Differential Topology

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Manifolds are particularly nice and important topological spaces. The study of manifolds is one of the oldest and largest branches of topology. Differential topology deals in first place with the study of so-called differentiable manifolds. Our main goal is to learn the basic notions of diffential topology. In the second half of the seminar, we will deal with the notion of cobordisms, an important relation between manifolds.

About this seminar

The requirements of this seminar is a regular participation in the seminar, a talk and a (possibly short) preliminary discussion of the talk with me about ten days before the actual talk. This preliminary discussion gives the opportunity to ask mathematical questions and also to fix the precise content of the talk.

Please feel free to contact me if you have any questions, both on formalities or on mathematical contents of your talk. Please plan your talk for 75-80 minutes. Take particular care of your target audience, consisting of your fellow students. For example, if you do not know a notion or do not understand a proof for a while, do not assume your audience to understand the same proof immediately or to know all the new notions. Personally, I would recommend to make a test run of your talk, either in front of a fellow student or even for yourself. The important point is to simulate your part as precisely as possible.

If you have any questions about the seminar, feel free to contact me in my office 4.014 or via email at

ozornova@math.uni-bonn.de.

I also recommend to read the following hints by Prof. M. Lehn on how to give a seminar talk (in German):

http://www.mathematik.uni-mainz.de/Members/lehn/le/seminarvortrag

Even if some of the expressed advices might be exaggerated, this tutorial is in any case helpful.

Talk 1: Manifolds

The main objective of this talk is to recollect the knowledge on manifolds from Analysis II-IV (and other lectures), and to create a common level in preliminaries. Be sure to (re)introduce the notions of a (differentiable) manifold and a submanifold and give some examples. Be sure to explain that every manifold embeds into some $\mathbb{R}^n$ (present the proof only if you have time; use §II.10 of [Bre97]). Moreover, introduce and discuss also manifolds with boundary. State also the classification theorem for 1-manifolds. For basics of manifolds, follow [BJ82], Ch. 1. You may also want to take a look at [GP10], §1.1. For manifolds with boundary, use §2.1 of [GP10] and have a look at Ch. 13 of [BJ82].

Talk 2: Tangent spaces

Give three definitions of a tangent space and explain how to compare them; you will have to decide which parts of the proof you want to present. Define also smooth maps between manifolds and their derivatives. Also, briefly discuss tangent spaces for manifolds with boundary. Follow [BJ82], Ch. 1-2 (and Ch. 13 for manifolds with boundary). You may also want to take a look at [GP10], Ch. 1.

Talk 3: Tangent bundles

Tangent spaces of a manifold at different points may be considered not only on their own, but also as a totality or family of vector spaces. Such a parametrized family of vector spaces is called a vector bundle. The main objective of this talk is to learn some vector bundle basics in the context of manifolds. The most important example will be the tangent bundle of a manifold, and later on the normal bundle of an embedding. Follow Ch. 3 of [BJ82]. If you have time left, give a short account on direct sums of vector bundles as in Ch. 4 of [BJ82].

Talk 4: Transversality basics

The aim of this talk is to give an appropriate generalization of the regular value theorem. Recall or introduce the regular value theorem for manifolds. Give a short account on Sard's Theorem (without proofs). Move on to the notion of transversality, which generalizes regular values, and discuss the corresponding theorem on preimages of manifolds. Explain the statement of the stability theorem. If you have time left, give also some ideas of the proof. Follow §§1.4-1.6 of [GP10] for most of the talk and §1.7 for Sard's Theorem. You might also want to take a look at Ch.5-6 of [BJ82].

Talk 5: Transversality Theorem

Your main objective is to prove that any smooth map can be deformed into one transversal to a given submanifold of the target. We will need also the ε-neighborhood theorem, and you should also state the tubular neighborhood theorem. Follow §2.3 of [GP10]. Prove Transversality Homotopy Theorem and state Extension Theorem, give an idea of its proof if you have time for it.

Talk 6: Intersection theory mod 2

Define and study intersection numbers mod 2 and the degree mod 2. You should prove that those are homotopy invariants. Use the degree mod 2 to give a proof of the Brouwer fixed point theorem. Explain the statement of the Jordan-Brouwer separation theorem and sketch the proof idea using winding numbers mod 2. Follow §§2.4-2.5 of [GP10] and §4 of [Mil97].

Talk 7: Orientability and intersection theory

Your task is to refine intersection numbers mod 2 to a more powerful invariant. To do so, we need a certain assumption on the manifolds we consider, called orientability. Discuss the notion of orientability and examples. Discuss the induced orientation of a boundary and of a preimage manifold. Introduce (oriented) intersection numbers and degree, and discuss their basic properties like homotopy invariance. Use this theory to give a proof of fundamental theorem of algebra. Follow §3.2 and 3.3 of [GP10]. You might also want to browse through §5 of [Mil97].

Talk 8: Lefschetz theory

Your aim is to apply intersection theory to some examples. The main topic of your talk is Lefschetz theory. Introduce Lefschetz numbers and explain their computation in terms of local Lefschetz numbers. Introduce the Euler number, one of the most famous topological invariants, and use Lefschetz theory to compute it in some examples. Follow §3.4 of [GP10]. If you have time, report on the homological definition of Lefschetz numbers, following e.g. §IV.23 of [Bre97].

Talk 9: Poincaré-Hopf theorem

Discuss vector fields on manifolds. Your main objective is to show that zeros of vector fields on a manifold are constrained by the topology of the manifold, in particular, by the Euler number of the manifold. Give a proof of the Poincaré-Hopf theorem and also some examples. Follow §3.5 of [GP10]. You should also look at [Mil97], §6. If you have time, report on the Hopf degree theorem, following §3.6 of [GP10].

Talk 10: Framed cobordism I

Your aim is to introduce framed manifolds and the cobordism relation for framed manifolds, which can be roughly described as "being boundary parts of the same manifold". This relation, which is much coarser than "being diffeomorphic", turns out to be both very interesting and rather complicated. Your task is to explain the statements of the theorems classifying framed bordism. If you have time, start the proof. Be sure to agree with the speaker of the next talk on the precise point to stop. Follow §7 of [Mil97].

Talk 11: Framed cobordism II

Your task is to identify framed cobordism classes with certain homotopy classes of maps between spheres. Explain Pontrjagin's construction and prove the theorems formulated in the last talk. Use the results to prove the Hopf degree theorem. Be sure to agree with the speaker of the previous talk on the precise subdivision of the topic. Follow §7 of [Mil97].

References

[BJ82] Theodor Bröcker and Klaus Jänich. *Introduction to differential topology.* Cambridge University Press, Cambridge-New York, 1982. Translated from the German by C. B. Thomas and M. J. Thomas.

[Bre97] Glen E. Bredon. *Topology and geometry*, volume 139 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1997. Corrected third printing of the 1993 original.

[GP10] Victor Guillemin and Alan Pollack. *Differential topology.* AMS Chelsea Publishing, Providence, RI, 2010. Reprint of the 1974 original.

[Mil97] John W. Milnor. *Topology from the differentiable viewpoint.* Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Based on notes by David W. Weaver, Revised reprint of the 1965 original.