Braid groups

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Braid groups appear in many different ways in topology and also other areas of mathematics. They were originally defined by Emil Artin in 1925 to study knots; the connection was investigated further by Markov. Moreover, braid groups turn out to be a special case of mapping class groups, which can be viewed as certain symmetry groups of surfaces. Furthermore, braid groups are also connected to configuration spaces. This makes them into interesting objects, both for topological and algebraic studies. Nowadays, there are also many generalizations of braid groups, both of topological and of group-theoretic flavor. In this seminar, we are going to study different aspects of braid groups and their analogs.

Dates

The seminar will take place Tuesdays, 12-14 in 0.011.

About this seminar

The requirements of this seminar are a regular participation in the seminar, a talk and a (possibly short) preliminary discussion of the talk with me about ten days before the actual talk. This preliminary discussion gives the opportunity to ask mathematical questions and also to fix the precise content of the talk.

Please feel free to contact me if you have any questions, both on formalities or on mathematical contents of your talk. Please plan your talk for 75-80 minutes. Take particular care of your target audience, consisting of your fellow students. For example, if you do not know a notion or do not understand a proof for a while, do not assume your audience to understand the same proof immediately or to know all the new notions. Personally, I would recommend to make a test run of your talk, either in front of a fellow student or even for yourself. The important point is to simulate your part as precisely as possible.

If you have any questions about the seminar, feel free to contact me in my office 4.014 or via email at

ozornova@math.uni-bonn.de.

I also recommend to read the following hints by Prof. M. Lehn on how to give a seminar talk (in German):

http://www.mathematik.uni-mainz.de/Members/lehn/le/seminarvortrag

Even if some of the expressed advice might be exaggerated, this tutorial is in any case helpful.

Talk 1: Definition of Braids and Basic Properties I- Oct 18 (AA)

Define geometric braids and braid diagrams. Discuss briefly the variations in the choice of equivalence relations on braids. Discuss the existence of good projections and different possible notions of braids. Introduce the group structure on the isotopy classes of braids. Introduce briefly configuration spaces of a manifold and identify their fundamental groups. Follow [KT08], §1.2, and [Bir74], §1.1.

Talk 2: Definitions of Braids and Basic Properties II- Oct 25 (TT)

Introduce Reidemeister moves for braids and give a very sketchy proof of Theorem 1.6 in [KT08]. Identify the standard presentation of braid groups. Identify the braid group B_3 with the fundamental group of the complement of the trefoil knot. Define the pure braid groups and identify them also as fundamental groups of configuration spaces. If you have time, explain and give a (sketch) proof of Theorem 1.5 of [Bir74], proven in [Bir69].

Talk 3: (Review of) Fiber Bundles - Nov 8 (GL)

For the next talks, we will need the notion of fiber bundle, which is a generalization of a covering where the assumption on the fiber to be discrete is omitted. Define fiber bundles, vector bundles, principal bundles and give example of these. Define classifying spaces for discrete/topological groups, and provide examples. Recall also higher homotopy groups and formulate the long exact sequence for the homotopy groups for a fibration/fiber bundle. Follow [DK01], Chapter 4 (and 6.2 and 6.13 for the long exact sequence), and [Hat02], pp. 376-383 and also Section 1B.

Talk 4: Configuration Spaces and Pure Braid Groups - Nov 15(JK)

Describe the fundamental sequence by Fadell and Neuwirth which relates different configuration spaces to each other. Use this to show the high connectivity of configuration spaces. Show moreover that pure braid groups have a structure of an iterated semidirect product, and describe the corresponding normal form. Use the original article [FN62], but also the discussion in [KT08], §§1.3-1.4 and [Bir74], §1.2.

Talk 5: Cohomology of Braid Groups - Nov 22 (BR)

Homology and cohomology of a group are defined as the corresponding invariants of its classifying space and encode interesting information about the group. As we already have a model for the classifying space of a braid group, your objective is to compute and describe its (co)homology with $\mathbb{Z}/2$-coefficients. For this, exhibit in particular the cell decomposition of configuration spaces of a plane and compute the additive structure of the cohomology. State the multiplicative structure and comment on its computation if you have time. Follow the original exposition in [Fuk70]. Beware that there are several mistakes in the translation.

Talk 6: Braid Groups as Mapping Class Groups I - Nov 29 (YS)

Your objective is to identify braid groups as certain symmetry groups of punctured discs. For this, define first more generally mapping class groups and half-twists, and identify braid groups as mapping class groups of a punctured disc (Section 1.6.2 of [KT08]). Show how this induces an action of braid groups on free groups. If you have time, explain also how the identification with the above mapping class group can be generalized to Birman's exact sequence ([Bir74], §4.1, and [FM11], §9.1). Follow otherwise §§1.5-1.6 of [KT08].

Talk 7: Braid Groups as Mapping Class Groups II - Dec 6 (RS)

Your main objective is to discuss Birman-Hilden Theorem, which connects braid groups with mapping class groups of surfaces of higher genus. Introduce Dehn twists and discuss briefly some of the relations for them. Formulate the theorem and discuss its proof. If you have time, discuss also how one can conclude relations for Dehn twists from those in braid groups. Follow §3.1, 3.5, 9.1 and 9.4 of [FM11].

Talk 8: Braids vs. Knots I - Dec 13 (DG)

Your main objective is to classify closed braids in solid tori. Explain first closing of braids and prove (large parts of) Theorem 2.1 of [KT08], which gives a classification of closed braids. Recall the notion of the link and prove also Alexander's theorem relating links and closed braids. Follow [KT08], §§2.1-2.3. You might also want to take a look at §§2.1-2.2 of [Bir74].

Talk 9: Braids vs. Knots II: Markov's Theorem - Dec 20(LH)

Your main objective is to prove how we can decide whether two closed braids in $\mathbb{R}^3$ are equivalent as links. Explain first an algorithmic version of Alexander's theorem, allowing to obtain a closed braid isotopic to a given link. Then explain as much of the proof of Markov's theorem as possible. Markov's theorem gives a precise statement on when two braids give the same closed braids. As the proof is rather long, you will have to choose the parts of the proof most suitable for a talk. Follow [KT08], §§2.4-2.7. You might also want to have a look into [BZH14], Chapter 10D.

Talk 10: Coxeter Groups I - Jan 10 (CB)

Your task is to give a geometric motivation for the definition of Coxeter groups. It arises from the description of the finite subgroups of the orthogonal groups generated by reflections. Define these and give first easy examples. Show moreover that any of these has a Coxeter-type presentation (Theorem 1.9 in [Hum90]). Describe also the Classification Theorem 2.7 (and the connection to Coxeter groups in Proposition 2.1). If you have time, give a sketch proof of the Classification Theorem. Follow [Hum90], Chapters 1+2.

Talk 11: Coxeter Groups II - Jan 17 (JQ)

Your objective is to introduce the general theory for Coxeter groups. Define general Coxeter groups and give first examples. Prove in particular Matsumoto's property. Explain also the geometric representation (Section 5.3 of [Hum90]) allowing to view Coxeter groups as generalized reflection groups. If you have time, describe also the fundamental domain of this group action ([Hum90], Section 5.13). Follow [DDG$^+$15], Section IX.1.1, and [Hum90], Chapter 5.

Talk 12: Artin Groups and Garside Theory I: Garside Theory - Jan 24(CF)

Your objective is to discuss the basic Garside theory which we will use in the next talk to solve word and conjugacy problems of Artin groups. Define general Artin groups and Artin monoids and give first examples. Define (comprehensive) Garside monoids and describe the solution of the word and conjugacy problems for them. *Warning:* As these notions are very recent, the precise definitions have changed several times in the last 15 years, so please make sure to work with consistent definitions. For the Garside theory, see [KT08], Sections 6.1-6.4, but also have a look at [DDG$^+$15], Sections III.1.1-III.1.3.

Talk 13: Artin Groups and Garside Theory II: Word and Conjugacy Problems in Braid Groups - Jan 31 (SD)

Your main goal is to apply the theory developed in the last talk to Artin groups of finite type, so in particular to braid groups. Prove then that Artin monoids satisfy assumptions of the Garside theory, and use this to solve the word and conjugacy problems for Artin groups of finite type. Follow §§6.5-6.6 of [KT08]. Identify Artin groups as fundamental groups of spaces constructed in previous talks, stating Brieskorn's theorem ([KT08], §6.6). You might also want to have a look at [DDG$^+$15], Sections III.1.2-III.1.3. If you have time, describe also the Birman-Ko-Lee ("dual") braid monoid inside braid groups and its Garside structure.

Talk 14: Dehornoy Ordering for Braid Groups - Feb 7 (AK)

Your main objective is to introduce the notion of orderable and biorderable groups and to analyze them for braid groups. Give the definitions and first examples. Describe the biordering on pure braid groups and show that braid groups are not biorderable. Describe the Dehornoy left order on braid groups. Show how this implies that the group ring of braid groups does not have zero-divisors. If you have time, discuss parts of the proof. Follow Chapter 7 of [KT08]. You might also want to have a look at [DDRW08], in particular, Chapter 3.

References

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