# LARGE DEVIATIONS FOR RANDOM MATRIX ENSEMBLES IN MESOSCOPIC PHYSICS

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Abstract. In his seminal 1962 paper on the "threefold way", Freeman Dyson classified the spaces of matrices that support the random matrix ensembles deemed relevant from the point of view of classical quantum mechanics. Recently, Heinzner, Huckleberry and Zirnbauer have obtained a similar classification based on less restrictive assumptions, thus taking care of the needs of modern mesoscopic physics. Their list is in one-to-one correspondence with the infinite families of Riemannian symmetric spaces as classified by Cartan. The present paper develops the corresponding random matrix theories, with a special emphasis on large deviation principles.

In the wake of the discovery of its connections with fields as diverse as combinatorics, analytic number theory, and free probability, random matrix theory has experienced a tremendous upsurge of interest during the last decade. Its classical period, however, began in the late 1950s and early 1960s, when physicists like Wigner and Dyson proposed to model the discrete part of the spectrum of the Hamiltonian of a complicated quantum system by the spectrum of a suitable random matrix ensemble. To be a good model, this ensemble had to share certain symmetries with the quantum system. In his famous paper [8], Dyson adopted a set of symmetry assumptions which was motivated by the framework of classical quantum mechanics, and classified those spaces of matrices which are compatible with the given symmetries. He ended up with the "threefold way" of hermitian matrices with real, complex, and quaternion entries, i.e., precisely with those spaces on which the familiar Gaussian orthogonal, unitary, and symplectic ensembles (GOE, GUE, GSE) of classical random matrix theory are supported. In recent years, however, it has emerged that Dyson's symmetry assumptions are too restrictive for the purposes of quantum chromodynamics or condensed matter physics. For instance, chiral symmetries are not taken into account, and still other symmetries arise in the Bogoliubov-de Gennes (BdG) mean field approximation to superconductor systems. In their recent paper [14], P. Heinzner, A. Huckleberry and M. Zirnbauer study a set of symmetry conditions which is flexible enough to capture the chiral random matrix ensembles, introduced by J. Verbaarschoot [23], and the BdG ensembles discovered by A. Altland and M. Zirnbauer [1]. It turns out that for each choice of symmetries imposed on a system, there exists a classical Riemannian symmetric space (RSS) such that each matrix model for a Hamiltonian subject to these symmetries belongs to the infinitesimal version of this RSS. Conversely, all classical RSS arise in this way.

Like Dyson's "threefold way", the "tenfold way" of [14] is established in geometrical terms, without reference to probability measures on the matrix spaces in question. It is the object of

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the present paper to provide a mathematical treatment of the corresponding random matrix theories. Besides being based on the systematic framework of [14], the present article differs from the existing literature on random matrix ensembles associated to symmetric spaces (see [6], [3]) in its focus on Large Deviations Principles. Thus, before going to business, let us review those aspects of the classical theory of the Wigner-Dyson ensembles which will be subsumed in the present analysis as instances of symmetry classes A, AI, AII. For convenience, we only mention the results in the GOE (AI) case. If $X = (X_{ij})_{1\le i,j\le n}$ is a symmetric matrix of real-valued centred Gaussian random variables such that

- $(X_{ij})_{1\le i\le j\le n}$ are independent,
- $\mathbb{E}(X_{ij}^2) = \frac{1}{2n}$ $(i \neq j)$, $\mathbb{E}(X_{ii}^2) = \frac{1}{n}$,

then its distribution is a probability measure on the space of symmetric $n\times n$ matrices, which is invariant under conjugation by matrices from the orthogonal group $\mathrm{O}_n$. The eigenvalues of $X$, $\lambda_1, \dots, \lambda_n$, say, are real valued random variables, and by orthogonal invariance, their joint distribution has a Lebesgue density $q_n$ that can be given explicitly:

$$q_n(x_1,\dots,x_n) = \frac{1}{Z_n}\prod_{1\le i<j\le n}|x_i - x_j|\exp\Big(-\frac{1}{2}n\sum_{i=1}^n x_i^2\Big) \tag{1}$$

where $Z_n$ is for normalization. As $n\to\infty$, the random measure

$$L_n := \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i}$$

tends to a nonrandom limit, namely, to Wigner's semicircle distribution $\sigma_1$, which is given by its density

$$\frac{1}{\pi}1_{\{|x|\le\sqrt{2}\}}\sqrt{2-x^2}. \tag{2}$$

In their paper [2], Ben Arous and Guionnet have carried out a finer analysis of $L_n$, and have proven that it satisfies a large deviation principle (LDP) in $\mathcal{M}_1(\mathbb{R})$ (the space of probability measures on the Borel sets of $\mathbb{R}$, endowed with the weak topology) with speed $n^2$ and good rate function

$$I(\mu) = \frac{1}{4}\int x^2\,\mu(dx) + \frac{1}{2}\int\int \log|x-y|^{-1}\,\mu(dx)\,\mu(dy) - \frac{3}{8},$$

whose unique minimizer is the semicircle distribution. Recall that a family of probability measures $(\mu_\varepsilon)_{\varepsilon>0}$ on some topological space $X$ is said to obey a Large Deviation Principle (LDP) with speed $\varepsilon^{-1}$ and good rate function $I : X \to [0,\infty]$ if

- $I$ is lower semi-continuous and has compact level sets $N_L := \{x \in X : I(x) \le L\}$, for every $L \in [0,\infty[$,
- $\liminf_{\varepsilon\to 0}\varepsilon\log\mu_\varepsilon(G) \ge -\inf_{x\in G} I(x) \quad \forall G\subseteq X$ open,
- $\limsup_{\varepsilon\to 0}\varepsilon\log\mu_\varepsilon(A) \le -\inf_{x\in A} I(x) \quad \forall A\subseteq X$ closed.

This paper is organized as follows: In Section 1 we review the symmetry classification of matrix Hamiltonians in mesoscopic physics, as given in [14], and describe the infinitesimal versions of classical symmetric spaces that turn out to be in one-to-one correspondence with these symmetry classes. Then, in Section 2, we introduce probability measures on these spaces which enjoy invariance properties that reflect those of quantum mechanical observables. The resulting random matrix ensembles are called Hamiltonian ensembles.

They can be viewed as generalizations of the Wigner-Dyson ensembles GOE, GUE, GSE. We use the geometric description of the underlying spaces to derive the induced joint eigenvalue densities in a uniform way, thus generalizing (1) above. In Section 3, we turn to the large deviations analysis of the empirical eigenvalue measure. We prove a generalization of the main result of [2], which covers not only the Hamiltonian ensembles introduced above, but also some matrix ensembles or particle systems of different origin that have been studied in recent years. In Section 4, then, we describe the Gaussian Hamiltonian ensembles in concrete terms, analogously to the construction of the GOE which was reviewed above. We make explicit what the results of Section 3 mean in these special cases.

## 1. Symmetries of Nambu space

Let $(W, b)$ be a complex vector space of dimension $2n$ ($n \in \mathbb{N}$) together with a nondegenerate symmetric bilinear form $b$. By polarization we may assume that $W = V \oplus V^*$ and that $b$ is the natural pairing of $V$ with its dual $V^*$, i.e., $b(x_1 + \varphi_1, x_2 + \varphi_2) = \varphi_1(x_2) + \varphi_2(x_1)$. Write $S := \bigwedge V^*$. To $\varphi \in V^*$ assign the wedge multiplication operator $\epsilon(\varphi) \in \mathrm{End}(S)$, and to $v \in V$ assign the contraction operator $\iota(v) \in \mathrm{End}(S)$. The $\epsilon(\varphi)$ and $\iota(v)$ satisfy the Canonical Anticommutation Relations (CAR) of creation and annihilation operators on fermionic Fock space, and if $\rho : W \to \mathrm{End}(S)$ is given by $\rho(\varphi + v) = \epsilon(\varphi) + \iota(v)$, then $(\mathrm{End}(S), \rho)$ is a Clifford algebra $\mathrm{Cliff}(W, b)$ for $(W, b)$. We regard $W$ as a subspace of the associative algebra $\mathrm{Cliff}(W, b)$, which we interpret in the usual way as a Lie algebra. One can embed $\mathfrak{so}(W, b)$ as a Lie subalgebra consisting of elements which are quadratic in the $w \in W$, and the adjoint action of $\mathfrak{so}(W, b)$ on $W$ turns out to be nothing else than the natural action of $\mathfrak{so}(W, b) \subset \mathrm{End}(W)$ on $W$ (see [12] for details). In physical terms, the embedding of $W$ into $\mathrm{Cliff}(W, b) = \mathrm{End}(S)$ suggests the interpretation of an element of $W$ as a field operator on fermionic Fock space. The dynamics of a system of field operators is given by Heisenberg's equation of motion

$$i\hbar \frac{dw}{dt} = [H, w],$$

the self-adjoint operator $H$ being the Hamiltonian of the system. Thus we have seen that if the dynamics of the system is governed by a quadratic Hamiltonian of a certain type, all relevant information is encoded in the structure $(W, b)$.

So far we have not yet made explicit that $V$, which plays the role of the space of single particle states, comes with an hermitian scalar product $\langle \cdot, \cdot \rangle$. It gives rise to a $\mathbb{C}$-antilinear bijection $C : V \to V^* : v \mapsto \langle v, \cdot \rangle$. $C$ and $\langle \cdot, \cdot \rangle$ can be extended to the entire space $W$ in such a way that $\langle w_1, w_2 \rangle = b(Cw_1, w_2)\ \forall w_1, w_2 \in W$. The triplet $(W, b, C)$ is called *Nambu space*.

Now suppose that a compact group $\mathcal{G}$ acts on $W$ by unitary or antiunitary transformations. It is inevitable to bring antiunitary transformations into play here, because time enters the formalism of quantum mechanics via $i\hbar \frac{d}{dt}$, and so time reversal is an antiunitary rather than a unitary transformation. Once $(W, b, C)$, $\mathcal{G}$ and its action on $W$ are fixed, the translation of Dyson's problem to the present context is as follows: Describe the space of all hermitian $H \in \mathfrak{so}(W, b)$ with the property that $Hg = gH$ for all $g \in \mathcal{G}$. It is convenient to call these

*H good Hamiltonians*. Denote by $\mathcal{G}_0$ the group of all elements of $\mathcal{G}$ which act by unitary transformations. The basic assumption of [14] is that $\mathcal{G}$ is generated by $\mathcal{G}_0$ together with at most two elements acting by antiunitary transformations (One may think of systems which are invariant under both time reversal and charge conjugation).

It is one of the key insights of Heinzner, Huckleberry and Zirnbauer in [14], that it is possible to reduce the problem to the case $\mathcal{G}_0 = 1$. But this comes at the price that the good Hamiltonians need not be elements of $\mathfrak{so}(W, b)$, but can belong to $\mathfrak{sp}(W, a)$ (for an alternating form $a$ on $W$) or to $\mathfrak{sl}(V)$ (diagonally embedded into $\mathrm{End}(V) \oplus \mathrm{End}(V^*)$). Write $\mathfrak{s}$ for any of these three Lie algebras. Although one is ultimately interested in hermitian operators, one first considers the skew hermitian elements of $\mathfrak{s}$, which make up a compact real form $\mathfrak{g}$ of $\mathfrak{s}$. Now let $T$ be an antilinear transformation of $W$ such that $T^2 = \pm\mathrm{id}$ and suppose that $\mathcal{G} = \langle T\rangle$. From the reduction step in [14] it emerges that $T$ may be assumed to fix $\mathfrak{g}$. If $\theta$ denotes conjugation (in $\mathrm{End}(W)$) by $T$, then $\theta$ restricts to an involutive Lie algebra automorphism of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition into the $+1$-eigenspace $\mathfrak{k}$ and the $-1$-eigenspace $\mathfrak{p}$ of $\theta$, the so-called *Cartan decomposition*. Then $i\mathfrak{p}$ consists precisely of those hermitian operators in $\mathfrak{g}$ which commute with $T$, hence it is the space of good Hamiltonians. If $G$ is a connected compact Lie group corresponding to $\mathfrak{g}$ and $K$ its closed subgroup with Lie algebra $\mathfrak{k}$, then $\mathfrak{p}$ can be thought of as an infinitesimal version of the RSS $G/K$. If $\mathcal{G} = 1$, then the space of good Hamiltonians is $i\mathfrak{g}$. Since compact Lie groups can be given the structure of an RSS (see [16], Ch. IV §6), this case fits into the overall picture.

It is the main result of [14] that the following is the complete list of spaces of good Hamiltonians that correspond to Nambu space with the kind of symmetries in question. The labels refer to Cartan's classification of the classical compact Lie algebras and their involutive automorphisms (see [16, Ch. X §2.3]), i.e., plainly, to his classification of classical symmetric spaces. Note that in order to obtain the case BDI below for the full range of parameters, and thus to obtain Cartan's full list from the symmetries of Nambu space, one has to refine the above argument in order to take care of the case $\mathcal{G} = \langle T, T_1\rangle$, $T_1$ being another antilinear transformation of $W$ with $T_1^2 = \pm\mathrm{id}$. Note that we deviate from standard practice in Lie theory in that we do not require the matrices to be trace-free, in order to recover the familiar Wigner-Dyson ensembles as classes A, AI, AII.

**Class A:**

$$i\mathfrak{g} = i\mathfrak{u}(n) = \{X \in \mathbb{C}^{n\times n} :\ X \text{ hermitian}\}$$

**Class AI:**

$$i\mathfrak{p} = \{X \in \mathbb{R}^{n\times n} :\ X \text{ symmetric}\}$$

**Class AII:**

$$i\mathfrak{p} = \left\{ \begin{pmatrix} X_1 & X_2 \\ -\overline{X_2} & \overline{X_1} \end{pmatrix} :\ \begin{matrix} X_i \in \mathbb{C}^{n\times n}, X_1 \text{ hermitian}, \\ X_2 \text{ skew symmetric} \end{matrix} \right\}$$

**Class AIII:**

$$i\mathfrak{p} = \left\{ \begin{pmatrix} 0 & X \\ \overline{X}' & 0 \end{pmatrix} :\ X \in \mathbb{C}^{s\times t} \right\}$$

**Class B:**

$$i\mathfrak{g} = i\mathfrak{so}(n) = \{X \in (i\mathbb{R})^{n\times n} :\ X \text{ skew symmetric}\} \quad (n \text{ odd})$$

**Class D:**

$$i\mathfrak{g} = i\mathfrak{so}(n) \quad (n \text{ even})$$

**Class BDI:**

$$i\mathfrak{p} = \left\{ \begin{pmatrix} 0 & X \\ -X' & 0 \end{pmatrix} : \; X \in (i\mathbb{R})^{s\times t} \right\} = \left\{ \begin{pmatrix} 0 & X \\ \overline{X'} & 0 \end{pmatrix} : \; X \in (i\mathbb{R})^{s\times t} \right\}$$

**Class DIII:**

$$i\mathfrak{p} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} : \; X_i \in (i\mathbb{R})^{n\times n} \text{ skew symmetric} \right\}$$

**Class C:**

$$i\mathfrak{g} = i\mathfrak{usp}(n) = \left\{ \begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} : \; \begin{array}{c} X_i \in (i\mathbb{R})^{n\times n}, \\ X_1 \text{ skew symmetric}, \; X_2 \text{ symmetric} \end{array} \right\}$$

**Class CI:**

$$i\mathfrak{p} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} : \; X_i \in \mathbb{R}^{n\times n} \text{ symmetric} \right\}$$

**Class CII:**

$$i\mathfrak{p} = \left\{ \begin{pmatrix} 0 & X_1 & 0 & X_2 \\ \overline{X_1}' & 0 & X_2' & 0 \\ 0 & \overline{X_2} & 0 & -\overline{X_1} \\ \overline{X_2}' & 0 & -X_1' & 0 \end{pmatrix} : \; X_i \in \mathbb{C}^{s\times t} \right\}$$

Remark 1.1. By analogy to Dyson's *threefold way*, this list is sometimes referred to as the *tenfold way*, even if it consists of 11 items. In fact, one counts classes B and D (i.e. the Lie algebras of the special orthogonal group for odd and even matrix size) only as one single item, which is reasonable for our purposes since we are going to study the eigenvalues of random elements of $\mathfrak{so}(n)$ as $n \to \infty$.

Remark 1.2. In random matrix theory, the space that supports the GSE($n$) ensemble is usually defined (e.g. in [11, Sec. 1.4]) as the space of hermitian $n\times n$ matrices with quaternion elements, i.e., an element of $\mathbb{C}^{2n\times 2n}$ that is hermitian and whose elements are grouped into $2\times 2$ blocks of the form $\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$ $(z, w \in \mathbb{C})$. A change of basis induced by the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2j-1 & 2j & \dots & 2n \\ 1 & n+1 & 2 & n+2 & \dots & j & n+j & \dots & 2n \end{pmatrix}$$

transforms this space into $i\mathfrak{p}$ of class AII. In case CII ($n = s+t$) change the basis via the permutation which maps $n+j$ $(1 \le j \le s)$ to $2j$, $j$ $(1 \le j \le s)$ to $2j-1$, $n+s+j$ $(1 \le j \le t)$ to $2s+2j$, and $s+j$ $(1 \le j \le t)$ to $2s+2j-1$. This yields a block matrix of the form

$$\begin{pmatrix} 0 & iX \\ \overline{iX}' & 0 \end{pmatrix},$$

where $X$ has quaternion entries. This takes CII to the form

$$\left\{ \begin{pmatrix} 0 & X \\ \overline{X}' & 0 \end{pmatrix} : \; X \in (i\mathbb{H})^{s\times t} \right\},$$

thus exhibiting the analogy with BDI (real case) and AIII (complex case). BDI, AIII and CII underlie what is known in physics as *chiral* (orthogonal, unitary, symplectic) *ensembles*, see [23]. B, D, DIII, C, CI have their physical rationale in the Bogolioubov-de Gennes mean field approximation to the superconductor Hamiltonian and will be called *superconductor* or *BdG* classes, see [1]. A, AI, AII, which underlie the much-studied GUE, GOE, GSE ensembles, will be referred to as *Wigner-Dyson classes*.

## 2. Hamiltonian ensembles

In this section we randomize the good Hamiltonians, i.e., we put probability measures on $\tilde{\mathfrak{g}} := i\mathfrak{g}$ resp. $\tilde{\mathfrak{p}} := i\mathfrak{p}$. Let $G$ be a connected compact Lie group with Lie algebra $\mathfrak{g}$, $K$ its closed subgroup with Lie algebra $\mathfrak{k}$. The adjoint representations $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$ and $\mathrm{Ad}_K : K \to \mathrm{GL}(\mathfrak{p})$ are given by conjugation of matrices. If $\mathfrak{v}$ is a (nonempty open subset of a) finite dimensional Euclidian vector space, write $\mathrm{m}_{\mathfrak{v}}$ for Lebesgue measure on $\mathfrak{v}$.
Now we study probability measures on $\tilde{\mathfrak{g}}$ resp. $\tilde{\mathfrak{p}}$, restricting our attention to those which are absolutely continuous w.r.t. $\mathrm{m}_{\tilde{\mathfrak{g}}}$ resp. $\mathrm{m}_{\tilde{\mathfrak{p}}}$. Since quantum mechanical observables are invariant under unitary similarity transformations, it is natural to assume that the measures are invariant under conjugation with unitary matrices. So we only consider $\mathrm{m}_{\tilde{\mathfrak{g}}}$-densities that are constant on the $\mathrm{Ad}_G$-orbits in $\tilde{\mathfrak{g}}$ resp. $\mathrm{m}_{\tilde{\mathfrak{p}}}$-densities that are constant on the $\mathrm{Ad}_K$-orbits in $\tilde{\mathfrak{p}}$.
Now we wish to compute the joint densities of the eigenvalues of a matrix $X \in \tilde{\mathfrak{g}}$ resp. $X \in \tilde{\mathfrak{p}}$ chosen according to such a measure. These are easy consequences of the infinitesimal version of Weyl's integration formula for $\mathfrak{g}$ and its analog, due to Harish-Chandra, for $\mathfrak{p}$. Here we use standard terminology of elementary Lie theory, see e.g. [20], [16], [7]. To state the formula for $\mathfrak{g}$, let $T$ be a maximal torus of $G$ with Lie algebra $\mathfrak{t}$, $W = \mathrm{N}_G(T)/T$ the corresponding Weyl group, $R^+$ resp. $\mathfrak{t}^+ \subset \mathfrak{t}$ compatible choices of a system of positive roots resp. of a positive Weyl chamber.

Proposition **2.1.** *There exists $c > 0$ such that for all $f \in \mathrm{L}^1(\mathfrak{g}, \mathrm{m}_{\mathfrak{g}})$ which are constant on $\mathrm{Ad}_G$-orbits there holds*

$$\begin{aligned} \int_{\mathfrak{g}} f \; d\mathrm{m}_{\mathfrak{g}} &= c \int_{\mathfrak{t}^+} f(T) \prod_{\alpha \in R^+} \alpha(T)^2 \; \mathrm{m}_{\mathfrak{t}^+}(dT) \\ &= \frac{c}{\#W} \int_{\mathfrak{t}} f(T) \prod_{\alpha \in R^+} |\alpha(T)|^2 \; \mathrm{m}_{\mathfrak{t}}(dT). \end{aligned}$$

*Proof.* [7] Cor. 3.14.2 (ii) □

To state the theorem for $\mathfrak{p}$, write $\mathfrak{a}$ for a maximal abelian subspace of $\mathfrak{p}$, $W = \mathrm{N}_K(\mathfrak{a})/\mathrm{C}_K(\mathfrak{a})$ for the Weyl group of $\mathfrak{g}$ w.r.t. $\mathfrak{a}$, $\Sigma^+$ resp. $\mathfrak{a}^+ \subset \mathfrak{a}$ for compatible choices of positive restricted roots (of $\mathfrak{g}$ w.r.t. $\mathfrak{a}$) resp. of a positive Weyl chamber. For $\rho \in \Sigma^+$, $m_\rho$ denotes the multiplicity of $\rho$, i.e. the dimension of the joint eigenspace, corresponding to the linear form $\rho$, of the commuting symmetric operators which are induced on $\mathfrak{g}$ by $\mathfrak{a}$. Alternatively, $m_\rho$ is the cardinality of the inverse image of $\rho$ w.r.t the restriction process described in [16], p. 263-4.

PROPOSITION **2.2.** *There exists $c > 0$ such that for all $f \in \mathrm{L}^1(\mathfrak{p}, \mathrm{m}_\mathfrak{p})$ which are constant on $\mathrm{Ad}_K$-orbits there holds*

$$\begin{aligned} \int_\mathfrak{p} f \; d\mathrm{m}_\mathfrak{p} &= c \int_{\mathfrak{a}^+} f(A) \prod_{\rho \in \Sigma^+} \rho(A)^{m_\rho} \mathrm{m}_{\mathfrak{a}^+}(dA) \\ &= \frac{c}{\# W} \int_\mathfrak{a} f(A) \prod_{\rho \in \Sigma^+} |\rho(A)|^{m_\rho} \mathrm{m}_\mathfrak{a}(dA) \end{aligned}$$

*Proof.* [15, Thm. I.5.17] □

In practical terms, if $f$ is an $\mathrm{Ad}_G$-invariant $\mathrm{m}_{\tilde{\mathfrak{g}}}$-density $\tilde{\mathfrak{g}} \to [0, \infty[$ or an $\mathrm{Ad}_K$-invariant $\mathrm{m}_{\tilde{\mathfrak{p}}}$-density $\tilde{\mathfrak{p}} \to [0, \infty[$, then $f(X)$ only depends on $X$ through its eigenvalues. Now, for all classes except A, AI, AII, the nonzero eigenvalues come in pairs $\pm\lambda$. Note that in the chiral classes BDI, AIII and CII, the number of positive eigenvalues is $s \wedge t$ resp. $2(s \wedge t)$, and later on, when we will let the matrix size tend to infinity, we will have to control the growth of $s = s(n)$ as well as of $n$. What is more, note that many of the matrices that make up the spaces of good Hamiltonians are necessarily of even size, so that there is no question of simply letting the matrix size tend to infinity. Hence we choose the following framework: $(n, s(n)) \in \mathbb{N}^2$, $s(n) \le n$, is the pair of parameters that will be controlled, $t(n) := n - s(n)$ by definition. The space of good Hamiltonians for symmetry class $\mathcal{C}$ and parameters $(n, s(n))$, written as $\mathbb{M}_\mathcal{C}^{(n)} := \mathbb{M}_\mathcal{C}^{(n,s(n))}$, is contained in $\mathbb{C}^{d(n) \times d(n)}$. The map

$$\pi_n : \; \mathbb{M}_\mathcal{C}^{(n)} \to \overline{\mathbb{R}^{p(n),+}},$$

where

$$\mathbb{R}^{d,+} := \{x \in \mathbb{R}^d : \; x_1 > x_2 > \ldots > x_d\},$$

assigns to each $X \in \mathbb{M}_\mathcal{C}^{(n)}$ the nonincreasing vector $(\lambda_1, \ldots, \lambda_{p(n)})$ of its eigenvalues (resp. of its nonnegative eigenvalues if $\mathcal{C}$ is not one of A, AI, AII). By construction, $\pi_n$ separates the adjoint orbits of $G$ resp. $K$, and it is easily seen that $\pi_n(X) \in \mathbb{R}^{p(n),+}$ a.s.

Plugging well-known facts about (restricted) root systems (see, e.g., the appendix to the monograph [20]) into Propositions 2.1, 2.2 above, one obtains the following

COROLLARY **2.3.** *Suppose that $X$ is a random element of $\mathbb{M}_\mathcal{C}^{(n)}$, whose distribution is given by a Lebesgue density $f$ such that $f = \tilde{f} \circ \pi_n$ for some measurable $\tilde{f} : \mathbb{R}^{p(n),+} \to [0, \infty[$.*

(a) *If $\mathcal{C}$ is A, AI or AII, then the joint density of the eigenvalues of $x$ (in nonincreasing order) is*

$$(x_1, \ldots, x_{p(n)}) \mapsto \text{const } \tilde{f}(x_1, \ldots, x_{p(n)}) \prod_{1 \le i < j \le p(n)} (x_i - x_j)^\beta,$$

*where $\beta$ and $p(n)$ are given in the table below.*

(b) *Otherwise, the joint density of the positive eigenvalues of $X$ (in nonincreasing order) is given by*

$$(x_1, \ldots, x_{p(n)}) \mapsto \text{const } \tilde{f}(x_1, \ldots, x_{p(n)}) \prod_{1 \le i < j \le p(n)} (x_i^2 - x_j^2)^\beta \prod_{1 \le i \le p(n)} x_i^\alpha,$$

*where $\alpha, \beta$ and $p(n)$ are given in the table below.*

| Class | $d(n)$ | $p(n)$ | $\alpha$ | $\beta$ |
|---|---|---|---|---|
| A | $n$ | $n$ | | 2 |
| AI | $n$ | $n$ | | 1 |
| AII | $2n$ | $n$ | | 4 |
| BDI | $n$ | $s(n) \wedge t(n)$ | $\lvert s(n) - t(n) \rvert$ | 1 |
| AIII | $n$ | $s(n) \wedge t(n)$ | $2\lvert s(n) - t(n) \rvert + 1$ | 2 |
| CII | $2n$ | $s(n) \wedge t(n)$ | $4\lvert s(n) - t(n) \rvert + 3$ | 4 |
| B | $2n+1$ | $n$ | 2 | 2 |
| D | $2n$ | $n$ | 0 | 2 |
| C | $2n$ | $n$ | 2 | 2 |
| CI | $2n$ | $n$ | 1 | 1 |
| DIII ($n$ even) | $2n$ | $n$ | 1 | 4 |
| DIII ($n$ odd) | $2n$ | $n$ | 5 | 4 |

## 3. An LDP for the empirical eigenvalue measure

In this section we prove a theorem which contains LDPs for the empirical eigenvalue measures of all Hamiltonian ensembles, assuming a product structure for $\tilde{f}$ in the notation of Corollary 2.3. Note that this subsumes the well-studied densities on $\mathbb{M}_{\mathcal{C}}^{(n)}$ of the form $\exp(-\mathrm{Tr}(V(X)))$, $V$ a polynomial with positive leading coefficient (see Remark 3.4 below). This is the set-up: Let $\beta > 0$, $\gamma \in \mathbb{N}$. Let $\Sigma$ be a closed subinterval of $\mathbb{R}$ if $\gamma$ is odd, of $[0, \infty[$ if $\gamma$ is even. Let $(w_n)_{n\in\mathbb{N}}$ be a family of continuous nonnegative real-valued functions on $\Sigma$. For $n \in \mathbb{N}$ consider random variables $\Lambda_n = (\lambda_1, \ldots, \lambda_{p(n)})$ with joint distribution $Q_n = \mathbb{P} \circ \Lambda_n^{-1} \in \mathcal{M}_1(\Sigma^{p(n)})$, given by its Lebesgue density

$$q_n(x_1, \ldots, x_{p(n)}) = \frac{1}{Z_n} \prod_{1\le i<j\le p(n)} |x_i^\gamma - x_j^\gamma|^\beta \prod_{j=1}^{p(n)} w_n(x_j)^n \, 1_{\Sigma^{p(n)}}(x_1, \ldots, x_{p(n)}), \tag{3}$$

where $Z_n$ is for normalization. In what follows we assume that $p(n) \to \infty$ for $n \to \infty$, satisfying

$$\lim_{n\to\infty} \frac{p(n)}{n} = \kappa \in ]0, \infty[. \tag{4}$$

Write $\mathcal{N}(f)$ for the set of zeros of a function $f$. We will make the following assumptions about the sequence $(w_n)_n$:

(a1) there exists a continuous function $w : \Sigma \to [0, \infty[$ such that
   - $\#\mathcal{N}(w) < \infty$, $\mathcal{N}(w_n) \subseteq \mathcal{N}(w)$ for large $n$.
   - As $n \to \infty$, $w_n \to w$ uniformly on compact sets.

(a2) If $\Sigma$ is unbounded, then there exists $n_0 \in \mathbb{N}$ such that

$$\lim_{x\to\pm\infty} |x|^{\gamma\kappa(\beta\vee 1)+\epsilon} \sup_{n\ge n_0} w_n(x) = 0$$

for some fixed $\epsilon > 0$.

For $x = (x_1, \ldots, x_{p(n)}) \in \Sigma^{p(n)}$ set

$$L_n(x) := \frac{1}{p(n)} \sum_{j=1}^{p(n)} \delta_{x_j}.$$

THEOREM **3.1.** *$\big(\mathbb{P} \circ (L_n \circ \Lambda_n)^{-1}\big)_n = (Q_n \circ L_n^{-1})_n$ satisfies an LDP on $\mathcal{M}_1(\Sigma)$ with respect to the weak topology with speed $n^2$ and good rate function*

$$I(\mu) = \frac{\beta}{2}\kappa^2 \int\int \log|x^\gamma - y^\gamma|^{-1} \mu(dx)\,\mu(dy) \tag{5}$$

$$- \kappa \int \log w(x)\,\mu(dx) - c, \tag{6}$$

*where $\mu \in \mathcal{M}_1(\Sigma)$ and*

$$c := \lim_{n\to\infty} \frac{1}{n^2} \log Z_n < \infty. \tag{7}$$

COROLLARY **3.2.** *If $I$ has a unique minimizer $\mu^*$, then*

$$\mathbb{P}\big(L_n \circ \Lambda_n \to \mu^*\big) = 1$$

*where $\to$ means weak convergence.*

*Proof of the corollary:* Using the upper bound of the LDP one obtains the strong law applying Borel-Cantelli's lemma, see [9, Theorem II B.3]. □

REMARK 3.3. If $\beta\kappa \geq 1$, then it follows from the theory of logarithmic potentials with external fields, applied to the weight function $x \mapsto w(x^{1/\gamma})$, that $\mu^*$ exists and that $I(\mu^*) + c$ is finite, see [21], Thm. I.1.3 and Ex. I.3.5. In this case (7) can be sharpened to $|c| < \infty$. There exists a vast literature that describes $\mu^*$ in more detail for various classes of weights, see e.g. [21], [4], [10, Thm. 3.1]. We will restrict ourselves to giving explicit formulae for $\mu^*$ for Gaussian Hamiltonian ensembles below in Section 4. In these cases, $\mu^*$ is "universal" in the sense of Wigner's Theorem. This is the content of the companion paper [19].

REMARK 3.4. Although the main focus of the present paper is on Hamiltonian ensembles associated to (infinitesimal) symmetric spaces, let us note that Theorem 3.1 contains an LDP for the much wider class of ensembles which is considered in [10, (1.6)]. Furthermore, Theorem 3.1 applies to Wishart matrices (see Remark 4.2 below) and to Jacobi ensembles, for which the LDP was first proven in [18]. As observed in [6], the latter class of ensembles includes the random matrix ensembles associated to compact symmetric spaces, except those of classes A, AI, AII. In the companion paper [22], a systematic treatment of the compact case will be provided.

The *proof* of Theorem 3.1 extends the approach of [2], [13], [17]. Define on $\Sigma \times \Sigma$ the function

$$F(x, y) := -\frac{\beta}{2}\kappa^2 \log|x^\gamma - y^\gamma| - \frac{\kappa}{2}\big(\log w(x) + \log w(y)\big),$$

with $F(x, y) = \infty$ for $x^\gamma = y^\gamma$ (hence $x = y$ by definition of $\Sigma$) or $\{x, y\} \cap \mathcal{N}(w) \neq \emptyset$, and its truncated versions

$$F^M(x, y) := F(x, y) \wedge M, \quad M > 0.$$

Moreover, consider the functions

$$F_n(x,y) := -\frac{\beta}{2}\big(\frac{p(n)}{n}\big)^2 \log|x^\gamma - y^\gamma| - \frac{p(n)}{2n}\big(\log w_n(x) + \log w_n(y)\big)$$

and their truncated versions

$$F_n^M(x,y) := F_n(x,y) \wedge M, \quad M > 0.$$

From the definition of $F_n$ it follows that

$$q_n(x_1,\dots,x_{p(n)}) = \frac{1}{Z_n}\exp\Bigg(-\frac{2n^2}{p(n)^2}\sum_{1\le i<j\le p(n)} F_n(x_i,x_j) + \frac{n}{p(n)}\sum_{i=1}^{p(n)}\log w_n(x_i)\Bigg). \tag{8}$$

Lemma **3.5.**

(i) *For any $M > 0$, $F_n^M(x,y)$ converges to $F^M(x,y)$ uniformly as $n \to \infty$.*
(ii) *$F$ is bounded from below.*

*Proof.* The estimate $\log|x-y| \le \log(|x|+1) + \log(|y|+1)$ implies

$$F_n(x,y) \ge -\frac{p(n)}{2n}\Big[\log\big((|x^\gamma|+1)^{\beta p(n)/n}\, w_n(x)\big) + \log\big((|y^\gamma|+1)^{\beta p(n)/n}\, w_n(y)\big)\Big]. \tag{9}$$

Observe that $\log\big((|x^\gamma|+1)^{\beta p(n)/n}\, w_n(x)\big)$ is bounded from above by (a2) and the continuity of $w_n$. Invoking the full strength of (a2), one sees that for each $M > 0$ there exist $n_0 \in \mathbb{N}$, $R_M > 0$, $\delta_{\nu,M} > 0$ ($\nu \in \mathcal{N}(w)$) such that $F_n(x,y) \ge M$ holds for all $n \ge n_0$ on

$$A_M := \{|x| \vee |y| > R_M\} \cup \bigcup_{\nu\in\mathcal{N}(w)} \{|x-\nu| \wedge |y-\nu| < \delta_{\nu,M}\}.$$

By compactness of $A_M^c$, (i) follows from (a1) and the definition of $F^M, F_n^M$. One also has $F \ge M$ on $A_M$. $F$ being continuous, it is bounded on $A_M^c$, and this proves (ii). □

Our strategy is to first consider the finite positive measures $P_n := Z_n Q_n$. As to the *upper bound*, note that

$$L_n \otimes L_n(\{(x,y) \in \Sigma^2 : x = y\}) = \frac{1}{p(n)}$$

$\mathrm{m}_{\mathbb{R}^{p(n)}}$-almost surely, since the eigenvalues are a.s. distinct under the product Lebesgue measure. Hence almost surely

$$\int\int_{x\neq y} F_n^M(x,y)\, L_n(dx)\, L_n(dy) = \int\int F_n^M(x,y)\, L_n(dx)\, L_n(dy) - \frac{M}{p(n)}. \tag{10}$$

Now let $A$ be a Borel set in $\mathcal{M}_1(\Sigma)$ and write

$$\tilde{A} := \Big\{x \in \Sigma^{p(n)} : L_n(x) \in A\Big\}.$$

Using (8) and Hölder's inequality we obtain

$$\begin{aligned}
P_n\big(L_n \in A\big) &= \int_{\tilde{A}} \exp\Big(-\frac{2n^2}{p(n)^2} \sum_{1\le i<j\le p(n)} F_n(x_i,x_j)\Big) \\
&\qquad \exp\Big(\frac{n}{p(n)} \sum_{i=1}^{p(n)} \log w_n(x_i)\Big) \mathrm{m}_{\mathbb{R}^{p(n)}}(dx) \qquad (11)\\
&\le \Big(\int \exp\Big(\frac{2n}{p(n)} \log w_n(t)\Big) \mathrm{m}_{\mathbb{R}}(dt)\Big)^{\frac{p(n)}{2}} \\
&\qquad \left(\int_{\tilde{A}} \exp\left(-\frac{2n^2}{p(n)^2} \sum_{i\neq j} F_n^M(x_i,x_j)\right) \mathrm{m}_{\mathbb{R}^{p(n)}}(dx)\right)^{1/2} \\
&= (\mathrm{I}) \times (\mathrm{II}).
\end{aligned}$$

Note that we have used that $2\sum_{i<j} F_n(x_i,x_j) = \sum_{i\neq j} F_n(x_i,x_j)$ by symmetry of $F_n$ in its arguments. Now, $\lim_{n\to\infty} \frac{1}{n^2}\log(\mathrm{I}) = 0$ by (a2). On the other hand, for any $M > 0$ (10) yields

$$\begin{aligned}
(\mathrm{II}) &\le \left\{\int_{\tilde{A}} \exp\left(-2n^2\left(L_n(x)^{\otimes 2}(F_n^M) - \frac{M}{p(n)}\right)\right) \mathrm{m}_{\mathbb{R}^{p(n)}}(dx)\right\}^{1/2} \\
&\le \left\{\exp\left(-2n^2\left(\inf_{\mu\in A} \mu^{\otimes 2}(F_n^M) - \frac{M}{p(n)}\right)\right)\right\}^{1/2} \\
&= \exp\left(-n^2 \inf_{\mu\in A} \mu^{\otimes 2}(F_n^M)\right) \exp\left(\frac{Mn^2}{p(n)}\right).
\end{aligned}$$

Using Lemma 3.5, we obtain

$$\lim_{n\to\infty} \left(\inf_{\mu\in A} \mu^{\otimes 2}(F_n^M)\right) = \inf_{\mu\in A} \mu^{\otimes 2}(F^M).$$

We have thus shown that for any Borel set $A \subset \mathcal{M}_1(\Sigma)$ one has

$$\limsup_{n\to\infty} \frac{1}{n^2} \log P_n\big(L_n \in A\big) \le -\inf_{\mu\in A} \int\int F^M(x,y)\,\mu(dx)\mu(dy). \qquad (12)$$

Setting

$$H(\mu) := \int F d\mu^{\otimes 2},\ \ H^M(\mu) := \int F^M d\mu^{\otimes 2},$$

one obtains well defined maps on $\mathcal{M}_1(\Sigma)$ (see Lemma 3.5). We will show that $H$ is a good rate function that governs the LDP for $(P_n)$. To this end, observe that, $F^M$ being bounded and continuous, $H^M$ is weakly continuous on $\mathcal{M}_1(\Sigma)$ for each $M > 0$. By monotone convergence, we have $\lim_{M\to\infty} H^M = H$ pointwise on $\mathcal{M}_1(\Sigma)$. As a limit of an increasing sequence of continuous functions, $H$ is lower semi-continuous, i.e. the level sets $\{H \le L\}$ are closed. We claim that they are compact. Indeed, let $m_F := |\inf F|$ and $a > 0$. Then for any $\mu \in \mathcal{M}_1(\Sigma)$

one has

$$\begin{aligned}\Big(\inf_{x,y\in[-a,a]^c}(F+m_F)(x,y)\Big)\mu\big([-a,a]^c\big)^2 &\leq \int\int(F+m_F)(x,y)\,\mu(dx)\,\mu(dy)\\ &\leq H(\mu)+m_F,\end{aligned}$$

hence $\{H\leq L\}\subset K_L$, $L\in]0,\infty[$, with

$$K_L:=\bigcap_{a>0}\Big\{\mu\in\mathcal{M}_1(\Sigma):\mu\big([-a,a]^c\big)\leq\Big(\frac{L+m_F}{\inf_{x,y\in[-a,a]^c}(F+m_F)(x,y)}\Big)^{1/2}\Big\}.$$

Since $\lim_{a\to\infty}\inf_{x,y\in[-a,a]^c}(F+m_F)(x,y)=\infty$, $K_L$ is weakly compact by Prohorov's theorem. Hence the rate function $H$ is good. Furthermore, this argument easily yields the exponential tightness of $(P_n\circ L_n^{-1})_n$. Fix $M>0$, and define $K_L$ $(L>0)$ as above, using $F^M$ in the place of $F$. For every $\mu\in K_L^c$ there exists $a=a_\mu>0$ such that

$$\Big(\inf_{x,y\in[-a,a]^c}F^M(x,y)+m_{F^M}\Big)\;\mu([-a,a]^c)^2>L+m_{F^M},$$

hence

$$\inf_{x,y\in[-a,a]^c}F^M(x,y)\;\mu([-a,a]^c)^2>L. \tag{13}$$

Then (12) implies

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n^2}\log P_n(L_n\in K_L^c)\leq-\inf_{\mu\in K_L^c}\int F^M d\mu^{\otimes 2}\\ =\;&-\inf_{\mu\in K_L^c}\Big(\int_{([-a_\mu,a_\mu]^c)^2}F^M d\mu^{\otimes 2}+\int_{\mathbb{R}^2\setminus([-a_\mu,a_\mu]^c)^2}F^M d\mu^{\otimes 2}\Big)\\ \leq\;&-\inf_{\mu\in K_L^c}\int_{([-a_\mu,a_\mu]^c)^2}\inf_{x,y\in[-a_\mu,a_\mu]^c}F^M(x,y)\;d\mu^{\otimes 2}-\inf_{\mu\in K_L^c}\int_{\mathbb{R}^2\setminus([-a_\mu,a_\mu]^c)^2}\inf F^M d\mu^{\otimes 2}\\ \leq\;&-L+m_{F^M}.\end{aligned}$$

Since $\inf F^M>-\infty$, we have shown that

$$\limsup_{L\to\infty}\limsup_{n\to\infty}\frac{1}{n^2}\log P_n(L_n\in K_L^c)=-\infty,$$

hence that $(P_n\circ L_n^{-1})_n$ is exponentially tight.

Now let $B(\mu,\delta)$ denote the ball centered at $\mu\in\mathcal{M}_1(\Sigma)$ with radius $\delta$ for a distance compatible with the weak topology. Since $\mu\mapsto H^M(\mu)$ is continuous, from (12) we obtain for any $\mu\in\mathcal{M}_1(\Sigma)$

$$\inf_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n^2}\log P_n\big(L_n\in B(\mu,\delta)\big)\leq-\int\int F^M(x,y)\,\mu(dx)\mu(dy).$$

Finally, letting $M$ go to infinity, we obtain the following upper bound

$$\inf_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n^2}\log P_n\big(L_n\in B(\mu,\delta)\big)\leq-H(\mu).$$

Turning to the *lower bound* for $(P_n \circ L_n^{-1})$, we show that for any $\mu \in \mathcal{M}_1(\Sigma)$

$$\inf_{\delta>0} \liminf_{n\to\infty} \frac{1}{n^2} \log P_n(L_n \in B(\mu,\delta)) \geq \frac{\beta}{2}\kappa^2 \int\int \log|x^\gamma - y^\gamma|\mu(dx)\mu(dy) + \kappa \int \log w d\mu. \quad (14)$$

Claim **3.6.** *We may assume, without loss of generality, that*

(i) *$\mu$ has no atoms*
(ii) *$\mathcal{S} := \operatorname{supp}(\mu)$ is a compact subset of $\Sigma$ such that $\mathcal{S} \cap (\mathcal{N} \cup \{0\}) = \emptyset$.*

*Proof.* With the notations of the proof of the upper bound, we know that

$$\frac{\beta}{2}\kappa^2 \int\int \log|x^\gamma - y^\gamma|\mu(dx)\mu(dy) + \kappa \int \log w(x)\mu(dx) = -\int F d\mu^{\otimes 2}$$

with $F$ bounded from below. If $\mu$ has an atom, then $\int F d\mu^{\otimes 2}$ is infinite, and there is nothing to prove. Set

$$A_k := [-k,k] \cap \Sigma \cap \left( \bigcup_{x\in\mathcal{N}\cup\{0\}} ]x - \frac{1}{k}, x + \frac{1}{k}[ \right)^c$$

and $\mu_k := \frac{1}{\mu(A_k)} \mu|_{A_k}$. Then

$$\int F d\mu^{\otimes 2} = \lim_{k\to\infty} \int F d\mu_k^{\otimes 2}.$$

Hence it suffices to prove (14) for $\mu_k$ in the place of $\mu$. Consequently, we may assume the support of $\mu$ to be contained in a finite union of compact intervals not meeting $\mathcal{N} \cup \{0\}$. This implies (ii). □

For $j = 1, \ldots, p(n)$ let $\xi_j = \xi_j^{(n)}$ be the $\frac{p(n)+1-j}{p(n)}$ quantile of $\mu$. Let $\gamma \in \mathbb{N}$ be as in (3) above. Write $\xi^\gamma = (\xi_{p(n)}^\gamma, \ldots, \xi_1^\gamma) \in \Sigma^{p(n),+}$. Set $\xi_{p(n)+1} := \inf \mathcal{S}$ and $\xi_0 := \xi_1 + 1$. Then, by Claim 3.6,

$$-\infty < \xi_{p(n)+1} < \xi_{p(n)} < \ldots < \xi_0 < \infty.$$

Set $\varphi_j^{(n)} := \inf\{w_n(x) : \ x \in [\xi_{j+1}, \xi_{j-1}]\}, \ j = 1, \ldots, p(n)$, and write $\psi_n$ for the step function which equals $\varphi_j^{(n)}$ on $]\xi_{j+1}, \xi_j]$ and is zero elsewhere. Set $C := \max\{\log w(x) : \ \xi_{p(n)+1} \leq x \leq \xi_1\} - \min\{\log w(x) : \ \xi_{p(n)+1} \leq x \leq \xi_1\}$.

For $\delta > 0$, $t \in \mathbb{R}^{p(n)}$ write

- $\pi_n(t) := \{i = 1, \ldots, p(n) : \ t_i \geq 0\}, \ \nu_n(t) := \{1, \ldots, p(n)\} \setminus \pi_n(t)$,
- $\mathcal{I}_j^{(n)}(t,\delta) := [t_{j+1}, t_{j-1}] \cap [t_j - \delta, t_j + \delta], \ j = 1, \ldots, p(n)$,
- $$\mathcal{J}_j^{(n)}(t,\delta) := \begin{cases} [t_j, t_{j-1}] \cap [t_j, t_j + \delta] & \text{for } j \in \pi_n(t), \\ [t_{j+1}, t_j] \cap [t_j - \delta, t_j] & \text{for } j \in \nu_n(t) \end{cases}$$
- $\mathbb{I}_n(t,\delta) := \prod_{j=1}^{p(n)} \mathcal{I}_j^{(n)}(t,\delta)$, $\mathbb{I}_n(t,\delta)^\gamma := \prod_{j=1}^{p(n)} \mathcal{I}_j^{(n)}(t,\delta)^\gamma$,
- $\mathbb{J}_n(t,\delta) := \prod_{j=1}^{p(n)} \mathcal{J}_j^{(n)}(t,\delta)$.

Here, for $M \subseteq \mathbb{R}$, we write $M^\gamma := \{m^\gamma : \ m \in M\}$. Fix $\delta > 0$, and write $G := B(\mu, \delta)$. Then for $n$ large enough we have that

$$\mathbb{I}_n(\xi, \frac{1}{n}) \subset \{x \in \Sigma^{p(n)} : \ L_n(x) \in G\}.$$

For such $n$, then,

$$\begin{aligned} P_n(L_n \in G) &\geq Z_n Q_n(\mathbb{I}_n(\xi, \frac{1}{n})) = \int_{\mathbb{I}_n(\xi,\frac{1}{n})} \prod_{i<j} |x_i^\gamma - x_j^\gamma|^\beta \prod_i w_n(x_i)^n \mathrm{m}_{p(n)}(dx) \\ &= \frac{1}{\gamma^{p(n)}} \int_{\mathbb{I}_n(\xi,\frac{1}{n})^\gamma} \prod_{i<j} |x_i - x_j|^\beta \prod_i w_n(x_i^{1/\gamma})^n |x_i|^{-\frac{\gamma-1}{\gamma}} \mathrm{m}_{p(n)}(dx). \end{aligned}$$

Observing that $\mathbb{I}_n(\xi, \frac{1}{n})^\gamma \supset \mathbb{J}_n(\xi^\gamma, \frac{1}{n^\gamma})$ and

$$\inf\left\{\prod_i w_n(x_i^{1/\gamma})^n : \ x \in \mathbb{I}_n(\xi, \frac{1}{n})^\gamma\right\} = \inf\left\{\prod_i w_n(x_i)^n : \ x \in \mathbb{I}_n(\xi, \frac{1}{n})\right\} \geq \prod_i (\varphi_i^{(n)})^n$$

we obtain that

$$P_n(L_n \in G) \geq \frac{1}{\gamma^{p(n)}} \prod_i \left(\left(\varphi_j^{(n)}\right)^n (|\xi_i|^\gamma + n^{-\gamma})^{-\frac{\gamma-1}{\gamma}}\right) \int_{\mathbb{J}_n(\xi^\gamma, n^{-\gamma})} \prod_{i<j} |x_i - x_j|^\beta \mathrm{m}_{p(n)}(dx).$$

Now, we can bound this last integral from below by

$$\begin{aligned} &\int_{\left([0,n^{-\gamma}]^{\#\pi_n(\xi^\gamma)} \times [-n^{-\gamma},0]^{\#\nu_n(\xi^\gamma)}\right) \cap \mathbb{R}^{p(n),+}} \prod_{i<j} |(\xi_i^\gamma + x_i) - (\xi_j^\gamma + x_j)|^\beta \ \mathrm{m}_{p(n)}(dx) \\ = &\int_{\left([0,n^{-\gamma}]^{\#\pi_n(\xi^\gamma)} \times [-n^{-\gamma},0]^{\#\nu_n(\xi^\gamma)}\right) \cap \mathbb{R}^{p(n),+}} \prod_{i<j} |(\xi_i^\gamma - \xi_j^\gamma) + (x_i - x_j)|^\beta \ \mathrm{m}_{p(n)}(dx) \\ \geq &\prod_{i<j-1} |\xi_i^\gamma - \xi_j^\gamma|^\beta \prod_{i=1}^{p(n)-1} |\xi_{i+1}^\gamma - \xi_i^\gamma|^{\frac{\beta}{2}} \\ \times &\int_{\left([0,n^{-\gamma}]^{\#\pi_n(\xi^\gamma)} \times [-n^{-\gamma},0]^{\#\nu_n(\xi^\gamma)}\right) \cap \mathbb{R}^{p(n),+}} \prod_{i=1}^{p(n)-1} |x_i - x_{i+1}|^{\frac{\beta}{2}} \ \mathrm{m}_{p(n)}(dx). \end{aligned} \tag{15}$$

By the change of variables $u_{p(n)} = x_{p(n)}, \ u_{i-1} = x_i - x_{i-1}$ $(i = p(n), \ldots, 2)$ one can bound (15) from below by

$$\int_{\left[0,\frac{1}{n^\gamma p(n)}\right]^{p(n)}} \prod_{i=2}^{p(n)} u_i^{\frac{\beta}{2}} \ \mathrm{m}_{p(n)}(du) = \left(\frac{2}{\beta+2}\right)^{p(n)-1} \left(\frac{1}{n^\gamma p(n)}\right)^{\frac{\beta+2}{2}(p(n)-1)+1}.$$

So far, it has been shown that

$$
\begin{aligned}
& P_n(L_n \in G) \\
\geq \; & \prod_{i<j-1} |\xi_i^\gamma - \xi_j^\gamma|^\beta \prod_{i=1}^{p(n)-1} |\xi_i^\gamma - \xi_{i+1}^\gamma|^{\frac{\beta}{2}} \\
\times \; & \frac{1}{\gamma^{p(n)}} \prod_i \left( \left(\varphi_i^{(n)}\right)^n (|\xi_i|^\gamma + n^{-\gamma})^{-\frac{\gamma-1}{\gamma}} \right) \left( \frac{2}{\beta+2} \right)^{p(n)-1} \left( \frac{1}{n^\gamma p(n)} \right)^{\frac{\beta+2}{2}(p(n)-1)+1}.
\end{aligned}
$$

So we obtain

$$
\begin{aligned}
& \frac{1}{n^2} \log P_n(L_n \in G) \\
\geq \; & \left( \frac{p(n)}{n} \right)^2 \frac{\beta}{p(n)^2} \sum_{i<j} \log |\xi_i^\gamma - \xi_{j+1}^\gamma| && (16) \\
+ \; & \left( \frac{p(n)}{n} \right)^2 \frac{\beta}{2p(n)^2} \sum_{i=1}^{p(n)-1} \log |\xi_{i+1}^\gamma - \xi_i^\gamma| && (17) \\
+ \; & \frac{1}{n} \sum_i \log \varphi_i^{(n)} && (18) \\
+ \; & \frac{1}{n^2} \sum_i -\frac{\gamma-1}{\gamma} \log(|\xi_i|^\gamma + n^{-\gamma}) && (19) \\
+ \; & \frac{1}{n^2} \left( -p(n) \log \gamma + (p(n)-1) \log \frac{2}{\beta+2} \right) && (20) \\
- \; & \frac{1}{n^2} \left( \frac{\beta+2}{2} ((p(n)-1)+1) \log(n^\gamma p(n)) \right). && (21)
\end{aligned}
$$

Now, for (16) and (17) observe that

$$
\begin{aligned}
& \int_{x<y} \log(y^\gamma - x^\gamma) \mu(dx)\mu(dy) \\
= \; & \sum_{i<j} \int_{(x,y)\in[\xi_{j+1},\xi_j]\times[\xi_{i+1},\xi_i]} \log(y^\gamma - x^\gamma)\mu(dx)\mu(dy) \\
+ \; & \frac{1}{2} \sum_i \int_{(x,y)\in[\xi_{i+1},\xi_i]^{\times 2}} \log(y^\gamma - x^\gamma)\mu(dx)\mu(dy) \\
\leq \; & \frac{1}{p(n)^2} \sum_{i<j} \log(\xi_i^\gamma - \xi_{j+1}^\gamma) + \frac{1}{2p(n)^2} \sum_i \log(\xi_i^\gamma - \xi_{i+1}^\gamma).
\end{aligned}
$$

As to (18), we claim

Claim **3.7.**

$$
\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{p(n)} \log \varphi_j^{(n)} = \kappa \int \log w \; d\mu.
$$

*Proof.* By definition, $\frac{1}{n}\sum_{j=1}^{p(n)} \log \varphi_j^{(n)} = \frac{p(n)}{n} \int \log \psi_n d\mu$. Let $\epsilon > 0$. $\log w$ being uniformly continuous on $[\xi_{p(n)+1}, \xi_1]$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|\log w(x) - \log w(y)| < \frac{\epsilon}{4}$. Furthermore, for $n$ large enough, we have $\|\log w - \log w_n\|_\infty < \frac{\epsilon}{4}$ and $\frac{\xi_1 - \xi_{p(n)+1}}{\delta}(C + \epsilon)\frac{1}{p(n)} < \frac{\epsilon}{4}$. Then if $|x - y| < \delta$ we have that $|\log w_n(x) - \log w_n(y)| \leq |\log w_n(x) - \log w(x)| + |\log w(x) - \log w(y)| + |\log w(y) - \log w_n(y)| \leq \frac{3\epsilon}{4}$. Hence

$$\begin{aligned}
& \int |\log \psi_n - \log w_n| d\mu \\
= \ & \sum_{\substack{j=1,\dots,p(n): \\ |\xi_{j-1} - \xi_{j+1}| < \delta}} \int_{[\xi_{j+1}, \xi_j]} |\log \psi_n - \log w_n| d\mu + \sum_{\substack{j=1,\dots,p(n): \\ |\xi_{j-1} - \xi_{j+1}| \geq \delta}} \int_{[\xi_{j+1}, \xi_j]} |\log \psi_n - \log w_n| d\mu \\
\leq \ & \#\{j = 1, \dots, p(n) : \ \xi_j - \xi_{j+1} < \delta\} \ \frac{1}{p(n)} \ \frac{3\epsilon}{4} \\
+ \ & \#\{j = 1, \dots, p(n) : \ \xi_j - \xi_{j+1} \geq \delta\} \ (C + \epsilon) \ \frac{1}{p(n)} \\
\leq \ & \frac{3\epsilon}{4} + \frac{\xi_1 - \xi_{p(n)+1}}{\delta}(C + \epsilon)\frac{1}{p(n)} \leq \epsilon.
\end{aligned}$$

(19), (20), (21) are easily seen to converge to zero. In the case of (19), notice that $\xi_i^\gamma \in [\xi_{p(n)+1}^\gamma, \xi_1^\gamma]$, a compact interval which doesn't meet zero. $\square$

Summing up, for any $\mu \in \mathcal{M}_1(\Sigma)$ we have obtained

$$\inf_{\delta > 0} \limsup_{n \to \infty} \frac{1}{n^2} \log P_n(L_n \in B(\mu, \delta)) \leq -H(\mu)$$

and

$$\inf_{\delta > 0} \liminf_{n \to \infty} \frac{1}{n^2} \log P_n(L_n \in B(\mu, \delta)) \geq -H(\mu).$$

Using the exponential tightness of $(P_n \circ L_n^{-1})_n$, we can apply [5, Thm. 4.1.11] to obtain an LDP for $(P_n \circ L_n^{-1})_n$ with rate $H$ and speed $n^2$. Setting $A = G = \mathcal{M}_1(\Sigma)$ in the lower and upper bound, we obtain

$$\lim_{n \to \infty} \frac{1}{n^2} \log Z_n = - \inf_{\mu \in \mathcal{M}_1(\Sigma)} \int F d\mu^{\otimes 2}.$$

By Lemma 3.5 (ii), the right-hand side is $< +\infty$. Now,

$$\frac{1}{n^2} \log Q_n(L_n \in A) = \frac{1}{n^2} \left(\log P_n(L_n \in A) - \log Z_n\right)$$

for any Borel set $A$ in $\mathcal{M}_1(\Sigma)$. Hence Theorem 3.1 is proven.

## 4. Application to Gaussian Hamiltonian ensembles

Let $\mathbb{M}_{\mathcal{C}}^{(n)} \subset \mathbb{C}^{d(n) \times d(n)}$ be the space of good Hamiltonians of symmetry class $\mathcal{C}$. We wish to define $\mathrm{Ad}_G$- resp. $\mathrm{Ad}_K$-invariant probability measures on $\mathbb{M}_{\mathcal{C}}^{(n)}$ with the additional property that all matrix entries should be Gaussian and as many entries as possible should be independent. Plainly, what we are looking for is an analog for class $\mathcal{C}$ of the GOE, which was reviewed in the Introduction and corresponds to $\mathcal{C} = \text{AI}$.

Let us look at an example: For $\mathcal{C} = \text{C}$, let us represent an element of $\mathbb{M}_{\text{C}}^{(n)}$ as

$$X = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

where $A = (ia_{ij}),\ B = (ib_{ij})$ are $n \times n$ matrices with purely imaginary entries, $A$ skew symmetric, $B$ symmetric. Then the diagonal entries of $A$ are zero, but the real parts of the strictly upper diagonal entries of $A$ can be chosen to be independent Gaussians. As to $B$, the real parts of all entries on or above the diagonal can be independent Gaussians. We thus obtain a family of random variables with joint Lebesgue density

$$\text{const} \times \prod_{k<l} \exp\left(-\frac{a_{kl}^2}{2\sigma_{a,k,l}^2}\right) \prod_{k\leq l} \exp\left(-\frac{b_{kl}^2}{2\sigma_{b,k,l}^2}\right).$$

It remains to choose the $\sigma_{a,k,l}, \sigma_{b,k,l}$ in such a way that the random element of $\mathbb{M}_{\mathcal{C}}^{(n)}$ which is made up from these coordinates has a distribution which is invariant under conjugation by elements of $\text{Sp}_{2n}$. To this end, compute

$$\begin{aligned} \text{Tr}(X^2) &= 2(\text{Tr}A^2 - \text{Tr}B^2) = 2\left(\sum_{kl} ia_{kl}\ ia_{lk} - \sum_{kl} ib_{kl}\ ib_{lk}\right) \\ &= 2\left(\sum_{kl} a_{kl}^2 + \sum_{kl} b_{kl}^2\right) = 4\sum_{k<l} a_{kl}^2 + 2\sum_{k} b_{kk}^2 + 4\sum_{k<l} b_{kl}^2. \end{aligned} \tag{22}$$

This means that in order to obtain an invariant distribution, we choose $\sigma^2 > 0$ and set $\sigma_{a,k,l}^2 = \sigma_{b,k,l} = \sigma^2\ (k<l),\ \sigma_{a,k,k} = 2\sigma^2$. We will write $\text{GE}_{\text{C}}^{(n)}(\sigma^2)$ for the probability distribution on $\mathbb{M}_{\text{C}}^{(n)}$ obtained in this way.

For a general symmetry class $\mathcal{C}$, $\text{GE}_{\mathcal{C}}^{(n)}(\sigma^2)$ can be defined in an analogous way. We simply indicate the places where the matrix entries have to be chosen of variance $2\sigma^2$ rather than $\sigma^2$, and where entries are identically zero. We use the description of the symmetry classes from Section 1.

| $\mathcal{C}$ | $2\sigma^2$ | $0$ |
|---|---|---|
| A | diagonal entries of $X$ | none |
| AI | diagonal entries of $X$ | none |
| AII | diagonal entries of $X_1$ | diagonal entries of $X_2$ |
| AIII | diagonal entries of $X$ | none |
| B/D | none | diagonal entries of $X$ |
| BDI | diagonal entries of $X$ | none |
| DIII | none | diagonal entries of $X_1, X_2$ |
| C | diagonal entries of $X_2$ | diagonal entries of $X_1$ |
| CI | diagonal entries of $X_1, X_2$ | none |
| CII | diagonal entries of $X_1, X_2$ | none |

Set $\varphi_{\mathcal{C}}(\sigma^2) = \frac{1}{4\sigma^2}$ for $\mathcal{C} =$ A, AI, B, D, $\varphi_{\mathcal{C}}(\sigma^2) = \frac{1}{8\sigma^2}$ otherwise. From calculations analogous to (22) one sees that $\mathrm{GE}_{\mathcal{C}}^{(n)}\left(\frac{\sigma^2}{n}\right)$ has Lebesgue density

$$\text{const} \times \exp\left(-n\varphi_{\mathcal{C}}(\sigma^2)\mathrm{Tr}(X^2)\right).$$

4.1. **Wigner-Dyson ensembles A, AI, AII.** From Corollary 2.3 we see that one can subsume the joint eigenvalue density induced by $\mathrm{GE}_{\mathcal{C}}^{(n)}\left(\frac{\sigma^2}{n}\right)$, $\mathcal{C} =$ A, AI, AII, under the general form of (3) by choosing $p(n) = n$, hence $\kappa = 1$, $\gamma = 1$, $\beta = 1, 2, 4$ according to $\mathcal{C} =$ AI, A, AII, and

$$w_n(x_j) = \exp\left(-\varphi_{\mathcal{C}}(\sigma^2)x_j^2\right), \tag{23}$$

independent of $n$. Then, under $\mathrm{GE}_{\mathcal{C}}^{(n)}\left(\frac{\sigma^2}{n}\right)$, $(L_n)_n$ satisfies an LDP with good rate function

$$I_{\mathcal{C}}(\mu) = I_{\mathcal{C},\sigma^2}(\mu) = \frac{\beta}{2}\int\int \log|x-y|^{-1}\mu(dx)\mu(dy) + \varphi_{\mathcal{C}}(\sigma^2)\int x^2\mu(dx) - \text{const}. \tag{24}$$

To apply the theory of logarithmic potentials with external fields, as presented in [21], we exploit the fact that the unique minimizer $\mu^*$ of $I_{\mathcal{C}}$ is also the unique minimizer of

$$J_{\mathcal{C}}(\mu) = \int\int \log|x-y|^{-1}\mu(dx)\mu(dy) - 2\int \log\exp\left(-\frac{\varphi_{\mathcal{C}}(\sigma^2)}{\beta}x^2\right)\mu(dx).$$

Setting $a := \left(\frac{\varphi_{\mathcal{C}}(\sigma^2)}{\beta}\right)^{-\frac{1}{2}}$, one reads off from p. 284 of [21] that $\mu^*$ has Lebesgue density

$$1_{[-a,a]}(x)\ \frac{2}{\pi a^2}\sqrt{a^2 - x^2},$$

the density of Wigner's *semicircle distribution* of radius $a$.

4.2. **Chiral ensembles BDI, AIII, CII.** We will freely use the notation introduced above and in Section 2. We have $\varphi_{\mathcal{C}}(\sigma^2) = \frac{1}{8\sigma^2}$ and $p(n) = s(n) \wedge t(n)$. For simplicity, we will assume that $s(n) \le t(n)$ for all $n \in \mathbb{N}$, hence $\alpha(n) = \beta(t(n) - s(n)) + \beta - 1$. Then we can subsume the joint eigenvalue density induced by $\mathrm{GE}_{\mathcal{C}}^{(n)}\left(\frac{\sigma^2}{n}\right)$ under the general form of (3) by setting $\gamma = 2$ and

$$w_n(x) = x^{\frac{\beta(t(n)-s(n))+\beta-1}{n}} e^{-\frac{x^2}{8\sigma^2}},$$

hence

$$w(x) = x^{\beta(1-2\kappa)}e^{-\frac{x^2}{8\sigma^2}}.$$

Then, by Theorem 3.1, the rate function of the LDP for $Q_n \circ L_n^{-1}$ is

$$I_{\mathcal{C}}(\mu) = \beta\frac{\kappa^2}{2}\int\int \log\frac{1}{|x^2 - y^2|}\mu(dx)\mu(dy) - \kappa\int \log\left(x^{\beta(1-2\kappa)}e^{-\frac{x^2}{8\sigma^2}}\right)\mu(dx) - c, \tag{25}$$

for $\mu \in \mathcal{M}_1([0,\infty[)$. For $r > 0, x \ge 0$ write $T_r(x) := x^r$. Now, $\mu^*$ is the unique minimizer of $I_{\mathcal{C}}$ if, and only if, $T_2(\mu^*)$ is the unique minimizer of

$$\beta\frac{\kappa^2}{2}\int\int \log\frac{1}{|x-y|}\nu(dx)\nu(dy) - \kappa\int \log\left(x^{\frac{\beta(1-2\kappa)}{2}}e^{-\frac{x}{8\sigma^2}}\right)\nu(dx) - c,$$

hence of

$$J_{\mathcal{C}}(\nu) := \int\int \log \frac{1}{|x-y|}\nu(dx)\nu(dy) - 2\int \log\left(x^{\frac{1-2\kappa}{2\kappa}} e^{-\frac{x}{8\beta\kappa\sigma^2}}\right)\nu(dx). \tag{26}$$

We can use the following facts from logarithmic potential theory with Laguerre weights:

**Lemma 4.1.** *For $s \geq 0,\ \lambda > 0$ the integral*

$$I_{s,\lambda}(\nu) := \int\int \log \frac{1}{|x-y|}\nu(dx)\nu(dy) - 2\int \log\left(x^s e^{-\lambda x}\right)\nu(dx), \quad \nu \in \mathcal{M}_1([0,\infty[)$$

*has a unique minimizer $\nu^*_{s,\lambda}$ with Lebesgue density*

$$1_{[a,b]}(x)\ \frac{\lambda}{\pi x}\sqrt{(x-a)(b-x)}, \tag{27}$$

*where*

$$a = a_{s,\lambda} = \frac{1}{\lambda}(s+1-\sqrt{2s+1}),\ b = b_{s,\lambda} = \frac{1}{\lambda}(s+1+\sqrt{2s+1}). \tag{28}$$

*Consequently, $T_{\frac{1}{2}}(\nu^*_{s,\lambda})$ has Lebesgue density*

$$1_{[\sqrt{a},\sqrt{b}]}(x)\ \frac{2\lambda}{\pi x}\sqrt{(x^2-a)(b-x^2)}. \tag{29}$$

*Proof.* [21], IV (1.31), IV (5.18) □

We have $s = \frac{1}{2\kappa} - 1$ and $\lambda = \frac{1}{8\sigma^2\beta\kappa}$. Note that $2\kappa \leq 1$ by our assumptions. This yields

$$a = 8\sigma^2\beta\left(\frac{1}{2} - \sqrt{\kappa(1-\kappa)}\right),\ b = 8\sigma^2\beta\left(\frac{1}{2} + \sqrt{\kappa(1-\kappa)}\right), \tag{30}$$

and from (29) we conclude that the minimizer $\mu^*$ of $I_{\mathcal{C}}$ has Lebesgue density

$$1_{[\sqrt{a},\sqrt{b}]}(x)\ \frac{1}{4\sigma^2\beta\kappa\pi x}\sqrt{(x^2-a)(b-x^2)} \tag{31}$$

with $a, b$ as in (30).

Remark 4.2. If

$$\begin{pmatrix} 0 & X \\ -X' & 0 \end{pmatrix} \quad (X \in i\mathbb{R}^{s(n)\times t(n)}),$$

is distributed according to $\mathrm{GE}^{(n)}_{\mathrm{BDI}}\left(\frac{\sigma^2}{n}\right)$, then $-XX' = (iX)(iX)'$ has a real Wishart distribution. Now, the LDP for the empirical eigenvalue distribution of Wishart matrices, first proven by Hiai and Petz in [17] and easily recognized as a special case of Theorem 3.1, leads to another energy integral with Laguerre weight, whose unique minimizer is a Marčenko-Pastur distribution. Nevertheless, our $T_2(\mu^*)$ is not exactly a Marčenko-Pastur law, because the natural scaling for Wishart matrices is not quite the same as for Gaussian Hamiltonian matrices of class BDI. In fact, in the set-up of the Hamiltonian ensembles one has an LDP of speed $n^2$ and a "typical" variance of $\frac{\sigma^2}{n}$ with $n = s(n) + t(n)$, while the interpretation of a BDI matrix as a Wishart matrix in disguise leads to $n = t(n)$.

4.3. **BdG ensembles B, D, DIII, C, CI.** Although there are five BdG ensembles, there are only four natural large $n$ limits for these ensembles. This is because B and D are $\mathfrak{iso}(n)$ for odd resp. even $n$, and, as $n$ tends to infinity, these series should be thought of as interlaced. It is consistent to do so, because we will see that the parameter $\alpha$, where B and D differ (see table in Corollary 2.3), does not affect the rate function. By the same token, it is in fact consistent to consider one rather than two large $n$ limits for DIII, because the even and odd cases of DIII only differ w.r.t. $\alpha$.

We subsume the joint eigenvalue density induced by $\mathrm{GE}_{\mathcal{C}}^{(n)}\left(\frac{\sigma^2}{n}\right)$, $\mathcal{C}$ = B, D, DIII, C, CI under the general form (3) by setting $p(n) = n$, hence $\kappa = 1$, $\gamma = 2$, $(\alpha, \beta)$ according to the table after Corollary 2.3, $w_n(x)^n = x^\alpha \exp(-n\varphi_{\mathcal{C}}(\sigma^2)x^2)$, hence

$$w_n(x) = x^{\alpha/n} \exp(-\varphi_{\mathcal{C}}(\sigma^2)x^2) \tag{32}$$

and

$$w(x) = \exp(-\varphi_{\mathcal{C}}(\sigma^2)x^2). \tag{33}$$

Then, under $\mathrm{GE}_{\mathcal{C}}^{(n)}\left(\frac{\sigma^2}{n}\right)$, $(L_n)_n$ satisfies an LDP with good rate function

$$I_{\mathcal{C}}(\mu) = \frac{\beta}{2} \int\int \log |x^2 - y^2|^{-1} \mu(dx)\mu(dy) + \varphi_{\mathcal{C}}(\sigma^2) \int x^2 \mu(dx) - \text{const.} \tag{34}$$

To determine the unique minimizer $\mu^*$ of $I_{\mathcal{C}}$, one can proceed as in Subsection 4.2 and apply Lemma 4.1 with $s = 0$ and $\lambda = \frac{\varphi_{\mathcal{C}}(\sigma^2)}{\beta}$. By (29), $\mu^*$ is a *quarter circle distribution*, given by the Lebesgue density

$$1_{[0,\sqrt{\frac{2}{\lambda}}]}(x)\ \frac{2\lambda}{\pi}\sqrt{\left(\frac{2}{\lambda} - x^2\right)},$$

where $\lambda^{-1} = 8\sigma^2$ for $\mathcal{C}$ = B, D, CI, $\lambda^{-1} = 16\sigma^2$ for $\mathcal{C}$ = C, and $\lambda^{-1} = 32\sigma^2$ for $\mathcal{C}$ = DIII.

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