism locally when $\mathcal{F} \simeq (\mathcal{O}^{top})^n$.

For the second part, we argue similarly that the map

$$\pi_p \mathcal{H}om_{\mathcal{O}^{top}}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_{\overline{\mathcal{M}}_{ell}}}(\pi_0\mathcal{F}, \pi_p\mathcal{G})$$

is an isomorphism as it is one locally when $\mathcal{F} \simeq (\mathcal{O}^{top})^n$. □

A related lemma to the following already appears in [4, Lemma 2.2.2].

Lemma 8.4. *Let $\mathcal{A}$ be a sheaf of $\mathcal{O}^{top}$-algebras on $\overline{\mathcal{M}}_{ell}$ that is locally free of rank n as an $\mathcal{O}^{top}$-module. There is a trace map*

$$\mathrm{tr}_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{O}^{top}$$

such that the composite $\mathrm{tr}_{\mathcal{A}}\, u$ with the unit map

$$u \colon \mathcal{O}^{top} \to \mathcal{A}$$

equals multiplication by n.

Proof. Consider the composite

$$\mathrm{tr}_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{H}om_{\mathcal{O}^{top}}(\mathcal{A}, \mathcal{A}) \xleftarrow{\simeq} \mathcal{A} \otimes_{\mathcal{O}^{top}} \mathcal{H}om_{\mathcal{O}^{top}}(\mathcal{A}, \mathcal{O}^{top}) \xrightarrow{\mathrm{ev}} \mathcal{O}^{top}.$$

Here, the middle map is an equivalence because $\mathcal{A}$ is locally free. We claim that the composite

$$\mathrm{tr}_{\mathcal{A}}\, u \colon \mathcal{O}^{top} \to \mathcal{O}^{top}$$

equals multiplication by n.

Note first that the map

$$\pi_0 \colon [\mathcal{O}^{top}, \mathcal{O}^{top}]^{\mathcal{O}^{top}} \to \mathrm{Hom}_{\pi_0\mathcal{O}^{top}}(\pi_0\mathcal{O}^{top}, \pi_0\mathcal{O}^{top})$$

is a bijection. Indeed, the source agrees with $\pi_0\Gamma(\mathcal{O}^{top}) = \pi_0 \mathrm{Tmf}$ and the morphism is the edge homomorphism of the descent spectral sequence for $\mathcal{H}om_{\mathcal{O}^{top}}(\mathcal{O}^{top}, \mathcal{O}^{top}) \simeq \mathcal{O}^{top}$. It can be deduced from [28, Section 3, Figure 11, Figure 26] that this edge homomorphism is an isomorphism, i.e. that the E_∞-term contains in the zeroth column only a $\mathbb{Z}$ in line 0 and nothing above it (see also the proof of [22, Lemma 4.9] for a different approach). Thus, it is enough to show that $\mathrm{tr}_{\mathcal{A}}\, u$ is multiplication by n on π_0.

As $\mathcal{A}$ is locally free,

$$\pi_0\mathcal{A} \otimes_{\mathcal{O}_{\overline{\mathcal{M}}_{ell}}} \pi_0\mathcal{H}om_{\mathcal{O}^{top}}(\mathcal{A}, \mathcal{O}^{top}) \to \pi_0(\mathcal{A} \otimes_{\mathcal{O}^{top}} \mathcal{H}om_{\mathcal{O}^{top}}(\mathcal{A}, \mathcal{O}^{top}))$$

is locally and hence globally an isomorphism. Note that the source is naturally isomorphic to $\pi_0\mathcal{A} \otimes_{\mathcal{O}_{\overline{\mathcal{M}}_{ell}}} \mathcal{H}om_{\mathcal{O}_{\overline{\mathcal{M}}_{ell}}}(\pi_0\mathcal{A}, \mathcal{O}_{\overline{\mathcal{M}}_{ell}})$ by the discussion before this lemma. Using these isomorphisms it can be checked that $\pi_0 \operatorname{tr}_{\mathcal{A}}\colon \pi_0\mathcal{A} \to \pi_0\mathcal{O}^{top}$ agrees with the trace map of $\pi_0\mathcal{A}$ over $\pi_0\mathcal{O}^{top} = \mathcal{O}_{\overline{\mathcal{M}}_{ell}}$. Its precomposition with $\pi_0 u$ equals n as it does locally (since we get exactly the trace of the identity map of a free module of rank n). This shows the claim. $\square$

Now we assume that $\frac{1}{2} \in R$. We will need the following variant of [36, Lemma 5.2.2]. Recall that we denote by f the natural map $\overline{\mathcal{M}}_1(2) \to \overline{\mathcal{M}}_{ell}$. By [21], we have a sheaf of E_∞-ring spectra $\mathcal{O}^{top}_{\overline{\mathcal{M}}_1(n)}$ on the étale site of every $\overline{\mathcal{M}}_1(n)$. We denote by $f_*f^*\mathcal{O}^{top}$ the sheaf $f_*\mathcal{O}^{top}_{\overline{\mathcal{M}}_1(n)}$ on $\overline{\mathcal{M}}_{ell}$, i.e. the one associating with every étale map $U \to \overline{\mathcal{M}}_{ell}$ the E_∞-ring spectrum $\mathcal{O}^{top}_{\overline{\mathcal{M}}_1(2)}(U \times_{\overline{\mathcal{M}}_{ell}} \overline{\mathcal{M}}_1(2))$. By the proof of [35, Theorem 3.5] the odd homotopy of $f_*f^*\mathcal{O}^{top}$ vanishes and $\pi_{2i} f_*f^*\mathcal{O}^{top} \cong f_*f^*\underline{\omega}^{\otimes i}$.

Lemma 8.5. *Let $\mathcal{F}$ be a locally free $\mathcal{O}^{top}$-module on $\overline{\mathcal{M}}_{ell}$ of finite rank. Let $g_{alg}\colon f_*f^*\underline{\omega}^{\otimes(-i)} \to \pi_0\mathcal{F}$ be a split injection. Then g_{alg} can be uniquely realized by a split map*

$$g\colon \Sigma^{2i} f_*f^*\mathcal{O}^{top} \to \mathcal{F}$$

with $\pi_0 g = g_{alg}$.

Proof. By Lemma 8.3,

$$\pi_k \mathcal{H}om_{\mathcal{O}^{top}}(\mathcal{F}, \mathcal{G}) \cong \mathcal{H}om_{\pi_0\mathcal{O}^{top}}(\pi_0\mathcal{F}, \pi_k\mathcal{G})$$

for an arbitrary $\mathcal{O}^{top}$-module $\mathcal{G}$.

Using Lemma 8.3 again and the projection formula, we reduce to the case $i = 0$. The dual of the vector bundle $f_*f^*\mathcal{O}_{\overline{\mathcal{M}}_{ell}} \cong f_*\mathcal{O}_{\overline{\mathcal{M}}_1(2)}$ is isomorphic to $\underline{\omega}^{\otimes 4} \otimes_{\mathcal{O}_{\overline{\mathcal{M}}_{ell}}} f_*\mathcal{O}_{\overline{\mathcal{M}}_1(2)} \cong f_*f^*\underline{\omega}^{\otimes 4}$ by Lemma 7.2.

This implies that

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}_{\overline{\mathcal{M}}_{ell}}}(f_*f^*\mathcal{O}_{\overline{\mathcal{M}}_{ell}}, \pi_k\mathcal{F}) &\cong f_*f^*\underline{\omega}^{\otimes 4} \otimes_{\overline{\mathcal{M}}_{ell}} \pi_k\mathcal{F} \\ &\cong f_*f^*(\underline{\omega}^{\otimes 4} \otimes_{\overline{\mathcal{M}}_{ell}} \pi_k\mathcal{F}).\end{aligned}$$

As f is affine and every quasi-coherent sheaf on $\overline{\mathcal{M}}_1(2)$ has cohomology at most in degrees 0 and 1 by [38, Proposition 2.4(4)], the descent spectral sequence

$$H^q(\overline{\mathcal{M}}_{ell}; \pi_p\mathcal{H}om_{\mathcal{O}^{top}}(f_*f^*\mathcal{O}^{top}, \mathcal{F})) \Rightarrow \pi_{p-q} \operatorname{Hom}_{\mathcal{O}^{top}}(f_*f^*\mathcal{O}^{top}, \mathcal{F})$$

is concentrated in the lines 0 and 1. Moreover, the E_2-term is zero for p odd and thus the edge homomorphism

$$[f_*f^*\mathcal{O}^{top}, \mathcal{F}]^{\mathcal{O}^{top}} = \pi_0 \operatorname{Hom}_{\mathcal{O}^{top}}(f_*f^*\mathcal{O}^{top}, \mathcal{F}) \to \operatorname{Hom}_{\mathcal{O}_{\overline{\mathcal{M}}_{ell}}}(f_*f^*\mathcal{O}_{\overline{\mathcal{M}}_{ell}}, \pi_0\mathcal{F})$$

is an isomorphism.

Similarly, one shows that

$$[\mathcal{F}, f_*f^*\mathcal{O}^{top}]^{\mathcal{O}^{top}} = \pi_0 \operatorname{Hom}_{\mathcal{O}^{top}}(\mathcal{F}, f_*f^*\mathcal{O}^{top}) \to \operatorname{Hom}_{\mathcal{O}_{\overline{\mathcal{M}}_{ell}}}(\pi_0\mathcal{F}, f_*f^*\mathcal{O}_{\overline{\mathcal{M}}_{ell}})$$

is an isomorphism. The lemma follows. $\square$

Recall that we denote the natural map $\overline{\mathcal{M}}_0(7) \to \overline{\mathcal{M}}_{ell}$ by h. By the work of [21], we have a sheaf of E_∞-ring spectra $\mathcal{O}^{top}_{\overline{\mathcal{M}}_0(7)}$ on the étale site of $\overline{\mathcal{M}}_0(7)$ and we denote by $h_*h^*\mathcal{O}^{top}$ its pushforward to $\overline{\mathcal{M}}_{ell}$ along h.

Theorem 8.6. *We can decompose* $\mathrm{Tmf}_0(7)_{(3)}$ *as*

$$\mathrm{Tmf}_{(3)} \oplus \Sigma^4 \mathrm{Tmf}_1(2)_{(3)} \oplus \Sigma^8 \mathrm{Tmf}_1(2)_{(3)} \oplus L,$$

where $L \in \mathrm{Pic}(\mathrm{Tmf}_{(3)})$*, i.e.* L *is an invertible* $\mathrm{Tmf}_{(3)}$*-module. There is a corresponding splitting*

$$h_* h^* \mathcal{O}^{top}_{(3)} \simeq \mathcal{O}^{top}_{(3)} \oplus \Sigma^4 f_* f^* \mathcal{O}^{top}_{(3)} \oplus \Sigma^8 f_* f^* \mathcal{O}^{top}_{(3)} \oplus \mathcal{L}$$

for a certain invertible $\mathcal{O}^{top}_{(3)}$*-module* $\mathcal{L}$*.*

Proof. Throughout this proof, we will implicitly localize at 3. By Lemma 8.4, the unit map $\mathcal{O}^{top} \to h_* h^* \mathcal{O}^{top}$ splits off as an $\mathcal{O}^{top}$-module; denote the cofiber by $\mathcal{F}$. Note that $\pi_k \mathcal{F} = 0$ for k odd. By Theorem 7.6,

$$\pi_0 \mathcal{F} \cong \underline{\omega}^{\otimes(-6)} \oplus f_* f^* \underline{\omega}^{\otimes(-2)} \oplus f_* f^* \underline{\omega}^{\otimes(-4)}.$$

By Lemma 8.5, we obtain a decomposition

$$\mathcal{F} \cong \mathcal{L} \oplus \Sigma^4 f_* f^* \mathcal{O}^{top} \oplus \Sigma^8 f_* f^* \mathcal{O}^{top}$$

with $\pi_0 \mathcal{L} \cong \underline{\omega}^{\otimes(-6)}$. As a summand of a locally free module, $\mathcal{L}$ is locally free as well and thus an invertible $\mathcal{O}^{top}$-module as it has rank 1. We obtain our result by taking global sections because the global sections of $h_* h^* \mathcal{O}^{top}$ are $\mathrm{Tmf}_0(7)$. To see that $L = \Gamma(\mathcal{L})$ is an invertible Tmf-module, we use that the global sections functor

$$\Gamma\colon \mathrm{QCoh}(\overline{\mathcal{M}}_{ell}, \mathcal{O}^{top}) \to \mathrm{Tmf}\,-\mathrm{mod}$$

is a symmetric monoidal equivalence of ∞-categories by one of the main results of [33]. $\square$

Remark 8.7. In [34], the Picard group $\mathrm{Pic}(\mathrm{Tmf}_{(3)})$ is identified with $\mathbb{Z} \oplus \mathbb{Z}/3$, where

$$\mathbb{Z} \to \mathrm{Pic}(\mathrm{Tmf}_{(3)})$$

is the map $k \mapsto \Sigma^k \mathrm{Tmf}_{(3)}$. The image of the generator of $\mathbb{Z}/3$ is called $\Gamma(\mathcal{J})$ [34, Construction 8.4.2]. As one can compute the homotopy groups of all $\Sigma^k \mathcal{J}^{\otimes l}$ from those of Tmf via a Mayer–Vietoris sequence, one can deduce the identity of L in the previous theorem by calculating $\pi_* \mathrm{Tmf}_0(7)$. This was done in unpublished work by Martin Olbermann and he shows that $L \simeq \Sigma^{36} \Gamma(\mathcal{J}^{\otimes 2})$.

Note that $\Gamma(\mathcal{J}^{\otimes l})$ is in the kernel of $\mathrm{Pic}(\mathrm{Tmf}_{(3)}) \to \mathrm{Pic}(\mathrm{TMF}_{(3)})$ so that L becomes $\Sigma^{36} \mathrm{TMF}_{(3)}$ after base changing to $\mathrm{TMF}_{(3)}$.

Appendix A. Modular forms and q-expansions

The aim of this appendix is to review several different definitions of modular forms (complex-analytic, in the sense of Katz and via stacks) and compare them via explicit isomorphisms. Moreover, we will repeat this for modular forms with respect to the congruence subgroup $\Gamma_1(n)$ and the corresponding algebraic definition via the moduli stack of elliptic curves with level structure. We have no claim of originality here. The main reason for writing this appendix anyhow is the existence of two different versions of level structures, often called *naive* and *arithmetic*, whose precise relationship has at least confused the authors in the past. In particular, we will deduce a q-expansion principle for the naive level structure, namely Theorem A.22.

We have based our treatment on [14], [13], [25] and [26, Section 2], of which we recommend especially the first two as an introduction to modular forms. We also refer to [10] for a thorough treatment of the geometry on the analytic side.

A.1. **Modular forms.** In this section, we will give three definitions of modular forms and compare them.

A.1.1. *Three definitions of modular forms.* We start with the classical definition and denote by $\mathrm{MF}_k(\mathrm{SL}_2(\mathbb{Z});\mathbb{C})$ the set of holomorphic functions $f\colon \mathbb{H} \to \mathbb{C}$ satisfying for every $z \in \mathbb{H}$ and every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ the compatibility condition

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \tag{A.1}$$

and meromorphic at ∞. To make this last condition precise, recall that the $\mathrm{SL}_2(\mathbb{Z})$-compatibility implies in particular that f is 1-periodic, and so there is a well-defined holomorphic function $g\colon \mathbb{D} \setminus \{0\} \to \mathbb{C}$ satisfying $f(z) = g(e^{2\pi i z})$, where $\mathbb{D}$ denotes the open unit disk. We require this g to be meromorphically extended to 0. We will say that the Laurent expansion of g at 0 is the *classical q-expansion of f* at ∞.

Elements of $\mathrm{MF}_k(\mathrm{SL}_2(\mathbb{Z});\mathbb{C})$ are called *meromorphic modular forms.* Denote by $\mathrm{MF}_k(\mathrm{SL}_2(\mathbb{Z});R_0)$ for a subring R_0 of $\mathbb{C}$ the subset of $\mathrm{MF}_k(\mathrm{SL}_2(\mathbb{Z});\mathbb{C})$ of modular forms with coefficients of classical q-expansion of f lying in R_0.

Note that the direct sum $\mathrm{MF}(\mathrm{SL}_2(\mathbb{Z});R_0) = \bigoplus_{k\in\mathbb{Z}} \mathrm{MF}_k(\mathrm{SL}_2(\mathbb{Z});R_0)$ carries a multiplication of functions, making it into a graded ring of modular forms. The q-expansion defines a ring homomorphism $\mathrm{MF}(\mathrm{SL}_2(\mathbb{Z});R_0) \to R_0((q))$.

For the algebro-geometric definitions of modular forms, we denote for a (generalized) elliptic curve $p\colon E \to T$ the quasi-coherent sheaf $p_*\Omega^1_{E/T}$ by ω_E. For the definition of a generalized elliptic curve see [12, Definition 1.12].

Proposition A.2 ([12, Proposition II.1.6])**.** *Let $p\colon E \to T$ be a generalized elliptic curve, and denote its chosen section by $e\colon T \to E$. Then the sheaf $\omega_E = p_*\Omega^1_{E/T}$ is a line bundle on T. Moreover, the adjunction counit*

$$p^*p_*\Omega^1_{E/T} \to \Omega^1_{E/T}$$

is an isomorphism, implying also $p_*\Omega^1_{E/T} \cong e^*\Omega^1_{E/T}$.

An *invariant differential* for E is a nowhere vanishing section of $\Omega^1_{E/T}$ or equivalently a trivialization of ω_E.

Our second definition of modular forms will define them as a certain kind of natural transformations. Fix a commutative ring R_0. For any R_0-algebra R, denote by $\mathrm{Ell}^1(R)$ the set of isomorphism classes of pairs (E,ω) consisting of an elliptic curve E over R together with an invariant differential. This defines (together with pullback of elliptic curves and of invariant differentials) a functor

$$\mathrm{Ell}^1(-)\colon (\mathrm{AffSh}\,/\,\mathrm{Spec}(R_0))^{\mathrm{op}} \to \mathrm{Sets}\,.$$

As in [25, Section 1.1], we can consider a notion of a modular form of level 1 and weight k over R_0 as the subset of the set of natural transformations $f \in$

$\mathrm{Nat}(\mathrm{Ell}^1(-), \Gamma(-))$ with the following scaling property: For any R_0-algebra R, elliptic curve with chosen invariant differential (E, ω) and any $\lambda \in R^\times$, we have

$$f(E, \lambda\omega) = \lambda^{-k} f(E, \omega). \tag{A.3}$$

Denote the set of such natural transformations by $\mathrm{Nat}_k(\mathrm{Ell}^1(-), \Gamma(-))$. Also here, the direct sum $\bigoplus_{k\in\mathbb{Z}} \mathrm{Nat}_k(\mathrm{Ell}^1(-), \Gamma(-))$ carries a multiplication by multiplying values in the target. This multiplication gives again a definition of a graded ring of modular forms.

For the third definition, let $\mathcal{M}_{ell,R_0}$ be the moduli stack of elliptic curves over $\mathrm{Spec}(R_0)$ (see e.g. [12] or [40]). On its big étale site, one defines a line bundle $\underline{\omega} = \underline{\omega}_{R_0}$ as follows. For a morphism $t\colon T \to \mathcal{M}_{ell,R_0}$ from a scheme T, let $p\colon E \to T$ be the corresponding elliptic curve with unit section e. We associate with (T, t) the line bundle ω_E on T. To check that this actually defines a line bundle consider a cartesian square

$$\begin{tikzcd} E' \arrow[r, "\widetilde{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ T' \arrow[r, "f"'] & T \end{tikzcd}$$

with unit section $e'\colon T' \to E'$. We obtain a chain of natural isomorphisms

$$f^*\omega_E \cong f^*e^*\Omega^1_{E/T} \cong (e')^*\tilde{f}^*\Omega^1_{E/T} \cong (e')^*\Omega^1_{E'/T'} \cong \omega_{E'} \tag{A.4}$$

as required.

The third definition of the meromorphic modular forms over R_0 of weight k is $H^0(\mathcal{M}_{ell,R_0}; \underline{\omega}_{R_0}^{\otimes k})$. Here, the direct sum $\bigoplus_{k\in\mathbb{Z}} H^0(\mathcal{M}_{ell,R_0}; \underline{\omega}_{R_0}^{\otimes k})$ carries a multiplication inherited from the tensor algebra $\bigoplus_{k\in\mathbb{Z}} \underline{\omega}_{R_0}^{\otimes k}$, defining also here a graded ring of modular forms. Sometimes it is convenient to reinterpret this ring as $H^0(\mathcal{M}^1_{ell,R_0}, \mathcal{O}_{\mathcal{M}^1_{ell,R}})$, where $\mathcal{M}^1_{ell,R_0}$ is the relative spectrum of $\bigoplus_{i\in\mathbb{Z}} \underline{\omega}_{R_0}^{\otimes i}$ [18, Section 12.1].

A.1.2. *Comparision of definitions of modular forms.* We start by comparing the last two definitions, both coming from algebraic geometry.

Proposition A.5. *There is a natural isomorphism*

$$\alpha\colon H^0(\mathcal{M}_{ell,R_0}, \underline{\omega}_{R_0}^{\otimes k}) \to \mathrm{Nat}_k(\mathrm{Ell}^1(-), \Gamma(-)).$$

Moreover, on the direct sum for all $k \in \mathbb{Z}$, the map α induces an isomorphism of graded rings.

Proof. There is an easy map

$$\alpha\colon H^0(\mathcal{M}_{ell,R_0}, \underline{\omega}_{R_0}^{\otimes k}) \to \mathrm{Nat}_k(\mathrm{Ell}^1(-), \Gamma(-)),$$

constructed as follows. Start with an element $f \in H^0(\mathcal{M}_{ell,R_0}, \omega^{\otimes k}_{\mathcal{M}_{ell,R_0}})$, an R_0-algebra R and an elliptic curve E/R together with an invariant differential ω. If E is classified by $\varphi\colon \mathrm{Spec}(R) \to \mathcal{M}_{ell,R_0}$, we have $\varphi^*(\underline{\omega}_{R_0}^{\otimes k}) \cong \omega_E^{\otimes k}$. By pulling back, f defines an element in $\Gamma(\varphi^*(\underline{\omega}_{R_0}^{\otimes k}))$, which via the previous isomorphism and via the isomorphism $\omega^{\otimes k}$ from $\mathcal{O}_R^{\otimes k}$ to $\omega_{E/R}^{\otimes k}$ is identified with

$$\Gamma(\varphi^*(\underline{\omega}_{R_0}^{\otimes k})) \cong \Gamma(\omega_E^{\otimes k}) \cong \Gamma(\mathcal{O}_R^{\otimes k}) \cong \Gamma(\mathcal{O}_R) = R.$$

Define $\alpha(f)(E,\omega)$ to be the image in R of the element defined by f in the left-hand side. The naturality of $\alpha(f)$ is clear. Replacing ω by $\lambda\omega$ for $\lambda \in R^\times$ multiplies the chosen isomorphism above by λ^k, so we obtain

$$\alpha(f)(E,\lambda\omega) = \lambda^{-k}\alpha(f)(E,\omega).$$

Let us sketch why α is an isomorphism. By definition, the section f corresponds to a compatible choice of sections in $H^0(T;\omega_E^{\otimes k})$ for all $T \to \mathcal{M}_{ell,R_0}$ classifying an elliptic curve E/T. As ω_E is locally trivial, f is uniquely determined by its values on those T where ω_E is already trivial and $T = \operatorname{Spec} R$ is affine and every coherent choice of values on such T induces a section of $\underline{\omega}^{\otimes k}_{\mathcal{M}_{ell,R_0}}$. For such T, a section of $\omega_E^{\otimes k}$ corresponds exactly to associating with each trivialization ω of ω_E an element $f(E,\omega)$ such that $f(E,\lambda\omega) = \lambda^{-k}f(E,\omega)$. This describes $\mathrm{Nat}_k(\mathrm{Ell}^1(-),\Gamma(-))$. $\square$

Next, we exhibit the map which will turn out to be an isomorphism between the algebraic geometric definitions and the complex analytic ones.

Proposition A.6. *For any subring R_0 of $\mathbb{C}$ define*

$$\beta\colon \mathrm{Nat}_k(\mathrm{Ell}^1(-),\Gamma(-)) \to \mathrm{MF}_k(\mathrm{SL}_2(\mathbb{Z}), R_0),$$

as follows. For any $f \in \mathrm{Nat}_k(\mathrm{Ell}^1(-),\Gamma(-))$ and any $\tau \in \mathbb{H}$, set

$$\beta(f)(\tau) = f(\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau, dz) \in \mathbb{C}.$$

Then β is a natural isomorphism, and induces an isomorphism of graded rings on the direct sum for all $k \in \mathbb{Z}$.

We will check (A.1) for $\beta(f)$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ be given. Observe that we have a biholomorphism

$$\begin{aligned} \psi\colon \mathbb{C}/\left(\mathbb{Z}\cdot 1 \oplus \mathbb{Z}\tau\right) \to &\quad \mathbb{C}/\left(\mathbb{Z}\cdot 1 \oplus \mathbb{Z}\frac{a\tau+b}{c\tau+d}\right), \\ [z] \mapsto &\quad \left[\frac{z}{c\tau+d}\right] \end{aligned}$$

and by GAGA thus an isomorphism of the associated algebraic curves. This shows that, since f is well-defined on isomorphism classes and the scaling property,

$$\begin{aligned} \beta(f)\left(\frac{a\tau+b}{c\tau+d}\right) &= f\left(\mathbb{C}/\left(\mathbb{Z}\cdot 1 \oplus \mathbb{Z}\frac{a\tau+b}{c\tau+d}\right), dz\right) \\ &= (c\tau+d)^k f(\mathbb{C}/\mathbb{Z}\cdot 1\oplus\mathbb{Z}\tau, dz) = (c\tau+d)^k\beta(f)(\tau). \end{aligned}$$

We will later sketch why this is holomorphic in the interior and meromorphic at the cusp and why β is an isomorphism.

A.1.3. *Holomorphic modular forms.* In each of the three definitions above, we can also restrict to modular forms that are "holomorphic at the cusps".

In the classical definition we say that a meromorphic modular form f is *holomorphic* if the associated meromorphic function g on $\mathbb{D}^2$ is actually holomorphic at 0 or, equivalently, that the classical q-expansion of f lies in $\mathbb{C}[\![q]\!]$. By requiring that the classical q-expansion is in $R_0[\![q]\!]$, we obtain the R_0-module $\mathrm{mf}_k(\mathrm{SL}_2(\mathbb{Z}); R_0)$ of *holomorphic modular forms.* An important example is $\Delta \in \mathrm{mf}_{12}(\mathrm{SL}_2(\mathbb{Z});\mathbb{Z})$ with q-expansion $q - 24q^2 + \cdots$. Thus for every meromorphic modular form f, there is a $k > 0$ such that $\Delta^k f$ is a holomorphic modular form. Moreover, Δ vanishes nowhere on the upper half plane [14, Corollary 1.4.2] so that Δ^{-1} is a meromorphic modular form over $\mathbb{Z}$ again. We see that $\mathrm{mf}(\mathrm{SL}_2(\mathbb{Z}); R_0)[\Delta^{-1}] \to \mathrm{MF}(\mathrm{SL}_2(\mathbb{Z}); R_0)$ is an isomorphism.

For the algebro-geometric version, we have to work with generalized elliptic curves instead[12, Definition II.1.12]. These allow to define the compactified moduli stack $\overline{\mathcal{M}}_{ell}$ (which is our notation for $\mathfrak{M}_1$ from [12, Remarque III.2.6]). As before we can use Proposition A.2 to show that the line bundles ω_E define a line bundle $\underline{\omega}$ on $\overline{\mathcal{M}}_{ell}$. Our algebro-geometric definition of holomorphic modular forms of weight k is $H^0(\overline{\mathcal{M}}_{ell,R_0}, \underline{\omega}^{\otimes k})$.

We will later sketch the comparison between these two definitions.

A.2. **Level structures.** Throughout this section, let R_0 be a $\mathbb{Z}[\frac{1}{n}]$-algebra.

We begin with the classical definition of modular forms with level structure. Let $\Gamma_1(n) \subset \mathrm{SL}_2(\mathbb{Z})$ be the subgroup of matrices that reduce to a matrix of the form $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ modulo n.

A *meromorphic/holomorphic modular form of level* n *and weight* k is a holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ satisfying the transformation formula (A.1) for matrices in $\Gamma_1(n)$ and is meromorphic/holomorphic at all cusps. We will say more about cusps later, but for the moment see [14, Section 1.2] for details. Note that modular forms of level n are still 1-periodic and thus the classical q-expansion still makes sense. Assume that $R_0 \subset \mathbb{C}$. We will denote by $\mathrm{MF}_k(\Gamma_1(n); R_0)$ the meromorphic modular forms of level n and weight k that have classical q-expansion with coefficients in R_0 and by $\mathrm{mf}_k(\Gamma_1(n); R_0)$ the analogue for holomorphic modular forms.

For the algebro-geometric definitions of modular forms with level structure, we have to distinguish between two different ways to phrase them, the *naive* and the *arithmetic* level structures.

A.2.1. *Naive level structures.*

Definition A.7 ([12, Construction 4.8])**.** For an R_0-algebra R, let $\mathrm{Ell}^1_{\Gamma_1(n)}(R)$ denote the set of isomorphism classes of triples (E, ω, j), where E is an elliptic curve over R, further ω is a chosen trivialization of the line bundle ω_E, and $j\colon \mathbb{Z}/n\mathbb{Z}_R \to E$ is a morphism of group schemes over $\mathrm{Spec}(R)$ and a closed immersion. This morphism j is called a $\Gamma_1(n)$*-level structure.*

Recall that $\mathbb{Z}/n\mathbb{Z}_R = \coprod_{\mathbb{Z}/n\mathbb{Z}} \mathrm{Spec}(R)$ as a scheme, with the obvious map to $\mathrm{Spec}\, R$ and group structure coming from the group structure on $\mathbb{Z}/n\mathbb{Z}$. The group structure on the elliptic curve is explained in [27, Section 2.1]. We can identify j with the image $P = j(1) \in E(R)$ since it determines j completely.

Remark A.8. We should remark that this variant of level structures is often called "naive" in the literature. Note also that the analogous definition in [13, Section 8.2], looks slightly different, but is equivalent by using that being closed immersion can be checked for proper schemes on geometric points.

Using again the scaling condition (A.3) we can define $\mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_1(n)}(-), \Gamma(-))$ analogously to our definition without level in Section A.1.

We can also define a moduli stack $\mathcal{M}_1(n)$ classifying elliptic curves over $\mathbb{Z}[\frac{1}{n}]$-schemes with $\Gamma_1(n)$-level structure. We obtain a morphism $f_n\colon \mathcal{M}_1(n) \to \mathcal{M}_{ell}$

by forgetting the level structure. As in Section A.1.2 we obtain a comparison isomorphism

$$\alpha\colon H^0(\mathcal{M}_1(n);\underline{\omega}^{\otimes k}) \to \mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_1(n)}(-),\Gamma(-));$$

here and in the following we will abuse notation to denote the pullback of $\underline{\omega}$ to $\mathcal{M}_1(n)$ by $\underline{\omega}$ as well.

There are different ways to compare modular forms with and without level structure. The particular form of compatibility is expressed in the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Nat}_k(\mathrm{Ell}^1(-),\Gamma(-)) \arrow[r, "(E,P)\mapsto E/\langle P\rangle"] \arrow[d, "(\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z},dz)"] & \mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_1(n)}(-),\Gamma(-)) \arrow[d, "(\mathbb{C}/\mathbb{Z}+n\tau\mathbb{Z},dz,\tau)"] \\
\mathrm{MF}_k(\mathrm{SL}_2(\mathbb{Z}),R_0) \arrow[r] & \mathrm{MF}(\Gamma_1(n),R_0)
\end{tikzcd}$$

We will denote the left vertical morphism by β_1. The reason for our particular choice of $\beta_1(n)$ might become clearer in the next subsection and even clearer when we discuss q-expansions. Note that we have not shown yet that the vertical morphism actually land in the indicated target, but we will do so later.

Remark A.9. The group $(\mathbb{Z}/n)^\times$ acts on $\mathrm{Ell}^1_{\Gamma_1(n)}(-)$ by multiplication on the point of order n. Moreover, $\Gamma_1(n)\setminus\Gamma_0(n)$ acts on $\mathrm{MF}(\Gamma_1(n),\mathbb{C})$ as follows. For $g \in \mathrm{MF}_k(\Gamma_1(n),\mathbb{C})$ and $\gamma\in\Gamma_0(n)$, we define the action by $g.[\gamma] = g[\gamma]_k$ in the sense of Section 3.2. The map

$$\Gamma_1(n)\setminus\Gamma_0(n) \to (\mathbb{Z}/n)^\times, \qquad \begin{pmatrix} a & b \\ c & d\end{pmatrix} \mapsto a$$

is an isomorphism and under this isomorphism β_1 is equivariant.

To be compatible with [14, Section 5.2], we will actually work with the *opposite* convention though. This means that we will act with the *inverse* of an element of $(\mathbb{Z}/n)^\times$ on $\mathrm{Ell}^1_{\Gamma_1(n)}(-)$ and $\mathcal{M}_1(n)$ and use the identification

$$\Gamma_1(n)\setminus\Gamma_0(n) \xrightarrow{\cong} (\mathbb{Z}/n)^\times, \qquad \begin{pmatrix} a & b \\ c & d\end{pmatrix} \mapsto d.$$

By the above, this makes β_1 into an equivariant map as well and this will the equivariance we will use throughout this document.

A.2.2. *Arithmetic level structures.* Now we would like to discuss a different variant of level structures, called “arithmetic” in the literature.

Definition A.10. For an R_0-algebra R, let $\mathrm{Ell}^1_{\Gamma_\mu(n)}(R)$ denote the set of isomorphism classes of triples (E,ω,ι), where E is an elliptic curve over R, again ω is a chosen trivialization of the line bundle ω_E, and $\iota\colon \mu_{n,R}\to E$ is a morphism of group schemes over $\mathrm{Spec}(R)$ and a closed immersion. Here, $\mu_{n,R}$ is a group scheme given by the spectrum of the bialgebra $R[t]/(t^n-1)$ with comultiplication determined by $t\mapsto t\otimes t$. The morphism ι is called an *arithmetic (or $\Gamma_\mu(n)$-) level structure* on E.

One can check that for a $\mathbb{Z}\left[\frac{1}{n},\zeta_n\right]$-algebra R, both group schemes $\mu_{n,R}$ and $\mathbb{Z}/n\mathbb{Z}(R)$ are isomorphic, but this is not true in general. Now we can define the set of weight k modular forms with arithmetic level structure to be $\mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_\mu(n)}(-),\Gamma(-))$ with the same scaling condition as before. Likewise, we can define a moduli stack $\mathcal{M}_\mu(n)$ of elliptic curves with $\Gamma_\mu(n)$-level structure (over bases with n invertible). As before we obtain a comparison isomorphism

$$\alpha\colon H^0(\mathcal{M}_\mu(n);\underline{\omega}^{\otimes k}) \to \mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_\mu(n)}(-),\Gamma(-)),$$

where we abuse notation again to denote the pullback of $\underline{\omega}$ to $\mathcal{M}_\mu(n)$ by $\underline{\omega}$ as well.

We need to discuss a relation between $\Gamma_1(n)$- and $\Gamma_\mu(n)$-level structures. This relation will be based by dividing out finite subschemes and we refer to [16, Example 4.40] for the fact that the quotient of an elliptic curve by a finite subscheme is an elliptic curve again. We have the following lemma from [26, Section 2.3].

Lemma A.11. *There is an equivalence $\varphi\colon \mathcal{M}_1(n) \to \mathcal{M}_\mu(n)$ sending $(E \to S, P)$ to $(E/\langle P\rangle \to S, \delta)$, where δ is an arithmetic level structure to be discussed in the proof.*

A choice of a primitive n-th root of unity $\zeta_n \in \Gamma(\mathcal{O}_S)$, if it exists, specifies an isomorphism $\mu_{n,S} \cong (\mathbb{Z}/n)_S$. Under this identification, we obtain a morphism $\delta\colon (\mathbb{Z}/n)_S \to E/\langle P\rangle$ that corresponds to $\pi(Q)$ for $Q \in E[n](S)$ a point with $e_n(P,Q) = \zeta_n^{-1}$. Here $\pi\colon E \to E/\langle P\rangle$ denotes the projection and e_n is the Weil pairing.

Proof. With notation as in the statement of the lemma, we define $\delta\colon \mu_{n,S} \to E/\langle P\rangle$ as follows: As explained in [27, Section 2.8] there is a bilinear pairing

$$\langle -,-\rangle_\pi\colon \ker(\pi)\times\ker(\pi^t) \to \mathbb{G}_{m,S} \tag{A.12}$$

of abelian group schemes for $\pi\colon E \to E/\langle P\rangle$ the projection and π^t the dual isogeny. By [27, 2.8.2.1] and because $\ker(\pi) = \langle P\rangle \cong (\mathbb{Z}/n)_S$ this induces a chain of isomorphisms

$$\ker(\pi^t) \to \mathrm{Hom}_{S-\mathrm{gp}}(\ker(\pi),\mathbb{G}_{m,S}) \xrightarrow{\mathrm{ev}_P} \mu_{n,S}. \tag{A.13}$$

The map δ is the composition of the inverse of this isomorphism with the natural inclusion $\ker(\pi^t) \to E/\langle P\rangle$ *composed with* $[-1]$. The reasons for composing with $[-1]$ will be apparent in the example below.

An analogous construction dividing out $\mu_{n,S}$ provides an inverse of φ. To see this, we are using that in the situation above, $(E/\langle P\rangle)/\delta \cong E/E[n]$, and the isomorphism $E/E[n] \cong E$ induced by $[n]$, the multiplication-by-n morphism. Thus, $\varphi\colon \mathcal{M}_1(n) \to \mathcal{M}_\mu(n)$ is an equivalence of stacks.

One can compute φ in terms of the Weil pairing as follows: As $\pi\pi^t = [n]$, we obtain from [27, 2.8.4.1] that $\langle P, \pi(Q)\rangle_\pi$ for $Q \in E[n](S)$ can be computed as $e_n(P,Q)$.Assume now the existence of a primitive n-th root of unity $\zeta_n \in \mu_n(S)$. The inverse of the composition (A.13) sends ζ_n to $\pi(Q)$ for some $Q \in E[n](S)$ with $e_n(P,Q) = \zeta_n$. We obtain $e_n(P,-Q) = \zeta_n^{-1}$ showing the result. □

Example A.14. Let $E = \mathbb{C}/(\mathbb{Z}+n\tau\mathbb{Z})$ be an elliptic curve over $\operatorname{Spec}\mathbb{C}$ with chosen n-torsion point τ. We claim that $\varphi(E,\tau) = (\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}, \zeta_n \mapsto \frac{1}{n})$ with $\zeta_n = e^{\frac{2\pi i}{n}}$.

Indeed, we have $e_n(\tau,\frac{1}{n}) = \zeta_n^{-1}$ by [27, 2.8.5.3] and thus $\langle \tau, \frac{1}{n}\rangle_\pi = \zeta_n^{-1}$.The claim follows.

Under the isomorphism

$$\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z} \to \mathbb{C}^\times/q^{\mathbb{Z}}, \qquad z \mapsto e^{2\pi i z}$$

with $q = e^{2\pi i\tau}$ the morphism α corresponds thus just to the obvious inclusion of μ_n.

The example implies directly the following lemma.

Lemma A.15. *The following diagram commutes:*

$$\begin{tikzcd}
H^0(\mathcal{M}_\mu(n)_{R_0}, \underline{\omega}^{\otimes k}) \arrow[r, "\varphi^*"] \arrow[d, "\alpha"] & H^0(\mathcal{M}_1(n)_{R_0}, \underline{\omega}^{\otimes k}) \arrow[d, "\alpha"] \\
\mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_\mu(n)}(-), \Gamma(-)) \arrow[r, "\varphi^*"] \arrow[dr, "(\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}, dz, \zeta_n \mapsto \frac{1}{n})"'] & \mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_1(n)}(-), \Gamma(-)) \arrow[d, "(\mathbb{C}/\mathbb{Z}+n\tau\mathbb{Z}, dz, \tau)"] \\
& \mathrm{MF}(\Gamma_1(n), R_0)
\end{tikzcd}$$

We will denote the diagonal arrow by β_μ. Moreover, we denote for an abelian group scheme G over a scheme S by $G[n]$ the n-torsion, i.e. the pullback $G \times_G S$, where we use the multiplication-by-n-map $[n]\colon G \to G$ and the unit map $S \to G$.

Lemma A.16. *Let E/S be an elliptic curve and n be invertible on S. Let $\iota\colon \mu_{n,S} \to E$ be a $\Gamma_\mu(n)$-structure. Then we have a short exact sequence*

$$1 \to \mu_{n,S} \xrightarrow{\iota} E[n] \xrightarrow{\kappa} (\mathbb{Z}/n)_S \to 1 \tag{A.17}$$

of étale sheaves of abelian groups.

Proof. The Weil pairing discussed in [27, Section 2.8] induces an isomorphism

$$E[n] \xrightarrow{\cong} \mathrm{Hom}_{S-\mathrm{gp}}(E[n], \mathbb{G}_{m,S}).$$

Postcomposing with ι^* induces a surjection

$$E[n] \to \mathrm{Hom}_{S-\mathrm{gp}}(\mu_{n,S}, \mathbb{G}_{m,S}) \cong (\mathbb{Z}/n)_S,$$

which we call κ. By [27, Theorem 2.3.1], the sequence (A.17) looks étale locally like

$$0 \to (\mathbb{Z}/n)_S \to (\mathbb{Z}/n \times \mathbb{Z}/n)_S \to (\mathbb{Z}/n)_S \to 0.$$

As exactness can be checked étale locally, it remains to check that the composition $\kappa\iota$ is zero, which is true by [27, 2.8.7]. $\square$

A.2.3. *Compactifications and comparison of algebraic and analytic theory.* In this section we discuss how to compactify $\mathcal{M}_1(n)$ and also the comparison of the algebraic and the analytic theory. The basic sources are [12] and [10] and we will just give a short summary.

The moduli stack $\mathcal{M}_1(n)$ has a compactification $\overline{\mathcal{M}}_1(n)$, which can be defined as the normalization of $\overline{\mathcal{M}}_{ell}$ in $\mathcal{M}_1(n)$ (see Section 5 for the normalization construction). It is shown in [12, Section IV] that $\overline{\mathcal{M}}_1(n) \to \operatorname{Spec} \mathbb{Z}[\frac{1}{n}]$ is proper and smooth of relative dimension 1. For $n \geq 5$, the stack $\overline{\mathcal{M}}_1(n)$ is representable by a projective scheme (see e.g. [38]). It is shown in [10, Thm 2.2.2.1] that the Riemann surface associated with $\overline{\mathcal{M}}_1(n)_\mathbb{C}$ is isomorphic to a more classical construction, namely the compactification $X_1(n)$ of the quotient $Y_1(n)$ of the upper half plane $\mathbb{H}$ by $\Gamma_1(n)$. Indeed, Conrad shows that both $\overline{\mathcal{M}}_1(n)_\mathbb{C}$ and $X_1(n)$ classify generalized elliptic curves over complex analytic spaces with $\Gamma_1(n)$-level structure. The family of elliptic curves $(\mathbb{C}/\mathbb{Z} + n\tau\mathbb{Z}, \tau)$ with $\Gamma_1(n)$-level structure over $\mathbb{H}$ descends to $Y_1(n)$ and extends to $X_1(n)$. (Indeed, Conrad considers the universal family $(\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}, \frac{1}{n})$ as in [10, Section 2.1.3], but the choice of $e^{2\pi i/n}$ as an n-th root of unity allows us to consider the automorphism $\mathcal{M}_1(n)_\mathbb{C} \xrightarrow{\varphi} \mathcal{M}_\mu(n)_\mathbb{C} \simeq \mathcal{M}_1(n)_\mathbb{C}$ that carries one family of elliptic curves into the other as follows from Example A.14.) This specifies an isomorphism $\overline{\mathcal{M}}_1(n)_{\mathbb{C}} \to X_1(n)$. More information about $X_1(n)$ can be found in [10] and in [14, Chapter 2].

We will abuse notation again and denote by $\underline{\omega}$ the line bundle on $X_1(n)$ corresponding to the analytification of $\underline{\omega}$ on $\overline{\mathcal{M}}_1(n)_{\mathbb{C}}$ under the isomorphism above and likewise its restriction to $Y_1(n)$. By GAGA [45, Théorème 1], the morphism $H^0(\overline{\mathcal{M}}_1(n)_{\mathbb{C}}; \underline{\omega}^{\otimes k}) \to H^0(X_1(n); \underline{\omega}^{\otimes k})$ is an isomorphism. Moreover, this restricts to an isomorphism $H^0(\mathcal{M}_1(n)_{\mathbb{C}}; \underline{\omega}^{\otimes k}) \to H^0(Y_1(n); \underline{\omega}^{\otimes k})$ as $H^0(\mathcal{M}_1(n)_{\mathbb{C}}; \underline{\omega}^{\otimes *}) \cong H^0(\overline{\mathcal{M}}_1(n)_{\mathbb{C}}; \underline{\omega}^{\otimes *})[\Delta^{-1}]$ and the corresponding statement is true for sections on $X_1(n)$ and $Y_1(n)$ as well.

Given a section of $\underline{\omega}^{\otimes k}$ on $Y_1(n)$ we can pull it back along $\pi \colon \mathbb{H} \to Y_1(n)$ and obtain a holomorphic function on $\mathbb{H}$ by trivializing $\pi^*\underline{\omega}$ via dz. It is shown in [10, 1.5.2.4 and Lemma 1.5.7.2] that the image consists exactly of the meromorphic modular forms of weight k for $\Gamma_1(n)$. Moreover, Conrad shows that the image of $H^0(X_1(n); \underline{\omega}^{\otimes k}) \hookrightarrow H^0(Y_1(n); \underline{\omega}^{\otimes k})$ consists exactly of the *holomorphic* modular forms of weight k for $\Gamma_1(n)$.

In summary, we obtain exactly that our comparison map

$$H^0(\mathcal{M}_1(n)_{\mathbb{C}}; \underline{\omega}^{\otimes k}) \xrightarrow{\cong} \mathrm{Nat}_k(\mathrm{Ell}_{\Gamma_1(n)}(-), \Gamma(-)) \to \mathrm{MF}(\Gamma_1(n); \mathbb{C})$$

is an isomorphism, where we restricted the domain of $\mathrm{Ell}_{\Gamma_1(n)}$ and Γ to $\mathbb{C}$-algebras. Moreover, this restricts to an isomorphism $H^0(\overline{\mathcal{M}}_1(n)_{\mathbb{C}}; \underline{\omega}^{\otimes k}) \to \mathrm{mf}(\Gamma_1(n); \mathbb{C})$.

For $n < 5$, $\overline{\mathcal{M}}_1(n)$ is no longer a scheme. In this case, one can analogously use a GAGA theorem for stacks as, for example, proven in [42]. In our situation the proof should be considerably simplified though as $\overline{\mathcal{M}}_1(n)_{\mathbb{C}}$ has a finite faithfully flat cover by a scheme (e.g. by $\overline{\mathcal{M}}_1(5n)_{\mathbb{C}}$) and one should be able to deduce a sufficiently strong GAGA theorem just by descent from the scheme case.

A.3. **The Tate curve.** In this section, we will discuss the Tate curve, which will give us an algebraic way to define q-expansions of modular forms. We first discuss the situation over the complex numbers.

Theorem A.18 ([47, Theorem V.1.1])**.** *For any $q, u \in \mathbb{C}$ with $|q| < 1$, define the following quantities:*

$$\begin{aligned}
\sigma_k(n) &= \sum_{d|n} d^k, \\
s_k(q) &= \sum_{n\geq 1} \sigma_k(n) q^n = \sum_{n \geq 1} \frac{n^k q^n}{1-q^n}, \\
a_4(q) &= -5 s_3(q), \\
a_6(q) &= -\frac{5s_3(q) + 7s_5(q)}{12}, \\
X(u,q) &= \sum_{n\in\mathbb{Z}} \frac{q^n u}{(1-q^n u)^2} - 2s_1(q), \\
Y(u,q) &= \sum_{n\in\mathbb{Z}} \frac{(q^n u)^2}{(1-q^n u)^3} + s_1(q).
\end{aligned}$$

(1) Then the equation

$$y^2 + xy = x^3 + a_4(q)x + a_6(q) \tag{A.19}$$

defines an elliptic curve E_q over $\mathbb{C}$, and X, Y define a complex analytic isomorphism

$$\begin{aligned} \mathbb{C}^\times/q^{\mathbb{Z}} \to \quad & E_q \\ u \mapsto \quad & \begin{cases} (X(u,q), Y(u,q)), & \text{if } u \notin q^{\mathbb{Z}}, \\ O, & \text{if } u \in q^{\mathbb{Z}} \end{cases} \end{aligned}$$

(2) *The power series $a_4(q)$ and $a_6(q)$ define holomorphic functions on the open unit disk $\mathbb{D}$.*
(3) *As power series in q, both $a_4(q), a_6(q)$ have integer coefficients.*
(4) *The discriminant of E_q is given by*

$$\Delta(q) = q \prod_{n \geq 1} (1-q^n)^{24} \in \mathbb{Z}[\![q]\!].$$

(5) *Every elliptic curve over $\mathbb{C}$ is isomorphic to E_q for some q with $|q| < 1$.*

Let $\mathrm{Conv} \subset \mathbb{Z}((q))$ be the subset of "convergent" Laurent series, i.e. those that define meromorphic functions on $\mathbb{D}$ that are holomorphic away from 0; in particular, $a_4, a_6 \in \mathrm{Conv}$. By the explicit description of the discriminant, we can use the Weierstraß equation (A.19) to define an elliptic curve $\mathrm{Tate}(q)$ over Conv. For our computations in Section 3 it will be convenient to consider the analogously defined ring $\mathrm{Conv}_{q^n} \subset \mathbb{Z}((q^n))$ with the Tate curve $\mathrm{Tate}(q^n)$ defined by $a_4(q^n)$ and $a_6(q^n)$ over it.

Let $q_0 \in \mathbb{D}$ be a nonzero point and consider the morphism $\mathrm{ev}_{q_0} \colon \mathrm{Conv} \to \mathbb{C}$. By the theorem above, we see that the analytic space associated with $\mathrm{ev}_{q_0}^* \mathrm{Tate}(q)$ is isomorphic to $\mathbb{C}^\times/q_0^{\mathbb{Z}}$. The invariant differential η^{can} associated to the Weierstraß equation corresponds under this isomorphism to $\frac{du}{u}$, as can be shown by elementary manipulations using [47, Section V.1].

Next, we want to describe a group homomorphism $\iota \colon \mu_{n,\mathrm{Conv}} \to \mathrm{Tate}(q)[n]$ for $n \geq 2$. For simplicity, we will only describe it over $\mathrm{Conv}[\frac{1}{n}]$. We first describe ι after base change to $\mathrm{Conv}[\frac{1}{n}, \zeta_n] = \mathrm{Conv} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{n}, \zeta_n]$. As μ_n is isomorphic to $\mathbb{Z}/n$ over this ring, it suffices to give an n-torsion point in $\mathrm{Tate}(q)[n](\mathrm{Conv}[\frac{1}{n}, \zeta_n])$; we take $[X(\zeta_n, q), Y(\zeta_n, q), 1]$. This is compatible with the Galois action and thus, we obtain a morphism $\mu_{n,\mathrm{Conv}[\frac{1}{n}]} \to \mathrm{Tate}(q)[n]_{\mathrm{Conv}[\frac{1}{n}]}$. Note that we can check that this is indeed a group homomorphism into the n-torsion by evaluating at infinitely many points in $\mathbb{D}$. For a nonzero $q_0 \in \mathbb{D}$, this ι corresponds under the isomorphism of $\mathrm{ev}_{q_0}^* \mathrm{Tate}(q)$ with $\mathbb{C}^\times/q_0^{\mathbb{Z}}$ exactly to the composite $\mu_n(\mathbb{C}) \to \mathbb{C}^\times \to \mathbb{C}^\times/q_0^{\mathbb{Z}}$. Note that ι defines a $\Gamma_\mu(n)$-structure on $\mathrm{Tate}(q)$.

We remark that there are other possible choices to define ι, corresponding to different constructions of the Tate curve. To avoid possible ambiguity, we show the following uniqueness statement.

Proposition A.20. *The morphism*

$$\mathbb{Z}/n \to \mathrm{Hom}(\mu_{n,\mathrm{Conv}}, \mathrm{Tate}(q)), \qquad k \mapsto k\iota$$

is a bijection.

Proof. The group $\mathrm{Tate}(q)[n]_{\mathrm{Conv}_{\mathbb{C}}[q^{1/n}]}$ is isomorphic to $(\mathbb{Z}/n)^2$ with

$$(X(\zeta^a q^{b/n}, q), Y(\zeta^a q^{b/n}, q), 1)$$

as the non-trivial torsion points, where $\zeta = e^{2\pi i/n}$; indeed these are all n-torsion points as we can check on infinitely many points in $\mathbb{D}$ (away from some chosen ray so that $q^{1/n}$ makes sense as a holomorphic function) and there cannot be more n-torsion points.

We see that only n of these torsion points have coordinates in $\mathrm{Conv}_{\mathbb{C}}$ and thus $\mathrm{Tate}(q)[n]_{\mathrm{Conv}_{\mathbb{C}}} \cong \mathbb{Z}/n$. We obtain that $\mathrm{Hom}(\mu_{n,\mathrm{Conv}_{\mathbb{C}}}, \mathrm{Tate}(q)_{\mathrm{Conv}_{\mathbb{C}}}) \cong \mathbb{Z}/n$ and the existence of ι shows that the injective map

$$\mathrm{Hom}(\mu_{n,\mathrm{Conv}}, \mathrm{Tate}(q)) \to \mathrm{Hom}(\mu_{n,\mathrm{Conv}_{\mathbb{C}}}, \mathrm{Tate}(q)_{\mathrm{Conv}_{\mathbb{C}}})$$

is also a surjection. $\square$

By Lemma A.16, we obtain for each $n \geq 1$ a short exact sequence

$$0 \to \mu_{n,\mathrm{Conv}[\frac{1}{n}]} \xrightarrow{\iota'} \mathrm{Tate}(q)[n] \xrightarrow{\kappa} (\mathbb{Z}/n\mathbb{Z})_{\mathrm{Conv}[\frac{1}{n}]} \to 0$$

of étale sheaves of abelian groups. We can normalize κ in the following way: For any $\mathrm{Conv}[\frac{1}{n}]$-algebra R, any $\zeta \in \mu_n(R)$ and $X \in \mathrm{Tate}(q)[n](R)$, the Weil pairing $e_n(\iota(\zeta), X)$ equals $\zeta^{\kappa(X)}$.

We remark that by comparing the explicit equations , one sees that our definition of the Tate curve agrees with the one discussed in [27, Section 8.8] (and e.g. in [12] before).

A.4. **q-expansions.** Our goal in this subsection is to define the q-expansion both in the holomorphic and in the algebraic context, to compare them and to obtain a q-expansion principle.

Consider a modular form g in $\mathrm{MF}(\Gamma_1(n);\mathbb{C})$ (possibly $n = 1$ so $\Gamma_1(n) = \mathrm{SL}_2(\mathbb{Z})$). We recall that $g(\tau) = g(\tau+1)$ and thus g factors through a meromorphic function $\widetilde{g}\colon \mathbb{D} \to \mathbb{C}$ with only pole in 0?, where $\mathbb{D}$ denotes the open disk with radius 1; more precisely, we have $\widetilde{g}(q) = g(\tau)$, where $q = q(t) = e^{2\pi i\tau}$. Taylor expansion of $\widetilde{g}$ at 0 yields a map

$$\Phi^{hol}\colon \mathrm{MF}_k(\Gamma_1(n);\mathbb{C}) \to \mathbb{C}((q)).$$

On the algebraic side, we obtain a map

$$\Phi^{\mu,R_0}\colon \mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_\mu(n)}(-), \Gamma(-)) \to R_0((q))$$

(for Ell living over a fixed $\mathbb{Z}[\frac{1}{n}]$-algebra R_0 again) by evaluating the natural transformation at the Tate curve $(\mathrm{Tate}(q), \eta_{can}, \iota)$ from the last section. More precisely, we evaluating on the pullback of the Tate curve to $R_0((q))$.

We want to show that Φ^{hol} and $\Phi^{\mu,\mathbb{C}}$ correspond to each other under β_μ. Both have actually image in $\widetilde{\mathrm{Conv}} \subset \mathbb{C}((q))$. Thus we can check the agreement of $\Phi^{hol}\beta_\mu(n)$ with $\Phi^{\mu,\mathbb{C}}$ after postcomposing these two maps with $\mathrm{ev}_{q_0}\colon \mathrm{Conv} \to \mathbb{C}$ for infinitely many $q_0 \in \mathbb{D} \setminus \{0\}$.

Choose $\tau_0 \in \mathbb{H}$ with $e^{2\pi i\tau_0} = q_0$. By definition, $\mathrm{ev}_{q_0} \Phi^{hol}(g) = \widetilde{g}(q_0) = g(\tau_0)$. Using that$\mathbb{C}/(\mathbb{Z}+\tau_0\mathbb{Z}) \cong \mathbb{C}^\times/q_0^{\mathbb{Z}}$, we observe that $\mathrm{ev}_{q_0} \Phi^{hol}\beta_\mu(g)$ (with $g \in \mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_\mu(n)}(-), \Gamma(-))$) equals $g(\mathbb{C}^\times/q_0^{\mathbb{Z}}, \frac{dq}{q}, \iota^{can})$, where ι^{can} denotes the composition $\mu_n(\mathbb{C}) \to \mathbb{C}^\times \to \mathbb{C}^\times/q_0^{\mathbb{Z}}$.

On the other hand, $\mathrm{ev}_{q_0} \Phi^{\mu,\mathbb{C}}(g)$ equals $(\mathrm{ev}^*_{q_0} \mathrm{Tate}(q), \mathrm{ev}^*_{q_0} \iota, \mathrm{ev}^*_{q_0} \eta^{can})$. We have seen in the last section that this triple is isomorphic to $(\mathbb{C}^\times/q_0^{\mathbb{Z}}, \iota^{can}, \frac{dq}{q})$, what was to be shown. Thus, the following triangle commutes:

$$\begin{tikzcd}
\mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_\mu(n)}(-),\Gamma(-)) \arrow[r, "\Phi^{\mu,\mathbb{C}}"] \arrow[d, "\beta_\mu"] & \mathbb{C}((q)) \\
\mathrm{MF}_k(\Gamma_1(n),\mathbb{C}) \arrow[ur, "\Phi^{hol}"'] &
\end{tikzcd}$$

We obtain the q-expansion morphism

$$\Phi^{1,R_0}\colon \mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_\mu(n)}(-),\Gamma(-)) \to \mathrm{Conv}\otimes R_0$$

as the composition $\Phi^{\mu,R_0}(\varphi^*)^{-1}$, where φ is as in Subsection A.2.2.

Lemma A.21. *Assume that $R_0 \subset \mathbb{C}$ and let $q_0 \neq 0$ be a point in the open unit disk. Evaluating at q_0 yields a morphism $\mathrm{ev}_{q_0}\colon \mathrm{Conv}[q^{-1}]\otimes R_0 \to \mathbb{C}$. Then*

$$\mathrm{ev}_{q_0}\Phi^{1,R_0}(g) = g(\mathbb{C}^\times/q^{n\mathbb{Z}}, \frac{dq}{q}, q)$$

for every $g \in \mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_\mu(n)}(-),\Gamma(-))$.

Proof. It suffices to show that

$$\varphi(\mathbb{C}^\times/q^{n\mathbb{Z}}, \frac{dq}{q}, q) = (\mathbb{C}^\times/q^{\mathbb{Z}}, \frac{dq}{q}, \iota^{can}).$$

This follows from Example A.14. □

Note that these discussions actually show that β_1 and β_μ actually have target $\mathrm{MF}(\Gamma_1(n);R_0)$, i.e. that the classical q-expansion ofβ_1 of a modular form over R_0 actually has coefficients in R_0 and similarly for β_μ.

Theorem A.22 (q-expansion principle)**.** *Let R_0 be a subring of $\mathbb{C}$. The morphisms*

$$\beta_\mu\colon \mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_\mu(n)}(-),\Gamma(-)) \to \mathrm{MF}(\Gamma_1(n);R_0)$$

and

$$\beta_1\colon \mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_1(n)}(-),\Gamma(-)) \to \mathrm{MF}(\Gamma_1(n);R_0)$$

are isomorphisms. In other words: If the coefficients of the q-expansion of a complex modular form are in R_0, it is actually already defined over R_0.

Proof. By the considerations above, it suffices to show the first statement. For $R_0 = \mathbb{C}$, this was discussed in Subsection A.2.3. The general case follows by the q-expansion principle as stated in [13, Theorem 12.3.4]. □

A.5. **Summary.** Let R be any ring. We can define holomorphic modular forms for $\Gamma_1(n)$ of weight k over R as $H^0(\overline{\mathcal{M}}_1(n);\underline{\omega}^{\otimes k})$ and meromorphic modular forms as $H^0(\mathcal{M}_1(n)_R;\underline{\omega}^{\otimes k})$. We have a morphism $\mathrm{Spec}\,\mathbb{C} \to \mathcal{M}_1(n)$ classifying the elliptic curve $\mathbb{C}/\mathbb{Z}+n\tau\mathbb{Z}$ with chosen point τ of order n. Pulling $f \in H^0(\mathcal{M}_1(n);\underline{\omega}^{\otimes k})$ back to $\mathrm{Spec}\,\mathbb{C}$ and using the trivialization $\underline{\omega}^{\otimes k}$ induced by the choice of differential dz, defines a holomorphic function of $\tau \in \mathbb{H}$ that is a meromorphic modular form for $\Gamma_1(n)$ in the classical sense. This defines an isomorphism

$$\beta_1\colon H^0(\overline{\mathcal{M}}_1(n)_{\mathbb{C}};\underline{\omega}^{\otimes k}) \to \mathrm{MF}_k(\Gamma_1(n);\mathbb{C}).$$

The q-expansion of $\beta_1(f)$ lies in $R \subset \mathbb{C}$ if and only if f is in the image of the injection

$$H^0(\overline{\mathcal{M}}_1(n)_R;\underline{\omega}^{\otimes k}) \to H^0(\overline{\mathcal{M}}_1(n)_{\mathbb{C}};\underline{\omega}^{\otimes k}).$$

Appendix B. (Potentially) Computation of homotopy groups of $\mathrm{Tmf}_0(7)$

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