Risk Sensitive Impulse Control and Deterministic Games

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Abstract. We investigate the relationship between stochastic impulse control problems and deterministic control problems. In particular, we study the problem of characterizing the limit point of the value function of a stochastic impulse control problem as the variance of the underlying diffusion process tends to zero. In the conventional setting the limit point is found to be the value function of an appropriate deterministic impulse control problem while in the risk-sensitive setting, where a logarithmic transform of the value function is the object of study, the limit point is found to be the value function of an appropriate zero-sum differential game involving impulse controls.

Keywords. Stochastic impulse control, Deterministic zero-sum differential games, Quasivariational inequalities, Viscosity solutions.

1 Introduction

We investigate the relationship between a class of stochastic optimal impulse control problems and deterministic control problems. In particular, we study the problem of characterization of the limit points of the value function of a class of stochastic impulse control problems when the variance of the underlying diffusion process tends to zero. In the conventional setting the value function converges to that obtained by consideration of an appropriate deterministic impulse control problem. In the risk-sensitive setting the value function converges to that of an appropriate zero-sum differential game involving impulse controls recently considered in [17]. The relationship between problems in risk-sensitive control and deterministic games has previously been studied in [3, 6, 7, 11, 16, 18].

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In Section 2 we consider the conventional stochastic optimal impulse control problem. A key role in our analysis is played by the concept of a viscosity solution to the relevant quasivariational inequality and use is made of established techniques in considering such solutions. In Section 3 we consider the risk-sensitive setting. Here, it is the limiting behaviour of a particular logarithmic transform of the value function we consider. Again, the concept and techniques of viscosity solutions play an important role.

Let $(\Sigma, F, P, F_t, B_t, t \geq 0)$ be a Wiener space with filtration $(F_t, t \geq 0)$ right continuous and F_0 containing all the P-negligible events in F. An impulse control is a sequence $(\theta_i, \xi_i)_{i\geq 1}$ where $\theta_i \leq \theta_{i+1}$ are stopping times with respect to F_t (that is, for each i, the event $\{\theta_i \leq t\}$ is an element of F_t for all $t \geq 0$) such that $\theta_i \to T$ and ξ_i are random variables measurable with respect to F_{θ_i}. We consider the adapted càdlàg stochastic process X which, on the interval $[\theta_i, \theta_{i+1})$, satisfies the equation

$$X_t = X_{\theta_i -} + \xi_i + \int_{\theta_i}^t g(X_s, s)ds + \int_{\theta_i}^t \sigma(X_s, s)dB_s \ . \tag{1}$$

Here g and σ are bounded and required to satisfy, for some constant k,

$$|g(x,t) - g(y,s)| + |\sigma(x,t) - \sigma(y,s)| \leq k(|x-y| + |t-s|) \ . \tag{2}$$

For a discussion of the construction of such processes we refer to [2, Chapter 6, Section 1]. We will denote the value of X at t by both X_t and $X(t)$.

2 The conventional setting

In this section we consider the convergence of the value function associated with a problem of impulse control when the variance of the underlying diffusion process tends to zero. We demonstrate that the value function tends to a unique limit point, given by the value function of an appropriate deterministic impulse control problem.

We consider the impulse control problem with value function given by

$$V(x,t) = \inf_{(\theta_i,\xi_i)_{i\geq 1}} E_{x,t}\Big[\int_t^T f(X(s),s)ds + \sum_{i\geq 1} c(\xi_i) + h(X(T))\Big] \ , \tag{3}$$

where we suppress the dependence of X upon the impulse control and the subscript x, t denotes that $X(t) = x$. The infimum is taken over the class of all admissible control strategies $\{(\theta_i, \xi_i), i \geq 1\}$: here $\theta_i \leq \theta_{i+1}$ for all $i \in \mathbb{N}$, θ_i is assumed to be F_t-measurable, $\theta_i \to T$ and ξ_i is assumed to be F_{θ_i}-measurable. Moreover we require that f and h are bounded and, for some constant k,

$$|c(x) - c(y)| + |f(x,t) - f(y,s)| + |h(x) - h(y)| \leq k(|x-y| + |t-s|) \ . \tag{4}$$

In addition we assume that

$$\inf_x c(x) > 0 \qquad \lim_{|x|\to\infty} c(x) = \infty \qquad \text{and} \qquad c(x+y) < c(x) + c(y)\ . \tag{5}$$

We remark that we may wish also to consider the control problem in which the random variables ξ_i are restricted to take values in some set $K \subset \mathbb{R}$. In this case, in the remainder of this section, when considering possible values for the ξ, attention should be limited to K.

In case the following quasivariational inequality (qvi) possesses a solution $V \in C^{2,1}[\mathbb{R}\times[0,T]]$, the space of functions $f : \mathbb{R} \times [0,T] \to \mathbb{R}$ such that $\partial^2 f/\partial x^2$ and $\partial f/\partial t$ exist and are continuous, it then follows, as in [14, Bemerkung 4.10.d] and [13, Theorem 4.3] (cf. also the proof of Theorem 4), that the solution coincides with value function (3):

$$\begin{aligned} \frac{\partial V}{\partial t} + a(x,t)\frac{\partial^2 V}{\partial x^2} + g(x,t)\frac{\partial V}{\partial x} + f(x,t) \ &\geq\ 0 \\ \inf_\xi [c(\xi) + V(x+\xi,t)] - V(x,t) \ &\geq\ 0 \\ \Big(\frac{\partial V}{\partial t} + a(x,t)\frac{\partial^2 V}{\partial x^2} + g(x,t)\frac{\partial V}{\partial x} + f(x,t)\Big)\Big(\inf_\xi [c(\xi) + V(x+\xi,t)] - V(x,t)\Big) \ &=\ 0 \\ V(x,T) = \inf_\xi \{c(\xi) + h(x+\xi)\}\ . \end{aligned} \tag{6}$$

Here $a(x,t) = \frac{1}{2}\sigma^2(x,t)$. We remark that on occasion impulse control problems are formulated with terminal conditions of a form different from that appearing here, seemingly in error (see, for example, [17] for a deterministic setting and [2] for a stochastic setting). Here, in the impulse control setting, the qvi plays the role of the Hamilton-Jacobi-Bellman equation in stochastic control (see Bensoussan and Lions [2]). Furthermore, with $(\theta_0, \xi_0) = (0,0)$ define, recursively,

$$\begin{aligned} \theta_i &= \inf\Big\{ t \in [\theta_{i-1}, T) : V(X(t-),t) = \inf_\xi [c(\xi) + V(X(t-)+\xi,t)] \Big\} \\ \xi_i &= \arg\min_\xi \{c(\xi) + V(X(\theta_i-)+\xi, \theta_i)\} \ \ ,\ \ i = 1,2,\ldots \end{aligned}$$

It follows that, almost surely, the sequence $(\theta_i, \xi_i)_{i\geq 1}$ is finite in length and that an optimal impulse control is provided by this sequence together with a single jump at time T of size $\xi = \arg\min_\xi\{c(\xi) + V(X(T-)+\xi,T)\}$. We remark that by our assumptions on c and [17, Lemma 4.4] we may replace the intervals $[\theta_{i-1}, T)$ by the open intervals (θ_{i-1}, T), $i = 2,3,\ldots$.

Before continuing we recall the concept of viscosity solution. In general one cannot guarantee sufficient regularity of the solutions of the qvi. A celebrated approach to overcome this problem was the introduction of viscosity solutions by Crandall and Lions [4] (see also [14, Definition 5.1] and [10]). A continuous function V is said to

be a *viscosity solution* of the above qvi in case $V(\cdot,T)=\inf_\xi\{c(\xi)+h(\cdot+\xi)\}$ and, for any $(x,t)\in \mathbb{R}\times[0,T)$, both of the following hold:
For each ϕ in $C^{2,1}[\mathbb{R}\times[0,T]]$ with $\phi(x,t)=V(x,t)$ and $V\le\phi$ it holds that

$$\min\left\{\frac{\partial\phi}{\partial t}+a(x,t)\frac{\partial^2\phi}{\partial x^2}+g(x,t)\frac{\partial\phi}{\partial x}+f(x,t),\inf_\xi[c(\xi)+V(x+\xi,t)]-V(x,t)\right\}\ge 0\ . \quad (7)$$

For each ϕ in $C^{2,1}[\mathbb{R}\times[0,T]]$ with $\phi(x,t)=V(x,t)$ and $V\ge\phi$ it holds that

$$\min\left\{\frac{\partial\phi}{\partial t}+a(x,t)\frac{\partial^2\phi}{\partial x^2}+g(x,t)\frac{\partial\phi}{\partial x}+f(x,t),\inf_\xi[c(\xi)+V(x+\xi,t)]-V(x,t)\right\}\le 0\ . \quad (8)$$

In case (7) (resp. (8)) holds, V is said to be a viscosity *subsolution* (resp. *supersolution*) of the qvi.

Analogous to the setting of second order partial differential equations (see, for example [8, Section II.6]), an equivalent formulation of a viscosity subsolution (resp. supersolution) consists in requiring that (7) (resp. (8)) hold for each ϕ in $C^{2,1}[\mathbb{R}\times[0,T]]$ such that $\phi-V$ has a strict local maximum (resp. minimum) at the point (x,t). A further equivalent formulation is obtained by dropping the requirement that the local maximum (resp. minimum) be strict.

In the first subsection we consider the case of smooth solutions of the relevant quasi-variational inequalities associated with the stochastic impulse control problem, while the case of viscosity solutions is considered in the next subsection.

2.1 Smooth solutions

We assume throughout the remainder of this subsection that the qvi (6) possesses a unique solution in $C^{2,1}[\mathbb{R}\times[0,T]]$ which coincides with the value function (3).

Replacing σ by $\sqrt{\epsilon}\sigma$ and denoting V by V^ϵ, it follows from the stability result of the following proposition (cf. also [8, pgs. 73-74 and 288]) that, in case $\{V^\epsilon\}_{\epsilon>0}$ is an equicontinuous family of functions, any limit function of the family is a viscosity solution of the qvi

$$\begin{aligned}\frac{\partial V}{\partial t}+g(x,t)\frac{\partial V}{\partial x}+f(x,t) &\ge 0\\ \inf_\xi[c(\xi)+V(x+\xi,t)]-V(x,t) &\ge 0\\ \left(\frac{\partial V}{\partial t}+g(x,t)\frac{\partial V}{\partial x}+f(x,t)\right)\left(\inf_\xi[c(\xi)+V(x+\xi,t)]-V(x,t)\right) &= 0\\ V(x,T)=\inf_\xi\{c(\xi)+h(x+\xi)\}\ .\end{aligned} \quad (9)$$

If this qvi has a unique viscosity solution, then we can say that the family $\{V^\epsilon\}_{\epsilon>0}$ converges to the solution of this qvi as $\epsilon\to 0$.

Proposition 1 *Suppose V^n is a viscosity solution of (6) with $a(\cdot,\cdot)$ replaced by $\beta^n a(\cdot,\cdot)$, where $\beta^n \searrow 0$. In case V^n converges uniformly on compact sets, the limit function, V, is a viscosity solution of (9).*

Proof: We show that V is a subsolution, the proof that V is a supersolution being similar. Suppose $\phi \in C^{2,1}[\mathbb{R} \times [0,T]]$ and $(x,t) \in \mathbb{R} \times [0,T)$ is a strict local maximum of $\phi - V$. Then, for n large enough, there exists points (x_n, t_n) with $(x_n, t_n) \to (x,t)$ such that $\phi - V^n$ has a local maximum at (x_n, t_n). Therefore, V^n satisfies (7) at (x_n, t_n) with $\tilde{\phi}^n(x,t) = \phi(x,t) + V^n(x_n,t_n) - \phi(x_n,t_n)$. Assume

$$\inf_\xi [c(\xi) + V^n(x_n + \xi, t_n)] - V^n(x_n, t_n) \;=\; 0$$

for infinitely many n. Noting that the V^n are uniformly bounded and recalling our assumptions on c it follows that

$$\inf_\xi [c(\xi) + V(x + \xi, t)] - V(x,t) \;=\; 0\;.$$

If, on the other hand,

$$\frac{\partial \tilde{\phi}^n}{\partial t} + \beta^n a(x_n,t_n)\frac{\partial^2 \tilde{\phi}^n}{\partial x^2} + g(x_n,t_n)\frac{\partial \tilde{\phi}^n}{\partial x} + f(x_n,t_n) \;=\; 0$$

for infinitely many n it holds, under our assumptions, that

$$\frac{\partial \phi}{\partial t} + g(x,t)\frac{\partial \phi}{\partial x} + f(x,t) \;=\; 0\;.$$

Hence, V is a subsolution of (9). ■

In the following theorem we show that indeed, under our assumptions, the functions V^n in the previous proposition are equicontinuous in the space $C[\mathbb{R} \times [0,T]]$ of continuous functions on $\mathbb{R} \times [0,T]$ equipped with the metric of uniform convergence.

Theorem 1 *The value function V is uniformly continuous on $\mathbb{R} \times [0,T]$.*

Proving the theorem we will apply the following exponential inequality which is [5, Lemma 5.6.18]:

Lemma 1 *Let b_t and σ_t be measurable processes and let*

$$dz_t = b_t\,dt + \sqrt{\varepsilon}\,\sigma_t\,dB_t,$$

where z_0 is deterministic. Let $\tau_1 \in [0,1]$ be a stopping time with respect to the filtration of $\{B_t, t \in [0,1]\}$. Suppose that σ is uniformly bounded, and for some constants M, B, ϱ and any $t \in [0, \tau_1]$,

$$|\sigma_t| \le M(\varrho^2 + |z_t|^2)^{1/2}$$

$$|b_t| \le B(\varrho^2 + |z_t|^2)^{1/2}.$$

Then for any $\delta > 0$ and any $\varepsilon \le 1$

$$\varepsilon \log P\Big(\sup_{t\in[0,\tau_1]} |z_t| \ge \delta \Big) \le K + \log\Big(\frac{\varrho^2 + |z_0|^2}{\varrho^2+\delta^2} \Big),$$

where $K = 2B + 3M^2$.

Proof of Theorem 1 : We first show that there exists a constant $C > 0$ such that we have

$$|V(x,t) - V(y,t)| \le C|x-y| \quad \forall\ x, y \in \mathbb{R},\ t \in [0,T]\ . \tag{10}$$

To see this, for any control $(\theta_i, \xi_i)_{i\ge 1}$ let X^x and X^y denote solutions to (1) with $X^x(t) = x$ and $X^y(t) = y$. We then have that

$$\begin{aligned} &\Big| E\Big[\int_t^T f(X^x(s), s)ds + \sum_{i\ge1} c(\xi_i) + h(X^x(T)) \Big] \\ &\qquad - E\Big[\int_t^T f(X^y(s), s)ds + \sum_{i\ge1} c(\xi_i) + h(X^y(T)) \Big] \Big| \\ \le\ & E\Big[\int_t^T |f(X^x(s), s) - f(X^y(s), s)|ds + |h(X^x(T)) - h(X^y(T))| \Big] \\ \le\ & C\Big[\Big(\int_t^T E|X^x(s) - X^y(s)|^2 ds \Big)^{1/2} + E|X^x(T) - X^y(T)| \Big]\ , \end{aligned} \tag{11}$$

for some constant $C > 0$ by our assumptions on f and h. By an application of Itô's formula followed by Gronwall's inequality (see, for example, [12]), we have, possibly increasing C, that

$$E|X^x(s) - X^y(s)|^2 \le C(x-y)^2 \quad \forall\ s \in [t,T]\ . \tag{12}$$

From (11) and (12), (10) readily follows.

Assuming $u > t$ we now show that

$$V(x,t) \le V(x,u) + \omega(u-t)\ ,$$

where $\lim_{u-t\to 0}\omega(u-t)=0$. To see this, note that, restricting the allowable controls in the following infimum to those which have no impulses in $[t,u)$, we have, for some K sufficiently large,

$$\begin{aligned} V(x,t) &\le \inf_{(\theta_i,\xi_i)_{i\ge1}} E_{x,t}\Big[\int_t^T f(X(s),s)ds+\sum_{i\ge1}c(\xi_i)+h(X(T))\Big] \\ &\le \inf_{(\theta_i,\xi_i)_{i\ge1}} E_{x,t}\Big[\int_u^T f(X(s),s)ds+\sum_{i\ge1}c(\xi_i)+h(X(T))\Big]+K(u-t) \qquad (13) \\ &= \inf_{(\theta_i,\xi_i)_{i\ge1}} E_{x,t}\Big[E_{X(u),u}\Big[\int_u^T f(X(s),s)ds+\sum_{i\ge1}c(\xi_i)\Big]+h(X(T))\Big]+K(u-t). \end{aligned}$$

Let, for $\delta>0$, $(\theta_i,\xi_i)_{i\ge1}$ be such that

$$E_{x,u}\Big[\int_u^T f(X(s),s)ds+\sum_{i\ge1}c(\xi_i)+h(X(T))\Big]\le V(x,u)+\delta\ .$$

We then have, for this control, using the previous part of the proof and possibly increasing K,

$$\begin{aligned} &E_{x,t}\Big[E_{X(u),u}\Big[\int_u^T f(X(s),s)ds+\sum_{i\ge1}c(\xi_i)+h(X(T))\Big]\Big] \qquad (14) \\ &\le V(x,u)+\delta+K|x-y|+KP_{x,t}(|X(u)-X(t)|\ge|x-y|)\ . \end{aligned}$$

From (13) and (14), as δ is arbitrary, we see that $\omega(\cdot)$ exists.

We now show, still with $u>t$,

$$V(x,u)\le V(x,t)+\kappa(u-t)\ ,$$

where $\lim_{u-t\to0}\kappa(u-t)=0$. To see this, fix $\delta>0$ and let $(\theta_i,\xi_i)_{\imath\ge1}$ be such that

$$V(x,t)\ge E_{x,t}\Big[\int_t^T f(X(s),s)ds+\sum_{i\ge1}c(\xi_i)+h(X(T))\Big]-\delta.$$

This last quantity is, for some $K>0$, due to the assumptions that f is bounded and $c>0$,

$$\ge E_{x,t}\Big[\int_t^{T-(u-t)} f(X(s),s)ds+\sum_{i\ge1}c(\xi_i)+h(X(T))\Big]-\delta-K(u-t)\ ,$$

which is, in turn, by our assumptions on f, g and σ and Lemma 1, increasing K if necessary,

$$\begin{aligned} \ge\ & E_{x,u}\Big[\int_u^T f(X(s),s)ds+\sum_{i\ge1}c(\xi_i)+h(X(T))\Big]-\delta \\ &-K\Big[(u-t)-\sqrt{u-t}\Big]-\exp\Big(K+\log\Big(\frac{(u-t)}{(u-t)+1}\Big)\Big) \\ \ge\ & V(x,u)-\delta-K\Big[(u-t)-\sqrt{u-t}\Big]-\exp\Big(K+\log\Big(\frac{(u-t)}{(u-t)+1}\Big)\Big). \end{aligned}$$

As $\delta > 0$ was arbitrary, the existence of $\kappa(\cdot)$ follows. ■

We now identify the solution of (9) with the value function of a particular deterministic optimal impulse control problem. To do so, consider the deterministic equation

$$\dot{y}(s) = g(y(s), s) + \dot{\xi}(s), \; s \in [t, T]$$

$$y(t-) = x \; ,$$

where $\xi(\cdot)$ is a control and it is required that the function $\xi(\cdot)$ be piecewise constant, so that we may write $\xi(\cdot) = \sum_{j=1}^{m} \xi_j 1_{[\tau_j, T]}(\cdot)$, for some $t \leq \tau_1 \leq \ldots \leq \tau_m \leq T$ and $m \geq 1$. A payoff function is given according to

$$J_{x,t}(\xi(\cdot)) = h(y(T)) + \int_t^T f(y(s), s) ds + \sum_{i \geq 1} c(\xi_j) \; .$$

Under our assumptions, [17, Theorem 3.4] states that $V(t, x) = \inf_{\xi(\cdot)} J_{x,t}(\xi(\cdot))$ is the unique viscosity solution of the qvi (9). Hence, together with Proposition 1 and Theorem 1 we conclude the following theorem.

Theorem 2 *For any sequence $\{V^{\epsilon_i}\}_{i \geq 1}$, with $\epsilon_i \downarrow 0$, the functions V^{ϵ_i} converges uniformly on compact sets to the unique viscosity solution of (9) or, what is the same, the value function of the deterministic control problem.*

2.2 The viscosity case

In general, it is asking too much that the value function (3) belong to $C^{2,1}[\mathbb{R} \times [0, T]]$ and one abandons hope of (6) holding. However, we will show that V is a viscosity solution of (6) in case the following Bellman principle holds:

$$V(x, t) = \inf_{(\theta_i, \xi_i)_{i \geq 1}} E_{x,t} \Big[\int_t^\tau f(X(s), s) ds + \sum_{i \geq 1} c(\xi_i) 1_{\{\theta_i \leq \tau\}} + V(X(\tau), \tau) \Big] \qquad (15)$$

for all $(F_t, t \geq 0)$-stopping times τ with $t \leq \tau \leq T$. Such a principle plays an important role in the control of Markov diffusion processes (cf. [8] and [15]) and is made use of by Korn in [13, Theorem 4.6] to prove that the value function of an infinite horizon impulse control problem is a viscosity solution of the relevant qvi.

Theorem 3 *Suppose (15) holds. The value function V is then a viscosity solution of (6).*

Remark: From the proof we note that V is a viscosity subsolution in case (15) holds for all stopping times which are constant.

Proof: Recalling (3), that $V(x,T) = \inf_\xi\{c(\xi) + h(x+\xi)\}$ is clear. We now show that V is a viscosity subsolution of (6). To do so, let $\phi \in C^{2,1}[\mathbb{R} \times [0,T]]$ with $\phi(x,t) = V(x,t)$ and $V \leq \phi$. We may assume that $t < T$. That

$$V(x,t) \leq \inf_\xi [c(\xi) + V(x+\xi,t)]$$

is clear and it remains to show that

$$\frac{\partial \phi}{\partial t} + a(x,t)\frac{\partial^2 \phi}{\partial x^2} + g(x,t)\frac{\partial \phi}{\partial x} + f(x,t) \geq 0 \; . \tag{16}$$

For any admissible policy it holds by our assumption that

$$\begin{aligned} \phi(x,t) = V(x,t) &\leq E_{x,t}\Big[\int_t^\tau f(X(s),s)ds + \sum_{i\geq 1} c(\xi_i)1_{\{\theta_i \leq \tau\}} + V(X(\tau),\tau)\Big] \\ &\leq E_{x,t}\Big[\int_t^\tau f(X(s),s)ds + \sum_{i\geq 1} c(\xi_i)1_{\{\theta_i \leq \tau\}} + \phi(X(\tau),\tau)\Big] \end{aligned} \tag{17}$$

for any stopping time τ. By Itô's formula we have that

$$\begin{aligned} \phi(X(\tau),\tau) = \phi(x,t) + \int_t^\tau \Big[\frac{\partial \phi}{\partial t} + a(X(s),s)\frac{\partial^2 \phi}{\partial x^2} + g(X(s),s)\frac{\partial \phi}{\partial x}\Big]ds \\ + \int_t^\tau \sigma(X(s),s)\frac{\partial \phi}{\partial x}dB_s + \sum_{t\leq s\leq \tau}[\phi(X(s),s) - \phi(X(s-),s)] \; . \end{aligned} \tag{18}$$

Setting $\tau = t+h$, substituting (18) into (17) and taking expectations yields

$$\begin{aligned} 0 \leq E_{x,t}\Big[\int_t^{t+h}\Big[\frac{\partial \phi}{\partial t} + a(X(s),s)\frac{\partial^2 \phi}{\partial x^2} + g(X(s),s)\frac{\partial \phi}{\partial x} + f(X(s),s)\Big]ds \\ + \sum_{i\geq 1} c(\xi_i)1_{\{\theta_i \leq t+h\}} + \sum_{t\leq s\leq t+h}[\phi(X(s),s) - \phi(X(s-),s)]\Big] \; . \end{aligned}$$

Choosing a policy which has no impulses on $[t, t+h]$, dividing by h and then letting $h \searrow 0$ we obtain (16).

To show that V is a viscosity supersolution let $\phi \in C^{2,1}[\mathbb{R} \times [0,T]]$ be such that $\phi(x,t) = V(x,t)$ and $V \geq \phi$. We proceed by contradiction. Assume therefore that

$$\inf_\xi [c(\xi) + V(x+\xi,t) - V(x,t)] \geq \gamma \tag{19}$$

and

$$\frac{\partial \phi}{\partial t} + a(x,t)\frac{\partial^2 \phi}{\partial x^2} + g(x,t)\frac{\partial \phi}{\partial x} + f(x,t) \geq \gamma \tag{20}$$

for some $\gamma > 0$. Since $\phi(x,t) = V(x,t)$ and as ϕ and V are continuous, decreasing γ if necessary, we have that (20) holds for all $(y,s) \in B_\delta(x,t) = \{(y,s) | d((y,s),(x,t)) \le \delta\}$ for some $\delta > 0$ and, moreover, that

$$c(\xi) + \phi(y+\xi, s) - \phi(y,s) \ge \gamma \tag{21}$$

holds whenever both $(y,s) \in B_\delta(x,t)$ and $(y+\xi, s) \in B_\delta(x,t)$. Continuing to suppress the dependence of X upon the control, we denote by θ the lesser of the exit time of X from $B_\delta(x,t)$ and T. By Itô's formula we have that (18) holds with τ replaced by θ. Taking expectations and taking (20) and the fact that $V \ge \phi$ into account yields

$$\begin{aligned} \phi(x,t) &= E_{x,t}\Big[\phi(X(\theta),\theta) - \int_t^\theta \Big[\frac{\partial \phi}{\partial t} + a(X(s),s)\frac{\partial^2 \phi}{\partial x^2} + g(X(s),s)\frac{\partial \phi}{\partial x}\Big] ds \\ &\quad - \sum_{t \le s \le \theta} [\phi(X(s),s) - \phi(X(s-),s)]\Big] \\ &\le E_{x,t}\Big[V(X(\theta),\theta) + \int_t^\theta f(X(s),s) ds - \sum_{t\le s\le\theta} [\phi(X(s),s) - \phi(X(s-),s)]\Big] \\ &\quad -\gamma\, E_{x,t}\Big[\int_t^\theta ds\Big] . \end{aligned}$$

As $V(x,t) = \phi(x,t)$ and $V \ge \phi$, by (21) we may write

$$V(x,t) \le E_{x,t}\Big[V(X(\theta),\theta) + \int_t^\theta f(X(s),s)ds + \sum_{i\ge1}[c(\xi_i) - \gamma]1_{\{\theta_i \le \theta\}}\Big] - \gamma\, E_{x,t}\Big[\int_t^\theta ds\Big]$$

$$\le E_{x,t}\Big[V(X(\theta),\theta) + \int_t^\theta f(X(s),s)ds + \sum_{i\ge1} c(\xi_i)1_{\{\theta_i\le\theta\}}\Big] - \gamma E_{x,t}[(\theta - t)1_{\{\theta_1 \ge \theta\}} + 1_{\{\theta_1 \le \theta\}}] .$$

However, we note that this last term is strictly negative independent of which control is chosen and hence (15) is contradicted. Therefore, V is in fact a viscosity supersolution. ■

Having established that V is a viscosity solution of (6), we now consider the question of uniqueness. For an alternative approach applied in an elliptic setting see [10].

Proposition 2 *The qvi (6) possesses a unique viscosity solution.*

Proof: Suppose V and $\hat{V}$ are two viscosity solutions of (6) and, without loss of generality, that

$$\sup_{(x,t)\in \mathbb{R}\times[0,T]} (V - \hat{V}) > 0 .$$

Thanks to (5), applying [17, Lemma 4.4] we may conclude that there exist $x_0 \in \mathbb{R}$ and $\delta > 0$ such that

$$\sup_{\Theta(x_0)} (V - \hat{V}) > 0$$

and

$$\hat{V}(x,t) < \inf_{\xi}[c(\xi) + \hat{V}(x+\xi,t)] \quad \forall \ (t,x) \in \Theta(x_0) \ ,$$

where

$$\Theta(x_0) = \{(x,t) \in (x_0 - \delta, x_0 + \delta) \times (T - \delta, T)\} \ .$$

Recalling (7) and (8) we now see that, in $\Theta(x_0)$, $\hat{V}$ is a viscosity solution of the partial differential equation

$$\frac{\partial V}{\partial t} + a(x,t)\frac{\partial^2 V}{\partial x^2} + g(x,t)\frac{\partial V}{\partial x} + f(x,t) = 0 \ ,$$

while V is a viscosity subsolution of this partial differential equation with the same terminal conditions. However, it follows by [8, Theorem V.9.1], upon noting that rather than the existence of various bounded derivatives required there it suffices that Lipschitz continuity holds, that

$$\sup_{\Theta(x_0)} (V - \hat{V}) \leq 0 \ ,$$

a contradiction. ■

As Proposition 1 and Theorems 1 and 2 continue to hold in this subsection, we have that the family $\{V^\epsilon\}_{\epsilon>0}$ converges to the viscosity solution of the qvi (9) or, what is the same, the value function of the deterministic control problem described at the end of Subsection 2.1.

In the following proposition we consider the difference between the value function (3) when σ is replaced by $\sqrt{\epsilon}\sigma$ and when σ is replaced by $\sqrt{\epsilon'}\sigma$, where $\epsilon, \epsilon' > 0$. For the treatment of this consideration for a class of elliptic equations see [9].

Proposition 3 *With V^δ denoting the value function (3) when σ is replaced by $\sqrt{\delta}\sigma$, where $\delta > 0$, it holds that*

$$\sup_{(x,t)\in\mathbb{R}\times[0,T]} |V^\epsilon(x,t) - V^{\epsilon'}(x,t)| \leq K|\sqrt{\epsilon} - \sqrt{\epsilon'}| \ ,$$

for some finite constant K.

Proof: Denote X^δ the solution of (1) when σ is replace by $\sqrt{\delta}\sigma$. For each fixed control $(\theta_i, \xi_i)_{i\geq 1}$, we have, by our assumptions on f and h, for some constant K,

$$\Big| E_{x,t}\Big[\int_t^T f(X^\epsilon(s),s)ds + \sum_{i\geq 1} c(\xi_i) + h(X^\epsilon(T))\Big]$$

$$- E_{x,t}\Big[\int_t^T f(X^{\epsilon'}(s),s)ds + \sum_{i\geq 1} c(\xi_i) + h(X^{\epsilon'}(T))\Big]\Big|$$

$$
\begin{aligned}
&\leq \sqrt{\int_t^T (E_{x,t}[f(X^\epsilon(s),s) - f(X^{\epsilon'}(s),s)])^2 ds} + \sqrt{E_{x,t}[h(X^\epsilon(T)) - h(X^{\epsilon'}(T))]^2} \\
&\leq K\sqrt{\int_t^T E_{x,t}(X^\epsilon(s) - X^{\epsilon'}(s))^2 ds} + K\sqrt{E_{x,t}(X^\epsilon(T) - X^{\epsilon'}(T))^2 ds}\,. \qquad (22)
\end{aligned}
$$

By Itô's formula we have that

$$
\begin{aligned}
(X^\epsilon(s) - X^{\epsilon'}(s))^2 &= \int_0^t 2(X^\epsilon(s) - X^{\epsilon'}(s))(g(X^\epsilon(s)) - g(X^{\epsilon'}(s)))\, ds \\
&+ \int_0^t 2(X^\epsilon(s) - X^{\epsilon'}(s))(\sqrt{\epsilon}\,\sigma(X^\epsilon(s)) - \sqrt{\epsilon'}\,\sigma(X^{\epsilon'}(s)))\, dB_s \\
&+ \frac{1}{2}\int_0^t 2(\sqrt{\epsilon}\sigma(X^\epsilon(s)) - \sqrt{\epsilon'}\sigma(X^{\epsilon'}(s)))^2\, ds\,.
\end{aligned}
$$

Taking expectations and using our assumptions on g and σ yields

$$
\begin{gathered}
E(X^\epsilon(s) - X^{\epsilon'}(s))^2 \\
\leq \int_0^t 2(X^\epsilon(s) - X^{\epsilon'}(s))^2\, ds + \int_0^t (\sqrt{\epsilon}\sigma(X^\epsilon(s)) - \sqrt{\epsilon'}\sigma(X^{\epsilon'}(s)))^2\, ds \\
= \int_0^t 2(X^\epsilon(s) - X^{\epsilon'}(s))^2\, ds \\
+ \int_0^t (\sqrt{\epsilon}\sigma(X^\epsilon(s)) - \sqrt{\epsilon'}\sigma(X^{\epsilon'}(s)) + \sqrt{\epsilon'}\sigma(X^\epsilon(s)) - \sqrt{\epsilon'}\sigma(X^\epsilon(s)))^2\, ds \\
\leq \int_0^t 2(X^\epsilon(s) - X^{\epsilon'}(s))^2\, ds \\
+ \int_0^t 2(\sqrt{\epsilon'}\sigma(X^\epsilon(s)) - \sqrt{\epsilon'}\sigma(X^{\epsilon'}(s)))^2\, ds + \int_0^t 2((\sqrt{\epsilon} - \sqrt{\epsilon'})\sigma(X^{\epsilon'}(s)))^2\, ds
\end{gathered}
$$

which is, for $\epsilon' \leq 1$ and M a bound on σ,

$$
\leq \int_0^t 4(X^\epsilon(s) - X^{\epsilon'}(s))^2\, ds + (\sqrt{\epsilon} - \sqrt{\epsilon'})^2 2M^2 T\,.
$$

An application of Gronwall's inequality yields, for some constant K,

$$
E_{x,t}(X^\epsilon(s) - X^{\epsilon'}(s))^2 \leq K(\sqrt{\epsilon} - \sqrt{\epsilon'})^2\,, \quad \forall\, s \in [t,T]\,. \qquad (23)
$$

The desired result follows upon combining (22) and (23). ■

3 The risk-sensitive setting

In this section we consider the convergence of the value function associated with a problem of impulse control, in which the value function is of risk-sensitive type, when the variance of the underlying diffusion process tends to zero. We shall demonstrate that a logarithmic transformation of the value function tends to a unique limit point, given by the value function of an appropriate zero-sum differential game involving impulse controls, recently considered in [17]. Throughout this section we restrict our attention to the autonomous case; i.e. the case in which both g and σ no longer possess a second argument.

In the setting of risk-sensitive impulse control the value function is given by

$$V(x,t) = \inf_{(\theta_i,\xi_i)_{i\geq 1}} E_{x,t}\Big[\exp\Big(\Big(\int_t^T f(X(s),s)ds + \sum_{i\geq 1} c(\xi_i) + h(X(T))\Big)/\epsilon\Big)\Big] . \qquad (24)$$

As before the infimum is taken over all admissible impulse controls. For a given impulse control the process X satisfies the equation (1) with f, g, h, c and σ satisfying (2), (4) and (5). The quantity $\epsilon > 0$ is a parameter.

In the first subsection we consider the case of smooth solutions of the relevant quasi-variational inequalities, while the case of viscosity solutions is considered in the next subsection.

3.1 Smooth solutions

As in the previous section, in case an appropriate qvi possesses a sufficiently smooth solution, the solution coincides with the value function and allows a means of representing an optimal control.

Theorem 4 *In case the following qvi possesses a solution $V \in C^{2,1}[\mathbb{R} \times [0,T]]$, the solution coincides with the value function:*

$$\frac{\partial V}{\partial t} + a(x)\frac{\partial^2 V}{\partial x^2} + g(x)\frac{\partial V}{\partial x} + f(x,t)\frac{1}{\epsilon}V \;\geq\; 0 \qquad (25)$$

$$\inf_{\zeta\in\mathbb{R}}[\exp(c(\zeta)/\epsilon)V(x+\zeta,t)] - V(x,t) \;\geq\; 0 \qquad (26)$$

$$\Big(\frac{\partial V}{\partial t} + a(x)\frac{\partial^2 V}{\partial x^2} + g(x)\frac{\partial V}{\partial x} + f(x,t)\frac{1}{\epsilon}V\Big)\cdot \Big(\inf_{\zeta\in\mathbb{R}}[\exp(c(\zeta)/\epsilon)V(x+\zeta,t)] - V(x,t)\Big) \;=\; 0$$

$$V(x,T) = \inf_{\xi}\{\exp((c(\xi)+h(x+\xi))/\epsilon)\} ,$$

where $a(x) = \sigma^2(x)/2$. Furthermore, with $(\theta_0,\xi_0) = (0,0)$ define, recursively,

$$\theta_i = \inf\Big\{t \in [\theta_{i-1},T) : V(X(t-),t) = \inf_{\zeta\in\mathbb{R}}[\exp(c(\zeta)/\epsilon)V(X(t-)+\zeta,t)]\Big\} \qquad (27)$$

$$\xi_i = \arg\min_{\xi}\{\exp(c(\zeta)/\epsilon)V(X(\theta_i-)+\zeta,\theta_i)\} , \; i = 1,2,\ldots$$

It follows that, almost surely, the sequence $(\theta_i,\xi_i)_{i\geq 1}$ is finite in length and that an optimal control is provided by this sequence together with a single jump at time T of size $\xi = \arg\min_\xi\{\exp(c(\xi)/\epsilon)V(X(T-)+\xi,T)\}$.

We note from the proof that it must be that $V > 0$. From this, our assumptions on c and [17, Lemma 4.4] we may replace the intervals $[\theta_{i-1}, T)$ by the open intervals (θ_{i-1}, T), $i = 2, 3, \ldots$.

Proof: Let $(\theta_i, \xi_i)_{i\geq 1}$ be a policy. By Itô's formula we have, for $i \geq 1$, with $f^\epsilon = f/\epsilon$,

$$\begin{aligned} V(X(\theta_i-), \theta_i-) e^{\int_{\theta_{i-1}}^{\theta_i} f^\epsilon(X(s),s)ds} &= V(X(\theta_{i-1}), \theta_{i-1}) \\ + \int_{\theta_{i-1}}^{\theta_i} \Big[e^{\int_{\theta_{i-1}}^{s} f^\epsilon(X(u),u)du} \Big(\frac{\partial V}{\partial t} + a(X(s)) \frac{\partial^2 V}{\partial x^2} &+ g(X(s)) \frac{\partial V}{\partial x} + f^\epsilon(X(s), s) V \Big) \Big] ds \\ + \int_{\theta_{i-1}}^{\theta_i} \Big[e^{\int_{\theta_{i-1}}^{s} f^\epsilon(X(u),u)du} \sigma(X(s)) \frac{\partial V}{\partial x} \Big] dB_s \, . \end{aligned} \tag{28}$$

This implies, using (25), that

$$\begin{aligned} & V(X(\theta_{i-1}), \theta_{i-1}) \\ \leq\; & V(X(\theta_i-), \theta_i-) e^{\int_{\theta_{i-1}}^{\theta_i} f^\epsilon(X(s),s)ds} - \int_{\theta_{i-1}}^{\theta_i} \Big[e^{\int_{\theta_{i-1}}^{s} f^\epsilon(X(u),u)du} \sigma(X(s)) \frac{\partial V}{\partial x} \Big] dB_s \, . \end{aligned} \tag{29}$$

Furthermore, by (26), we have

$$V(X(\theta_i-), \theta_i) \leq V(X(\theta_i-), \theta_i) e^{c(\xi_i)/\epsilon} \, . \tag{30}$$

Hence, using (29) and (30), we may write that

$$\begin{aligned} V(x,t) &\leq E_{x,t}\Big[1_{\{\theta_1 > 0\}} V(X(\theta_1-), \theta_1-) e^{\int_0^{\theta_1} f^\epsilon(X(s),s)ds} \Big] \\ &\quad + E_{x,t}\Big[1_{\{\theta_1 = 0\}} V(X(\theta_1), \theta_1) e^{c(\xi_1)/\epsilon} \Big] \\ &\leq E_{x,t}\Big[1_{\{\theta_1 > 0\}} V(X(\theta_1), \theta_1) e^{(\int_0^{\theta_1} f(X(s),s)ds + c(\xi_1))/\epsilon} \Big] \\ &\quad + E_{x,t}\Big[1_{\{\theta_1 = 0\}} V(X(\theta_1), \theta_1) e^{c(\xi_1)/\epsilon} \Big] \\ &= E_{x,t}\Big[V(X(\theta_1), \theta_1) e^{(\int_0^{\theta_1} f(X(s),s)ds + c(\xi_1))/\epsilon} \Big] \, . \end{aligned} \tag{31}$$

Continuing to apply (29) and (30) and making use of the dominated convergence theorem and the fact that $V(x,T) = \inf_\xi \{\exp((c(\xi) + h(x+\xi))/\epsilon)\}$ we conclude that

$$V(x,t) \leq E_{x,t}\Big[e^{(\int_0^T f(X(s),s)ds + \sum_{i\geq 1} c(\xi_i) + h(X(T)))/\epsilon} \Big] \, .$$

From this we see that the solution of the qvi provides a lower bound to the value function (24).

In case we had chosen the control as indicated in the second part of the statement of the theorem we would have obtained equalities in the derivation up to (31). Continuing to apply (29) and (30) would then allow to conclude that the sequence $(\theta_i, \xi_i)_{i\geq 1}$ is almost surely finite in length, an application of the dominated convergence theorem then yielding the desired conclusion. ∎

Theorem 5 *For each $\epsilon > 0$ fixed, the function $\epsilon \log V(x,t)$ is uniformly continuous in $\mathbb{R} \times [0,T]$.*

Proof: Setting $W(x,t) = \epsilon \log V(x,t)$, we first show that

$$|W(x,t) - W(y,t)| \leq \omega(x-y) \ , \tag{32}$$

where $\lim_{x-y\to 0} \omega(x-y) = 0$. To see this, for any control $(\theta_i, \xi_i)_{i\geq 1}$ let X^x and X^y denote solutions to (1) with $X^x(t) = x$ and $X^y(t) = y$. For $\delta > 0$ fixed, choose a control such that

$$E_{x,t}\Big[\exp\Big(\Big(\int_t^T f(X(s),s)ds + \sum_{i\geq 1} c(\xi_i) + h(X(T))\Big)/\epsilon\Big)\Big] \leq V(x,t) + \delta \ .$$

Fixing $M > 0$, consider the control obtained by restricting this control so that no jumps occur on the set of sample paths $A = \{\sup_{t\in[0,\theta_1]} |X^x(t) - X^y(t)| \geq M|x-y|\}$. For this control, we have

$$\begin{aligned}
& E_{x,t}\Big[\exp\Big(\Big(\int_t^T f(X(s),s)ds + \sum_{i\geq 1} c(\xi_i) + h(X(T))\Big)/\epsilon\Big)1_{A^c}\Big] \\
& \quad + E_{x,t}\Big[\exp\Big(\Big(\int_t^T f(X(s),s)ds + h(X(T))\Big)/\epsilon\Big)1_A\Big] \\
\leq \ & V(x,t) + \delta + E_{x,t}\Big[\exp\Big(\Big(\int_t^T f(X(s),s)ds + h(X(T))\Big)/\epsilon\Big)1_A\Big] \\
\leq \ & V(x,t) + \delta + e^{\bar{f}h/\epsilon} P(A) \\
\leq \ & V(x,t) + \delta + e^{\bar{f}h/\epsilon} e^{L(M)/\epsilon},
\end{aligned}$$

where $\bar{f}h = T \sup_{(x,t)\in\mathbb{R}\times[0,T]} |f(x,t)| + \sup_{x\in\mathbb{R}} |h(x)|$, $L(M) = 2k + 3k^2 - \log(M)$ and the last inequality follows from Lemma 1. Hence, as δ was arbitrary, with $\tilde{V}(x,t)$ defined as equal to the right hand side of (24) with the infimum restricted to such controls, we have

$$V(x,t) \leq \tilde{V}(x,t) \leq V(x,t) + e^{(\bar{f}h + L(M))/\epsilon} \ . \tag{33}$$

Continuing, fix $\delta > 0$ and choose a control amongst the restricted controls such that

$$\tilde{V}(y,t) \geq E_{y,t}\Big[\exp\Big(\Big(\int_t^T f(X(s),s)ds + \sum_{i\geq 1} c(\xi_i) + h(X(T))\Big)/\epsilon\Big)\Big] - \delta \ . \tag{34}$$

Alter this control so that in case a jump occurs on the set A^c the quantity $(y-x)$ is added to the jump. We have, in particular for this control,

$$\begin{aligned}
\tilde{V}(x,t) \leq \ & E_{x,t}\Big[\exp\Big(\Big(\textstyle\int_t^T f(X(s),s)ds + \sum_{i\geq 1} c(\xi_i) + h(X(T))\Big)/\epsilon\Big)1_{A^c}\Big] \\
& + E_{x,t}\Big[\exp\Big(\Big(\textstyle\int_t^T f(X(s),s)ds + h(X(T))\Big)/\epsilon\Big)1_A\Big] \\
\leq \ & E_{x,t}\Big[\exp\Big(\Big(\textstyle\int_t^T f(X(s),s)ds + \sum_{i\geq 1} c(\xi_i) + h(X(T))\Big)/\epsilon\Big)1_{A^c}\Big] + e^{(\bar{f}h + L(M))/\epsilon}.
\end{aligned}$$

But, given the modification to the control, we can write

$$\begin{aligned}
& E_{x,t}\Big[\exp\Big(\Big(\int_t^T f(X(s),s)ds + \sum_{i\geq 1} c(\xi_i) + h(X(T))\Big)/\epsilon\Big)1_{A^c}\Big] \\
\leq\ & \exp(TkM|x-y|/\epsilon)E_{y,t}\Big[\exp\Big(\Big(\int_t^T f(X(s),s)ds + \sum_{i\geq 1} c(\xi_i) + h(X(T))\Big)/\epsilon\Big)1_{A^c}\Big] \\
\leq\ & \exp(TkM|x-y|/\epsilon)[\tilde{V}(y,t)+\delta] \\
\leq\ & \exp(TkM|x-y|/\epsilon)\Big[V(y,t) + e^{(\bar{f}h+L(M))/\epsilon} + \delta\Big] ,
\end{aligned}$$

where in the first inequality we use again the unaltered control and the last two inequalities follow from (34) and (33). Hence, as δ was arbitrary,

$$\tilde{V}(x,t) \leq e^{TkM|x-y|/\epsilon}V(y,t) + e^{TkM|x-y|/\epsilon}e^{(\bar{f}h+L(M))/\epsilon} + e^{(\bar{f}h+L(M))/\epsilon} ,$$

and we may write

$$\begin{aligned}
& \epsilon \log V(x,t) - \epsilon \log V(y,t) \\
\leq\ & \epsilon \log\Big[e^{TkM|x-y|/\epsilon}V(y,t) + e^{TkM|x-y|/\epsilon}e^{(\bar{f}h+L(M))/\epsilon} + e^{(\bar{f}h+L(M))/\epsilon}\Big] \\
& -\epsilon\log\Big[e^{TkM|x-y|/\epsilon}V(y,t)\Big] + \epsilon\log\Big[e^{TkM|x-y|/\epsilon}V(y,t)\Big] - \epsilon\log V(y,t) \\
\leq\ & \epsilon \log\Big[e^{TkM|x-y|/\epsilon}V(y,t) + e^{TkM|x-y|/\epsilon}e^{(\bar{f}h+L(M))/\epsilon} + e^{(\bar{f}h+L(M))/\epsilon}\Big] \\
& -\epsilon\log\Big[e^{TkM|x-y|/\epsilon}V(y,t)\Big] + TkM|x-y| .
\end{aligned}$$

Next, $V(y,t) \geq e^{-\bar{f}h/\epsilon}$ implies

$$\begin{aligned}
& \epsilon \log V(x,t) - \epsilon \log V(y,t) \\
& \leq \epsilon \log\Big[1 + \tfrac{e^{TkM|x-y|/\epsilon}e^{(\bar{f}h+L(M))/\epsilon}+e^{(\bar{f}h+L(M))/\epsilon})}{e^{TkM|x-y|/\epsilon}V(y,t)}\Big] + TkM|x-y| \\
& \leq \epsilon\log\Big[1 + (e^{TkM|x-y|/\epsilon}e^{(\bar{f}h+L(M))/\epsilon} + e^{(\bar{f}h+L(M))/\epsilon})(e^{-TkM|x-y|/\epsilon}e^{\bar{f}h/\epsilon})\Big] + TkM|x-y| \\
& \leq \epsilon(e^{TkM|x-y|/\epsilon}e^{(\bar{f}h+L(M))/\epsilon} + e^{(\bar{f}h+L(M))/\epsilon})(e^{-TkM|x-y|/\epsilon}e^{\bar{f}h/\epsilon}) + TkM|x-y| , \quad (35)
\end{aligned}$$

from which the existence of ω readily follows.

Assuming $u > t$ we now show that

$$W(x,t) \leq W(x,u) + \tilde{\omega}(u-t) ,$$

where $\lim_{u-t\to 0}\tilde{\omega}(u-t) = 0$. To see this, note that, restricting the allowable controls in the following infimum to those which have no impulses in $[t,u)$, we have, for

$K \geq \max_{x \in \mathbb{R}} |f(x)|$,

$$V(x,t) \leq \inf_{(\theta_i,\xi_i)_{i\geq 1}} E_{x,t}\Big[\exp\Big(\Big(\int_t^T f(X(s),s)ds + \sum_{i\geq 1} c(\xi_i) + h(X(T))\Big)/\epsilon\Big)\Big]$$
$$\leq \inf_{(\theta_i,\xi_i)_{i\geq 1}} E_{x,t}\Big[\exp\Big(\Big(\int_u^T f(X(s),s)ds + K(u-t) + \sum_{i\geq 1} c(\xi_i) + h(X(T))\Big)/\epsilon\Big)\Big]$$
$$= \inf_{(\theta_i,\xi_i)_{i\geq 1}} E_{x,t}\Big[E_{X(u),u}\Big[\exp\Big(\Big(\int_u^T f(X(s),s)ds + \sum_{i\geq 1} c(\xi_i) + h(X(T))\Big)/\epsilon\Big)\Big]\Big] e^{K(u-t)/\epsilon}.$$

Now,

$$\begin{aligned}
&\inf_{(\theta_i,\xi_i)_{i\geq 1}} E_{x,t}\Big[E_{X(u),u}\Big[\exp\Big(\Big(\int_u^T f(X(s),s)ds + \sum_{i\geq 1} c(\xi_i) + h(X(T))\Big)/\epsilon\Big)\Big]\Big] \\
\leq\ & V(x,u)e^{\omega(x-y)/\epsilon} + e^{\bar{f}h/\epsilon} P_{x,t}(|X_u - X_t| \geq |x-y|)\ .
\end{aligned}$$

Next, with $M \geq \max_{x\in\mathbb{R}} |g(x)|\sqrt{T}$,

$$\begin{aligned}
& P_{x,t}(|X_u - X_t| \geq |x-y|) \\
=\ & P_{x,t}\Big(\Big|\int_t^u g(X_s)ds + \sqrt{\epsilon}\int_t^u \sigma(X_s)dB_s\Big| \geq |x-y|\Big) \\
\leq\ & P_{x,t}\Big(\Big|\frac{1}{\sqrt{\epsilon(u-t)}}\int_t^u \sigma(X_s)dB_s\Big| \geq \frac{|x-y|}{\epsilon\sqrt{(u-t)}} - \frac{M}{\epsilon}\Big) \\
\leq\ & E_{x,t}\Big[\exp\Big(\Big|\frac{1}{\sqrt{\epsilon(u-t)}}\int_t^u \sigma(X_s)dB_s\Big|\Big)\Big]\exp\Big(-\frac{|x-y|}{\epsilon\sqrt{(u-t)}} + \frac{M}{\epsilon}\Big).
\end{aligned}$$

However, with $\tilde{M} \geq \max_{x\in\mathbb{R}} |\sigma(x)|$, we have

$$E_{x,t}\Big[\exp\Big(\Big|\tfrac{1}{\sqrt{\epsilon(u-t)}}\int_t^u \sigma(X_s)dB_s\Big|\Big)\Big]$$
$$\leq E_{x,t}\Big[\exp\Big(\tfrac{1}{\sqrt{\epsilon(u-t)}}\int_t^u \sigma(X_s)dB_s\Big)\Big] + E_{x,t}\Big[\exp\Big(-\tfrac{1}{\sqrt{\epsilon(u-t)}}\int_t^u \sigma(X_s)dB_s\Big)\Big]$$
$$= E_{x,t}\Big[\exp\Big(\tfrac{1}{\sqrt{\epsilon(u-t)}}\int_t^u \sigma(X_s)dB_s - \tfrac{1}{\epsilon(u-t)}\int_t^u \sigma^2(X_s)ds + \tfrac{1}{\epsilon(u-t)}\int_t^u \sigma^2(X_s)ds\Big)\Big]$$
$$+E_{x,t}\Big[\exp\Big(-\tfrac{1}{\sqrt{\epsilon(u-t)}}\int_t^u \sigma(X_s)dB_s - \tfrac{1}{\epsilon(u-t)}\int_t^u \sigma^2(X_s)ds + \tfrac{1}{\epsilon(u-t)}\int_t^u \sigma^2(X_s)ds\Big)\Big]$$
$$\leq E_{x,t}\Big[\exp\Big(\tfrac{1}{\sqrt{\epsilon(u-t)}}\int_t^u \sigma(X_s)dB_s - \tfrac{1}{\epsilon(u-t)}\int_t^u \sigma^2(X_s)ds\Big)\Big]\exp\Big(\tfrac{\tilde{M}}{\epsilon}\Big)$$
$$+E_{x,t}\Big[\exp\Big(-\tfrac{1}{\sqrt{\epsilon(u-t)}}\int_t^u \sigma(X_s)dB_s - \tfrac{1}{\epsilon(u-t)}\int_t^u \sigma^2(X_s)ds\Big)\Big]\exp\Big(\tfrac{\tilde{M}}{\epsilon}\Big)\ .$$

Using the fact that both $\Big\{\exp\Big(\tfrac{1}{\sqrt{\epsilon(u-t)}}\int_t^v \sigma(X_s)dB_s - \tfrac{1}{\epsilon(u-t)}\int_t^v \sigma^2(X_s)ds\Big),\ v \geq t\Big\}$ and $\Big\{\exp\Big(-\tfrac{1}{\sqrt{\epsilon(u-t)}}\int_t^v \sigma(X_s)dB_s - \tfrac{1}{\epsilon(u-t)}\int_t^v \sigma^2(X_s)ds\Big),\ v \geq t\Big\}$ are martingales (cf. [12], Theorem 3.5.1 and Proposition 3.5.12) yields

$$E_{x,t}\Big[\exp\Big(\Big|\frac{1}{\sqrt{\epsilon(u-t)}}\int_t^u \sigma(X_s)dB_s\Big|\Big)\Big] \leq 2\exp\Big(\frac{\tilde{M}}{\epsilon}\Big) .$$

Putting these inequalities together, we have, possibly increasing M and recalling that $V(x,u) \geq e^{-\bar{f}h/\epsilon}$,

$$\begin{aligned} V(x,t) &\leq V(x,u)e^{\frac{K(u-t)}{\epsilon}}e^{\frac{\omega(x-y)}{\epsilon}} + e^{\frac{K(u-t)}{\epsilon}}e^{\frac{\bar{f}h+M}{\epsilon}}e^{-\frac{|x-y|}{\epsilon\sqrt{(u-t)}}} \\ &\leq V(x,u)\Big[e^{\frac{\omega(x-y)}{\epsilon}} + e^{\frac{2\bar{f}h+M}{\epsilon}}e^{-\frac{|x-y|}{\epsilon\sqrt{(u-t)}}}\Big]e^{\frac{K(u-t)}{\epsilon}} \end{aligned}$$

and therefore,

$$W(x,t) \leq W(x,u) + \epsilon\log\Big[e^{\frac{\omega(x-y)}{\epsilon}} + e^{\frac{2\bar{f}h+M}{\epsilon}}e^{-\frac{|x-y|}{\epsilon\sqrt{(u-t)}}}\Big] + K(u-t) .$$

From this we readily see the existence of $\tilde{\omega}(\cdot)$.

We now show, still with $u > t$,

$$W(x,u) \leq W(x,t) + \kappa(u-t) ,$$

where $\lim_{u-t\to 0}\kappa(u-t) = 0$. To see this, fix $\delta > 0$ and let $(\theta_i,\xi_i)_{\imath\geq 1}$ be such that

$$W(x,t) \geq \epsilon\log E_{x,t}\Big[\exp\Big(\Big(\int_t^T f(X(s),s)ds + \sum_{i\geq 1}c(\xi_i) + h(X(T))\Big)/\epsilon\Big)\Big] - \delta .$$

This last quantity is, for $K \geq \max_{x\in\mathbb{R}}|f(x)|$, due to the assumption that $c > 0$,

$$\geq \epsilon\log E_{x,t}\Big[\exp\Big(\Big(\int_t^{T-(u-t)} f(X(s),s)ds + \sum_{i\geq 1}c(\xi_i) + h(X(T))\Big)/\epsilon\Big)\Big] - \delta - K(u-t) ,$$

which is, in turn, increasing K if necessary

$$\begin{aligned} &\geq \epsilon\log E_{x,t}\Big[\exp\Big(\Big(\int_u^T f(X(s),s)ds + \sum_{i\geq 1}c(\xi_i) + h(X(T))\Big)/\epsilon\Big)\Big] - \delta - K(u-t) \\ &\geq W(x,u) - \delta - K(u-t) . \end{aligned}$$

As $\delta > 0$ was arbitrary, the desired result follows. ■

3.2 The viscosity case

In what follows we denote the qvi in Theorem 4 by qvi (26). As mentioned in the previous section, in general it is asking too much that $V \in C^{2,1}[\mathbb{R} \times [0,T]]$ and one abandons hope of the qvi (26) holding. In this subsection we show that V is a viscosity solution of the qvi (26) in case the following Bellman principle holds:

$$V(x,t) = \inf_{(\theta_i,\xi_i)_{i\geq 1}} E_{x,t}\Big[\exp\Big(\Big(\int_t^\tau f(X(s),s)ds + \sum_{i\geq 1} c(\xi_i)1_{\{\theta_i \leq \tau\}}\Big)/\epsilon\Big)V(X(\tau),\tau)\Big] \quad (36)$$

for all $(F_t, t \geq 0)$-stopping times τ with $t \leq \tau \leq T$. Analogous to the definition in Section 2 V is said to be a viscosity solution of the qvi (26) in case $V(x,T) = \inf_\xi\{\exp((c(\xi)+h(x+\xi))/\epsilon)\}$ when (7) is replaced by

$$\min\Big\{\frac{\partial\phi}{\partial t} + a(x)\frac{\partial^2\phi}{\partial x^2} + g(x)\frac{\partial\phi}{\partial x} + f(x,t)\frac{\phi}{\epsilon}, \inf_\xi[\exp(c(\xi)/\epsilon)V(x+\xi,t)] - V(x,t)\Big\} \geq 0 \ . \quad (37)$$

and (8) is replaced by

$$\min\Big\{\frac{\partial\phi}{\partial t} + a(x)\frac{\partial^2\phi}{\partial x^2} + g(x)\frac{\partial\phi}{\partial x} + f(x,t)\frac{\phi}{\epsilon}, \inf_\xi[\exp(c(\xi)/\epsilon)V(x+\xi,t)] - V(x,t)\Big\} \leq 0 \ . \quad (38)$$

Theorem 6 *Suppose (36) holds. The value function V is then a viscosity solution of the qvi (26).*

Remark: From the proof we note that V is a viscosity subsolution in case (36) holds for all stopping times which are constant.

Proof: That $V(x,T) = \inf_\xi\{\exp((c(\xi)+h(x+\xi))/\epsilon)\}$ is clear. We now show that V is a viscosity subsolution of the qvi (26). To do so, let $\phi \in C^{2,1}[\mathbb{R} \times [0,T]]$ with $\phi(x,t) = V(x,t)$ and $V \leq \phi$. We may assume that $t < T$. That

$$V(x,t) \leq \inf_\zeta[\exp(c(\zeta)/\epsilon)V(x+\zeta,t)]$$

is clear and it remains to show that

$$\frac{\partial\phi}{\partial t} + a(x)\frac{\partial^2\phi}{\partial x^2} + g(x)\frac{\partial\phi}{\partial x} + \frac{f(x,t)}{\epsilon}\phi \geq 0 \ . \quad (39)$$

For any admissible policy it holds by our assumption that

$$\begin{aligned}\phi(x,t) &= V(x,t) \\ &\leq E_{x,t}\Big[\exp\Big(\Big(\int_t^\tau f(X(s),s)ds + \sum_{i\geq 1} c(\xi_i)1_{\{\theta_i\leq\tau\}}\Big)/\epsilon\Big)V(X(\tau),\tau)\Big] \quad (40) \\ &\leq E_{x,t}\Big[\exp\Big(\Big(\int_t^\tau f(X(s),s)ds + \sum_{i\geq 1} c(\xi_i)1_{\{\theta_i\leq\tau\}}\Big)/\epsilon\Big)\phi(X(\tau),\tau)\Big]\end{aligned}$$

for any stopping time τ. By Itô's formula we have that

$$\phi(X(\tau),\tau) = \phi(x,t) + \int_t^\tau \left[\frac{\partial\phi}{\partial t} + a(X(s))\frac{\partial^2\phi}{\partial x^2} + g(X(s))\frac{\partial\phi}{\partial x}\right] ds \qquad (41)$$
$$+ \int_t^\tau \sigma(X(s))\frac{\partial\phi}{\partial x} dB_s + \sum_{t\le s\le\tau} \left[\phi(X(s),s) - \phi(X(s-),s)\right].$$

Considering only policies that take no action on $[t, t+h]$, setting $\tau = t+h$, and substituting (41) into (40) and taking expectations yields

$$\phi(x,t) \le E_{x,t}\Big[\exp\Big(\int_t^{t+h} f(X(s),s)ds/\epsilon\Big)\Big(\phi(x,t) + \int_t^{t+h}\Big[\frac{\partial\phi}{\partial t} + a(X(s))\frac{\partial^2\phi}{\partial x^2} \qquad (42)$$
$$+g(X(s))\frac{\partial\phi}{\partial x}\Big] ds\Big) + \exp\Big(\int_t^{t+h} f(X(s),s)ds/\epsilon\Big)\int_t^{t+h}\sigma(X(s))\frac{\partial\phi}{\partial x}\, dB_s\Big].$$

By Itô's formula we have that

$$E_{x,t}\Big[\exp\Big(\int_t^{t+h} f(X(s),s)ds/\epsilon\Big)\int_t^{t+h}\sigma(X(s))\frac{\partial\phi}{\partial x} dB_s\Big] \qquad (43)$$
$$= E_{x,t}\Big[\int_t^{t+h}\exp\Big(\int_t^s f(X(u),u)du/\epsilon\Big)\Big(\int_t^s \sigma(X(u))\frac{\partial\phi}{\partial x}dB_u\Big) f(X(s),s)/\epsilon\, ds\Big].$$

From this, we see that, upon dividing (42) by h and letting $h \searrow 0$, the term containing the stochastic integral disappears and hence we have that

$$0 \le \lim_{h\searrow 0} E_{x,t}\Big[\Big(\exp\Big(\int_t^{t+h} f(X(s),s)ds/\epsilon\Big) - 1\Big)\,\phi(x,t)$$
$$+\exp\Big(\int_t^{t+h} f(X(s),s)ds/\epsilon\Big)\Big(\int_t^{t+h}\Big[\frac{\partial\phi}{\partial t} + a(X(s))\frac{\partial^2\phi}{\partial x^2} + g(X(s))\frac{\partial\phi}{\partial x}\Big] ds\Big)\Big]/h ,$$

from which (39) follows.

To show that V is a viscosity supersolution let $\phi \in C^{2,1}[\mathbb{R}\times[0,T]]$ be such that $\phi(x,t) = V(x,t)$ and $V \ge \phi$. We proceed by contradiction. Assume therefore that

$$\inf_\zeta [\exp(c(\zeta)/\epsilon)V(x+\zeta,t)] - V(x,t) \ge \gamma \qquad (44)$$

and

$$\frac{\partial\phi}{\partial t} + a(x)\frac{\partial^2\phi}{\partial x^2} + g(x)\frac{\partial\phi}{\partial x} + f(x,t)\frac{1}{\epsilon}V \ge \gamma \qquad (45)$$

for some $\gamma > 0$. Since $\phi \in C^{2,1}[\mathbb{R}\times[0,T]]$, $\phi(x,t) = V(x,t)$ and V is continuous, decreasing γ if necessary, we have that

$$\inf_\zeta [\exp(c(\zeta)/\epsilon)V(y+\zeta,s)] - \phi(y,s) \ge \gamma\ , \quad \forall (y,s) \in B_\delta(x,t)\ , \qquad (46)$$

where $B_\delta(x,t) = \{(y,s) | d((y,s),(x,t)) \leq \delta\}$ for some $\delta > 0$ sufficiently small. Continuing to suppress the dependence of X upon the control, we denote by θ the lesser of the exit time of X from $B_\delta(x,t)$ and θ_1. From Itô's formula (we have in mind (28) here with V replaced by ϕ) and (45) we have that

$$\phi(x,t) = E_{x,t}\Big[\phi(X(\theta-),\theta-)e^{\int_t^\theta f(X(s),s)ds/\epsilon}\Big]$$

$$-E_{x,t}\Big[\int_t^\theta \Big[e^{\int_t^s f^\epsilon(X(u),u)du}\Big(\frac{\partial\phi}{\partial t} + a(X(s))\frac{\partial^2\phi}{\partial x^2} + g(X(s))\frac{\partial\phi}{\partial x} + f^\epsilon(X(s),s)\phi\Big)\Big]\, ds\Big]$$

$$\leq E_{x,t}\Big[e^{\int_t^\theta f(X(s),s)ds/\epsilon}\phi(X(\theta-),\theta-)\Big] - E_{x,t}\Big[M\gamma\int_t^\theta ds\Big] ,$$

for some $M > 0$. Recalling that $V \geq \phi$ and (46), this last quantity is seen to be

$$\leq E_{x,t}\Big[V(X(\theta),\theta)e^{(\int_t^\theta f(X(s),s)ds + c(\xi_1)1_{\{\theta_1\leq\theta\}})/\epsilon} - \gamma e^{\int_t^\theta f(X(s),s)ds/\epsilon}1_{\{\theta_1\leq\theta\}}\Big] - E_{x,t}\Big[M\gamma\int_t^\theta ds\Big] ,$$

which in turn is, possibly decreasing $M > 0$,

$$\leq E_{x,t}\Big[V(X(\theta),\theta)e^{(\int_t^\theta f(X(s),s)ds + c(\xi_1)1_{\{\theta_1\leq\theta\}})/\epsilon}\Big] - \gamma M E_{x,t}[(\theta - t)1_{\{\theta_1\geq\theta\}} + 1_{\{\theta_1\leq\theta\}}] .$$

However, the last term is strictly negative independent of which control is chosen and hence (36) is contradicted. Therefore, V is in fact a viscosity supersolution. ■

Having established that V is a viscosity solution of the qvi (26), we now consider the question of uniqueness.

Proposition 4 *The qvi (26) possesses a unique strictly positive viscosity solution.*

Proof: From the previous theorem we see that the value function is a viscosity solution of the qvi (26). That it is strictly positive is noticed by letting $\tau = T$ in (36) from which it is seen that the right hand side is bounded below by a strictly positive number. Suppose V and $\hat{V}$ are two viscosity solutions of the qvi (26) and, without loss of generality, that

$$\sup_{(x,t)\in\mathbb{R}\times[0,T]} (V - \hat{V}) > 0 .$$

Thanks to (5), applying [17, Lemma 4.4] we may conclude that there exist $x_0 \in \mathbb{R}$ and $\delta > 0$ such that

$$\sup_{\Theta(x_0)} (V - \hat{V}) > 0$$

and

$$\hat{V} < \inf_\xi[\exp(c(\xi)/\epsilon)\hat{V}(x+\xi,t)] \quad \forall \ (t,x) \in \Theta(x_0) ,$$

where

$$\Theta(x_0) = \{(x,t) \in (x_0 - \delta, x_0 + \delta) \times (T - \delta, T)\} \ .$$

Recalling (7) and (8) we now see that, in $\Theta(x_0)$, $\hat{V}$ is a viscosity solution of the partial differential equation

$$\frac{\partial V}{\partial t} + a(x)\frac{\partial^2 V}{\partial x^2} + g(x)\frac{\partial V}{\partial x} + f(x,t)\frac{1}{\epsilon}V = 0 \ ,$$

$$V(x,T) = \inf_{\xi}\{\exp((c(\xi) + h(x+\xi))/\epsilon)\} \ ,$$

while V is a viscosity subsolution of this partial differential equation with identical terminal conditions. However, it follows by [8, Theorem V.9.1], upon noting that rather than the existence of various bounded derivatives required there it suffices that Lipschitz continuity holds, that

$$\sup_{\Theta(x_0)} (V - \hat{V}) \leq 0 \ ,$$

a contradiction. ■

Throughout the remainder of this section we consider the setting in which, for each $\epsilon > 0$, not only does the value function depend upon ϵ, but we also replace σ by $\sqrt{\epsilon}\sigma$ in (1). Given the previous proposition, the following proposition is then easily proven.

Proposition 5 *Suppose (36) holds. The function $W = \epsilon \log V$ is then the unique bounded uniformly continuous viscosity solution of the qvi*

$$\begin{aligned} \frac{\partial W}{\partial t} + \epsilon a(x)\frac{\partial^2 W}{\partial x^2} + a(x)(\frac{\partial W}{\partial x})^2 + g(x)\frac{\partial W}{\partial x} + f(x,t) \ &\geq \ 0 \\ \inf_{\zeta \in \mathbb{R}}[c(\zeta) + W(x+\zeta,t)] - W(x,t) \ &\geq \ 0 \qquad (47) \\ \Big(\frac{\partial W}{\partial t} + \epsilon a(x)\frac{\partial^2 W}{\partial x^2} + a(x)(\frac{\partial W}{\partial x})^2 + g(x)\frac{\partial W}{\partial x} + f(x,t)\Big)\cdot & \\ \Big(\inf_{\zeta \in \mathbb{R}}[c(\zeta) + W(x+\zeta,t)] - W(x,t)\Big) \ &= \ 0 \\ W(x,T) = \inf_{\xi}\{c(\xi) + h(x+\xi)\} \ . & \end{aligned}$$

Proof: Suppose first that $\tilde{\phi} \in C^{2,1}[\mathbb{R} \times [0,T]]$ is such that $W \leq \tilde{\phi}$ and $W(x,t) = \tilde{\phi}(x,t)$. With $V = \exp(W/\epsilon)$ and $\phi = \exp(\tilde{\phi}/\epsilon)$, using that V is a viscosity subsolution of qvi (26), we have that (37) holds. However, as $W = \epsilon \log V$ and $\tilde{\phi} = \epsilon \log \phi$, it follows that

$$\min\left\{\frac{\partial\tilde{\phi}}{\partial t} + \epsilon a(x)\frac{\partial^2\tilde{\phi}}{\partial x^2} + a(x)(\frac{\partial\tilde{\phi}}{\partial x})^2 + g(x)\frac{\partial\tilde{\phi}}{\partial x} + f(x,t),\right. \tag{48}$$

$$\left.\inf_{\zeta\in\mathbb{R}}[c(\zeta) + W(x+\zeta,t)] - W(x,t)\right\} \geq 0\ .$$

Likewise, in case $\tilde{\phi} \in C^{2,1}[\mathbb{R}\times[0,T]]$ is such that $W \geq \tilde{\phi}$ and $W(x,t) = \tilde{\phi}(x,t)$, it follows that

$$\min\left\{\frac{\partial\tilde{\phi}}{\partial t} + \epsilon a(x)\frac{\partial^2\tilde{\phi}}{\partial x^2} + a(x)(\frac{\partial\tilde{\phi}}{\partial x})^2 + g(x)\frac{\partial\tilde{\phi}}{\partial x} + f(x,t),\right.$$

$$\left.\inf_{\zeta\in\mathbb{R}}[c(\zeta) + W(x+\zeta,t)] - W(x,t)\right\} \leq 0\ .$$

From this it follows that W is a viscosity solution of qvi (47).

To see the uniqueness, suppose W_1 and W_2 are bounded viscosity solutions of (47). Then, with $V_i = \exp(W_i/\epsilon)$, $i = 1, 2$, we have that V_1 and V_2 satisfy (37) and (38) for all $\phi \in C^{2,1}[\mathbb{R}\times[0,T]]$ of the form $\phi = \exp(\tilde{\phi}/\epsilon)$ for some $\tilde{\phi} \in C^{2,1}[\mathbb{R}\times[0,T]]$. However, as the partial differential operator in the definition of a viscosity solution is a local one, this suffices to show that V_1 and V_2 are both bounded, uniformly continuous viscosity solutions of qvi (26), a contradiction. ■

Superscripting W by ϵ to denote the dependence upon ϵ, Proposition 5 shows that W^ϵ is the unique bounded uniformly continuous viscosity solution of (47) while Theorem 5 shows that $\{W^\epsilon\}_{\epsilon>0}$ is an equicontinuous family of functions. By the Arzelà-Ascoli theorem it follows that any sequence $\{W^{\epsilon_i}\}_{i\geq 1}$, with $\epsilon_i \downarrow 0$, has a convergent subsequence, where convergence is in the sense of uniform convergence on compact sets. The following proposition characterizes the possible limit points (cf. also [8, pgs. 73-74 and 288]).

Proposition 6 *Suppose the sequence* $\{W^{\epsilon_i}\}_{i\geq 1}$*, with* $\epsilon_i \downarrow 0$*, is such that* W^{ϵ_i} *converges uniformly on compact sets to some function* W*. The function* W *is then a viscosity solution of*

$$\begin{aligned} \frac{\partial W}{\partial t} + a(x)(\frac{\partial W}{\partial x})^2 + g(x)\frac{\partial W}{\partial x} + f(x,t) \ &\geq\ 0 \\ \inf_{\zeta\in\mathbb{R}}[c(\zeta) + W(x+\zeta,t)] - W(x,t) \ &\geq\ 0 \\ \left(\frac{\partial W}{\partial t} + a(x)(\frac{\partial W}{\partial x})^2 + g(x)\frac{\partial W}{\partial x} + f(x,t)\right)\cdot\left(\inf_{\zeta\in\mathbb{R}}[c(\zeta) + W(x+\zeta,t)] - W(x,t)\right) \ &=\ 0 \\ W(x,T) = \inf_{\xi}\{c(\xi) + h(x+\xi)\}\ . \end{aligned} \tag{49}$$

Proof: We show that W is a subsolution, the proof that W is a supersolution being similar. Denote W^{ϵ_n} by W^n and suppose (x,t) is a strict local maximum of $\phi - W$, where $\phi \in C^{2,1}[\mathbb{R} \times [0,T]$. Then there exists points (x_n,t_n) with $(x_n,t_n) \to (x,t)$ such that $\phi - W^n$ has a local maximum at (x_n,t_n). Therefore, W^n satisfies (48) at (x_n,t_n) with $\tilde{\phi}^n(x,t) = \phi(x,t) + W^n(x_n,t_n) - \phi(x_n,t_n)$, $\epsilon = \epsilon^n$ and $W = W^n$. In case

$$\inf_\xi [c(\xi) + W^n(x_n+\xi,t_n)] - W^n(x_n,t_n) \ = \ 0$$

for infinitely many n it holds, under our assumptions, that

$$\inf_\xi [c(\xi) + W(x+\xi,t)] - W(x,t) \ = \ 0 \, .$$

If, on the other hand,

$$\frac{\partial \tilde{\phi}^n}{\partial t} + \epsilon_n a(x_n) \frac{\partial^2 \tilde{\phi}^n}{\partial x^2} + a(x_n)(\frac{\partial \tilde{\phi}^n}{\partial x})^2 + g(x_n)\frac{\partial \tilde{\phi}^n}{\partial x} + f(x_n,t_n) \ = \ 0$$

for infinitely many n it holds, under our assumptions, that

$$\frac{\partial \tilde{\phi}}{\partial t} + a(x)(\frac{\partial \tilde{\phi}}{\partial x})^2 + g(x)\frac{\partial \tilde{\phi}}{\partial x} + f(x,t) \ = \ 0$$

Hence, W is a subsolution of (49). ∎

Consider now the (zero-sum) deterministic game driven by

$$y(s) = x + \int_t^s \alpha(y(u),\eta(u))du + \xi(s) \ , \quad s \in [t,T] \ ,$$

and having lower value function

$$J(x,t) = h(y(T)) + \inf_{\xi^\cdot} \sup_{\eta \in U^K} [\int_t^T \beta(s,y(s),\eta(s))ds + \sum_{i \geq 1} 1_{\{t \geq t_i^\eta\}} c(\xi_i^\eta)] \ ,$$

where the dependence of y upon η and ξ^η is suppressed. Here α and β are measurable functions and U^K consists of all measurable functions η such that $|\eta| \leq K$ for some $0 < K \leq \infty$. The infimum is taken over all functions $\xi^\cdot$ mapping U^K to piecewise constant functions, and for each measurable function $\eta \in U^K$ the points $\{t_i^\eta\}_{i \geq 1}$ and $\{\xi_i^\eta\}_{i \geq 1}$ are chosen so that the representation $\xi^\eta(u) = \sum_{i \geq 1} 1_{\{u \geq t_i^\eta\}} \xi_i^\eta$, $u \in [t,T]$, holds. η and ξ are the controls taken by two different players. The control η is bounded and measurable, and the control ξ is a piecewise constant function (impulse control). In the game, player η wants to maximize the payoff by choosing a proper control η, whereas player ξ would like to minimize the resulting payoff by choosing a suitable impulse control.

Before stating the next theorem we note that by equation (35) any limit point of the family $\{W^\epsilon\}_{\epsilon > 0}$ is locally Lipschitz continuous and furthermore that we may choose a Lipschitz constant, $K > 0$ say, which is independent of the limit point chosen.

Theorem 7 *Let* $\alpha(x,\nu) := \sigma(x)\nu$ *and* $\beta(t,x,\nu) := -\nu^2/2 + g(x)\nu/\sigma(x) + f(x,t) - g^2(x)/2\sigma^2(x)$ *and assume that*

1) β *is continuous and bounded on* $[0,T] \times \mathbb{R} \times [-K,K]$.

2) A continuous function ω *with* $\omega(0) = 0$ *exists with*

$$|\beta(t,x,\nu) - \beta(x,y,\nu)| \le \omega(|x-y| + |t-s|) \ .$$

Then, for any sequence $\{W^{\epsilon_i}\}_{i\ge 1}$, *with* $\epsilon_i \downarrow 0$, *the functions* W^{ϵ_i} *converge uniformly on compact sets to the unique viscosity solution of (49),* W *say, and furthermore* $J = W$.

Proof: Under our assumptions it follows from Theorem 3.4 of [17] that J is the unique viscosity solution of the system

$$\frac{\partial J}{\partial t} + \sup_{\nu\in[-K,K]} \left\{ \frac{\partial J}{\partial x}\alpha(x,\nu) + \beta(t,x,\nu) \right\} \ \ge \ 0$$

$$\inf_{\zeta\in\mathbb{R}} [c(\zeta) + J(x+\zeta,t)] - J(x,t) \ \ge \ 0 \tag{50}$$

$$\left(\frac{\partial J}{\partial t} + \sup_{\nu\in[-K,K]} \left\{ \frac{\partial J}{\partial x}\alpha(x,\nu) + \beta(t,x,\nu) \right\} \right) \left(\inf_{\zeta\in\mathbb{R}} [c(\zeta) + J(x+\zeta,t)] - J(x,t) \right) \ = \ 0 \ .$$

We now show that any viscosity subsolution of (49), W say, is a viscosity subsolution of (50); the supersolution case is similar and is omitted. Letting ϕ be such that $\phi(x,t) = W(x,t)$ and $W \le \phi$, we have, by the Lipschitz continuity of W, that

$$-K|h| \le W(x+h,t) - W(x,t) \le \phi(x+h,t) - \phi(x,t), \ \ \forall (t,x,h) \in [0,T] \times \mathbb{R} \times \mathbb{R} \ .$$

From this it follows that $|\partial\phi/\partial x| \le K$. Solving the simple optimization problem $\sup_{\nu\in[-K,K]} \{ \frac{\partial\phi}{\partial x}\alpha(x,\nu) + \beta(t,x,\nu) \}$ allows to write

$$\frac{\partial\phi}{\partial t} + a(x)(\frac{\partial\phi}{\partial x})^2 + g(x)\frac{\partial\phi}{\partial x} + f(x,t) =$$

$$\frac{\partial\phi}{\partial t} + \sup_{\nu\in[-K,K]} \left\{ \frac{\partial\phi}{\partial x}\alpha(x,\nu) + \beta(t,x,\nu) \right\} .$$

From this it follows that the viscosity subsolutions of (49) are also viscosity subsolutions of (50) and, as can be shown similarly, the viscosity supersolutions of (49) are viscosity supersolutions of (50). Due to the uniqueness of the viscosity solution of (50) we have the desired result. ∎

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