A new fluctuation test for constant variances with applications to finance

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Abstract

We present a test to determine whether variances of time series are constant over time. The test statistic is a suitably standardized maximum of cumulative first and second moments. We apply the test to time series of various assets and find that the test performs well in applications. Moreover, we propose a portfolio strategy based on our test which hedges against potential financial crises and show that it works in practice.

Keywords: Variance, Econometric Modeling, Finance, Portfolio Optimization, Structural Breaks
JEL: C12, C14, G01, G11

1 Introduction

It is well known, in particular in empirical finance, that variances among many time series cannot be assumed to remain constant over longer stretches of time (Krishan et al., 2009). Especially, variances of stock indices seem to vary over time. A good example was the recent financial crisis, in which capital market volatilities and correlations raised quite dramatically. As a consequence, risk figures increased significantly as diversification effects were overestimated (Bissantz et al., 2010). In literature, this phenomenon is sometimes referred to as "Diversification Meltdown" (Campbell et al., 2008) and is well known also from other contexts.

A change in market parameters has serious consequences in practice, in particular for portfolio optimization which is based on diversification effects between several assets. If the relevant market parameters (e.g. volatilities) change, the optimization is no longer valid and the risk incorrectly calculated. Similar problems occur to applications in risk management and the valuation of financial instruments. Surprisingly, there is a lack of methods to formally test for changes in market parameters.

Most existing procedures either require strong parametric assumptions (Dias and Embrechts, 2004), assume that potential break points are known (Pearson and Wilks, 1933; Jennrich, 1970; Goetzmann et al., 2005), or simply estimate correlations from moving windows without giving a formal decision rule (Longin and Solnik, 2002). Only recently, Galeano and Peña (2007), Aue et al. (2009), Wied et al. (2010) and Arnold et al. (2010) have proposed formal tests in this context. Galeano and Peña (2007) and Aue et al. (2009) proposed a test to detect changes in the covariance structure, while Wied et al. (2010) and Arnold et al. (2010) presented a method to test for changes in the correlation structure between assets. These tests do not build upon prior knowledge as to the timing of potential shifts. While Galeano and Peña (2007) work in a parametric environment, the other authors propose complete nonparametric tests. They are based on cumulated sums of second order empirical cross moments (in the vain of Ploberger et al., 1989) and reject the null of constant covariance or correlation structure if these cumulated sums fluctuate too much.

This paper considers a non-parametric fluctuation test for constant variances over time. On the one hand, this test can be regarded as a special case of the Aue et al. (2009)-test for the one-dimensional case, on the other hand it goes beyond it by rigorously proving the asymptotic null distribution for the case that the expected values are estimated by arithmetic means basing on the first j observations (so that we compare successively estimated empirical variances). We use proving methods that were also used for the test for constant correlation described in Wied et al. (2010) and Arnold et al. (2010). Our second contribution is the application to financial data and the derivation of an investment strategy. We analyze the volatility structure of four indices including stocks, bonds and commodities and see that the test performs very well throughout the whole empirical application. The resulting dates of rejection seem to be reasonable. Besides, we derive a simple investment strategy based on the test and evaluate it by an out-of-sample study.

The paper is organized as follows. First, we describe the test statistic and the required theory. After that, we perform several tests based on real data and discuss the results. Proofs are given below the summary in Section 5.

2 Model and Test Statistic

Let $(X_t, t = 1, 2...)$ be a sequence of random variables with finite absolute $(4+\delta)$th moments. We want to test whether the variance of X_t,

$$\mathsf{Var}(X_t) = \mathsf{E}(X_t^2) - (\mathsf{E}(X_t))^2,$$

is constant over time, i.e. we test

$$H_0 : \mathsf{Var}(X_t) = \sigma^2 \ \forall t \in \{1, \ldots, T\} \text{ vs. } H_1 : \exists t \in \{1, \ldots, T-1\} : \mathsf{Var}(X_t) \neq \mathsf{Var}(X_{t+1})$$

for a constant σ^2. Our test statistic is

$$Q_T(X) = \max_{1 \le j \le T} \left| \hat{D} \frac{j}{\sqrt{T}} \left([\mathsf{Var}X]_j - [\mathsf{Var}X]_T \right) \right| \qquad (1)$$

where

$$[\mathsf{Var}X]_l = \frac{1}{l} \sum_{i=1}^{l} X_i^2 - \left(\frac{1}{l} \sum_{i=1}^{l} X_i \right)^2 =: \overline{X_l^2} - \left(\overline{X_l} \right)^2$$

is the empirical variance from the first l observations. Furthermore,

$$\hat{D} = \left(1 - 2\overline{X_T}\right)^{-1} (\hat{D}_1)^{-1/2}$$

with

$$\hat{D}_1 = \frac{1}{T} \sum_{i=1}^{T} \hat{U}_i' \hat{U}_i + 2 \sum_{j=1}^{T} k\left(\frac{j}{\gamma_T}\right) \frac{1}{T} \sum_{i=1}^{T-j} \hat{U}_i' \hat{U}_{i+j}$$

and

$$\hat{U}_l = \begin{pmatrix} X_l^2 - \overline{X_T^2} \\ X_l - \overline{X_T} \end{pmatrix},$$

$$k(x) = \begin{cases} 1 - |x|, & |x| \le 1 \\ 0, & otherwise \end{cases},$$

$$\gamma_n = \sqrt{T}.$$

The scalar $\hat{D}$ is needed for the asymptotic null distribution. It mainly captures the long-run-dependence and the fluctuations resulting from estimating the expected value. The test rejects the null hypothesis of constant variance if the empirical variances fluctuate too much, as measured by $\max_{1\leq j\leq T}|[\mathsf{Var}X]_j - [\mathsf{Var}X]_T|$, with the weighting factor $\frac{j}{\sqrt{T}}$ scaling down deviations at the beginning of the sample where the $[\mathsf{Var}X]_j$ are more volatile.

Of course, other functionals of the $[\mathsf{Var}X]_j$-series are likewise possible as suitable test statistics, such as some standardized version of

$$\max_{1\leq j\leq T}([\mathsf{Var}X]_j - [\mathsf{Var}X]_T) - \min_{1\leq j\leq T}([\mathsf{Var}X]_j - [\mathsf{Var}X]_T),$$

or simply some suitable average (see Krämer and Schotman, 1992, or Ploberger and Krämer, 1992), but for ease of exposition we stick to expression (1) for the purpose of the present paper.

The following technical assumptions are required for the limiting null distribution:

(A1) The sequence $(X_t, t = 1, 2...)$ is weak-sense stationary.

(A2) For

$$U_i = \begin{pmatrix} X_i^2 - \mathsf{E}(X_1^2) \\ X_i - \mathsf{E}(X_1) \end{pmatrix}$$

and $S_j := \sum_{i=1}^{j} U_i$, we have

$$\lim_{T\to\infty} \mathbb{E}\left(\frac{1}{T} S_T S_T'\right) =: D_1 \text{ is finite and positive definite.}$$

(A3) The r-th absolute moments of the components of U_i are uniformly bounded for some $r > 2$.

(A4) The sequence $(X_t, t = 1, 2...)$ is L_2-NED (near-epoch dependent) with size $-\frac{r-1}{r-2}$, with r from (A3), and constants $(c_i), i = 1, 2, \ldots$ on a sequence $(V_i), i = 1, 2, \ldots$, which is α-mixing of size $\phi^* := -\frac{r}{r-2}$, such that

$$c_t \leq 2\left(\left\{\mathsf{E}|X_i^2 - \mathsf{E}(X_1^2)|^2 + \mathsf{E}|X_i - \mathsf{E}(X_1)|^2\right\}\right)^{\frac{1}{2}}.$$

Assumption (A4) guarantees that

$$U_i^* := \left(X_i^2, \quad X_i\right)'$$

is L_2-NED with size $\frac{1}{2}$, see Davidson (1994). It could be modified to ϕ-mixing, requiring only finite 4-th moments, but this would admit less dependence than we allow here. In particular, assumption (A4) allows for GARCH-effects (see e.g. Hansen, 1991 or Carrasco and Chen, 2002), which are observed in financial data. Note that Assumption (A1) is already partly fulfilled because we assume constant variances under the null. The assumption of constant expected values is in line with Aue et al. (2009). To investigate large sample properties, we make the transformation

$$Q_T(X) = \sup_{z\in[0,1]}\left|\hat{D}\frac{\tau(z)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(z)} - [\mathsf{Var}X]_T\right)\right|$$

with $\tau(z) = [1 + z(T-1)]$.

Theorem 2.1. *Under H_0,*

$$Q_T(X) \to \sup_{z\in[0,1]} |B(z)|,$$

where $B(z)$ is a one-dimensional Brownian Bridge.

The limit distribution of $Q_T(X)$ is well known, see Billingsley (1968), and its quantiles provide an asymptotic test.

3 Applications

3.1 Historical rejection dates

In order to evaluate the quality of the test it is applied to several assets: two stock indices (S&P 500, DAX), a commodity index (CRB Spot Index) and a government bond index (REX), using daily data (final quote) for the time span 01.01.1988 - 01.04.2010. The procedure for the test is as follows. We start at the 20-th available data point and increase the period of time successively for one day. The choice of the starting point is due to the fact that approximately 20 data points are required for a reliable estimation of the volatility. For each of these time intervals the test is applied for $\alpha = 5\%$ and $\alpha = 1\%$, respectively. This procedure is performed until the tests rejects the null hypothesis of constant volatility. Then, the 20-th day after rejection is the new starting point and the procedure is repeated for the remaining time span. We have to wait this 20 days as the volatility cannot be assumed to be constant anymore, if the null hypothesis is rejected. A new reliable estimation requires another 20 data points after the point in time, where the volatility changed. Otherwise, the estimator would be biased as data of two different phases were mixed.
The following tables include the rejection dates of the null hypothesis for both confidence levels:

-Insert Table 1 and 2 about here-

The results seem to be reasonable. For example, the test rejects the null hypothesis during financial crises for all assets ($\alpha = 5\%$) and for both stock indices even for $\alpha = 1\%$. Besides, large differences of the market parameters between the break points can be observed. Figure 1 and Table 3 illustrate this phenomenon for the DAX. Table 3 includes the annualized market parameters (returns and volatilites) for the respective period between two structural breaks. Figure 1 shows the average and the rolling 250-day volatility of the DAX. Besides, the rejection dates are given for $\alpha = 1\%$ and $\alpha = 5\%$.

-Insert Figure 1 and Table 3 about here-

Our results show that the chosen confidence level plays an important role for both, rejection frequency and rejection dates. Consequently, the confidence level has to be chosen carefully in practical applications. We suggest $\alpha = 1\%$ for long-run applications, whereas $\alpha = 5\%$ might be more reasonable for an online risk management.

3.2 A Trading Strategy

The results above show that changes in market parameters can be detected reasonably. In order to investigate the possibility to derive trading strategies, which are based on the proposed test, we perform an out of sample study. In this study, we investigate a simple strategy which applies the proposed test.

The strategy is as follows. The available time span since the last detected change in volatility is used to calculate the historical return which is used as an estimator for the future. Moreover, an asset is allowed to be bought if the last structural break lies 20 days or more in the past. Finally, the capital is uniformly distributed between all allowed assets, whose expected future return is positive.

Portfolio shiftings are done the day after the test rejected in order to design the study as realistic as possible. Moreover, we choose $\alpha = 5\%$ for the test and neglect transaction costs. Besides, we assume a daily rebalancing and neglect currency fluctuations. The results can be found in Figure 2 and Table 4.

The average return of the strategy is higher than three of the underlying assets. Moreover, the portfolio development is relatively stable and only a little money is lost during financial crisis. This result is very remarkable as three risky assets are considered throughout the study. Compared to the average return and volatility of the underlying assets, the return of the strategy is 19,37% higher, while the volatility is 27,68% lower. This hints to singnificant improvement in performance in contrast to a naive strategy with uniformly distributed portfolio weights.

-Insert Figure 2 and Table 4 about here-

Figure 3 shows the resulting portfolio weights of the strategy over time. Especially during crises, a lot of fluctuation can be observed. This ensures the good performance of the strategy, because most of the downward movement in bear markets is automatically avoided.

-Insert Figure 3 about here-

If the suggested proceeding is not applicable (e.g. because it is not allowed to dismiss an asset completely), or the volatility is required to determine the Value at Risk or the price of a financial instrument, other strategies have to be found. For example, intra-day data could be used to estimate the volatility (Barndorff-Nielsen and Shephard, 2004) or subjective but conservative assumptions concerning the volatilities could be made.

4 Summary

In this paper, we introduced and proofed a new test to determine wether variances of time series are constant over time. Thereby, the test statistic is a suitably standardized maximum of cumulative first and second moments. We applied the test to several time series of assets which are relavant for applications in finance and found that the test performs well in these applications. The market parameters fluctuate a lot comparing the different periods between structural breaks.
Moreover, we derived a simple trading strategy, which outperforms a strategy based on uniformly distributed portfolio weights significantly. More precisely, the return increased by 19,37% while the volatility decreased by 27,68%. Nevertheless, the trading strategy is not sophisticated enough for practical applications, yet. This topic will be in focus of our ongoing research. Besides, the question arises if volatility estimators, based on the new test, will improve the performance of portfolio optimization, which depends of course on a reliable estimation of market parameters.

5 Proofs

For the proof of Theorem 2.1, we need a lemma and some notation: Let I be some index, e.g. $I = [\epsilon, 1]$ for some $\epsilon \in [0, 1)$. For an integer $k \geq 1$, let $l_\infty(I, \mathbb{R}^k)$ be the set of all bounded functions $\theta : I \to \mathbb{R}^k$, equipped with supremum norm

$$||\theta||_\infty := \sup_{i \in I} ||\theta(i)||,$$

where $|| \cdot ||$ denotes Euclidean norm.

Lemma 5.1. *Under H_0, in $l_\infty([0, 1], \mathbb{R})$,*

$$\hat{D}\frac{\tau(\cdot)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(\cdot)} - \sigma^2\right) \to_d W(\cdot)$$

where $\sigma^2 = \mathsf{E}(X_1^2) - (\mathsf{E}(X_1))^2$ and $W(z)$ is a one-dimensional Brownian Motion.

Lemma 5.1 requires

Lemma 5.2. *Under H_0, for arbitrary $\epsilon > 0$, in $l_\infty([\epsilon, 1], \mathbb{R})$,*

$$\hat{D}\frac{\tau(\cdot)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(\cdot)} - \sigma^2\right) \to_d W_1(\cdot)$$

where $\sigma^2 = \mathsf{E}(X_1^2) - (\mathsf{E}(X_1))^2$ and $W_1(z)$ is a one-dimensional Brownian Motion.

Lemma 5.2 is proved with a basic theorem on a modified functional delta method.

Theorem 5.3. *Consider a sequence $(\theta_T)_T$ of functions in $l_\infty(I, \mathbb{R}^k)$ converging uniformly to a function $\theta \in l_\infty(I, \mathbb{R}^k)$. Furthermore, let $(s_T)_T$ be a sequence of functions $s_T : I \to \mathbb{R}\backslash\{0\}$ such that $||s_T^{-1}||_\infty \to 0$, and let M_T be stochastic processes on I with values in $\mathbb{R}^k$ and bounded sample paths such that*

$$||Z_T||_\infty = O_p(1) \text{ with } Z_T := s_T(M_T - \theta_T).$$

Furthermore, let $f : \mathbb{R}^k \to \mathbb{R}^l$ be a mapping which is continuously differentiable on an open set $\Omega \subset \mathbb{R}^k$ with derivative Df. Suppose that

$$\overline{\theta(I)} \text{ is a compact subset of } \Omega,$$

where $\overline{\theta(I)}$ stands for the closure of the set $\{\theta(i) : i \in I\}$ in $\mathbb{R}^k$. Then it holds

1. $s_T(\cdot)\left(f(M_T(\cdot)) - f(\theta_T(\cdot))\right) = Df(\theta(\cdot))Z_T(\cdot) + R_T$
   with a stochastic process such that

$$||R_T||_\infty = o_p(1).$$

2. *If Z_T even converges in distribution (in $l_\infty(I, \mathbb{R}^k)$) to a stochastic process Z, then*

$$s_T(\cdot)\left(f(M_T(\cdot)) - f(\theta_T(\cdot))\right) \to_d Df(\theta(\cdot))Z(\cdot).$$

Proof. Assertion 2 directly follows from Assertion 1 with the usual continuous mapping theorem. To prove the expansion from Assertion 1, note that for any $i \in I$,

$$\begin{aligned} R_T(i) &:= s_T(i)\left(f(M_T(i)) - f(\theta_T(i))\right) - Df(\theta(i))Z_T(i) \\ &= s_T(i)\left(f\left(\theta_T(i) + s_T^{-1}(i)Z_T(i)\right) - f(\theta_T(i))\right) - Df(\theta(i))Z_T(i) \\ &= \int_0^1 Df\left(\theta_T(i) + us_T^{-1}(i)Z_T(i)\right)Z_T(i)du - Df(\theta(i))Z_T(i) \\ &= \int_0^1 \left(Df\left(\theta_T(i) + us_T^{-1}(i)Z_T(i)\right) - Df(\theta(i))\right)du \cdot Z_T(i), \end{aligned} \tag{2}$$

provided that

$$r_n := ||\theta_T - \theta||_\infty + ||s_T^{-1}||_\infty ||Z_T||_\infty = o_p(1)$$

is smaller than

$$\rho := \inf_{x \in \overline{\theta(I)}, y \in \mathbb{R}^k \setminus \Omega} ||x - y|| > 0.$$

The latter condition is needed such that (2) is well defined.
Hence

$$||R_T||_\infty \leq \sup\left\{||Df(y) - Df(x)|| : x \in \overline{\theta(I)}, y \in \mathbb{R}^k, ||y - x|| \leq r_T\right\} \cdot ||Z_T||_\infty. \tag{3}$$

Here $||Df(y) - Df(x)||$ is the usual operator norm of the matrix $Df(y) - Df(x)$ in case of $y \in \Omega$. (In case of $y \notin \Omega$ define $||Df(y) - Df(x)|| = \infty$.) One can easily deduce from continuity of $Df(\cdot)$ on Ω, compactness of $\overline{\theta(I)} \in \Omega$ and $r_T = o_p(1)$ that the right hand side of (3) converges to zero in probability. □

Proof of Lemma 5.2

For

$$U_i = \begin{pmatrix} X_i^2 - \mathsf{E}(X_1^2) \\ X_i - \mathsf{E}(X_1) \end{pmatrix}$$

we get with a common multivariate invariance principle, in $l_\infty([\epsilon, 1], \mathbb{R})$,

$$\frac{1}{\sqrt{T}} \sum_{i=1}^{\tau(\cdot)} U_i = \frac{\tau(\cdot)}{\sqrt{T}} \begin{pmatrix} \overline{X^2_{\tau(\cdot)}} - \mathsf{E}(X_1^2) \\ \overline{X_{\tau(\cdot)}} - \mathsf{E}(X_1) \end{pmatrix} \to_d D_1^{1/2} W_2(\cdot).$$

Here, $W_2(z)$ is a two-dimensional Brownian Motion with independent components and $D_1 = \mathsf{E}(U_1 U_1') + 2 \sum_{j=1}^{\infty} \mathsf{E}(U_1 U_{1+j}')$.

Applying Theorem 5.3 with the function $f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = x - y^2$, yields the convergence

$$\frac{\tau(\cdot)}{\sqrt{T}} \left(\overline{X^2_{\tau(\cdot)}} - \left(\overline{X_{\tau(\cdot)}}\right)^2 - \sigma^2 \right) \to_d \begin{pmatrix} 1 & -2\mathsf{E}(X_1) \end{pmatrix} D_1^{1/2} W_2(\cdot) =: B \; W_2(\cdot)$$

resp.

$$\frac{\tau(\cdot)}{\sqrt{T}} \left([\mathsf{Var} X]_{\tau(\cdot)} - \sigma^2 \right) \to_d (BB')^{1/2} W_1(\cdot).$$

The lemma then follows with the continuous mapping theorem and the fact that D_1 can be consistently estimated with a kernel estimator from Davidson and de Jong (2000). □

Proof of Lemma 5.1

With $W_T(z) = \hat{D} \frac{\tau(z)}{\sqrt{T}} \left([\mathsf{Var} X]_{\tau(z)} - \sigma^2 \right)$ we define the following functions:

$$W_T^\epsilon(z) = \begin{cases} W_T(z), & z \geq \epsilon \\ 0 & z < \epsilon \end{cases},$$

$$W^\epsilon(z) = \begin{cases} W_1(z), & z \geq \epsilon \\ 0 & z < \epsilon \end{cases}.$$

Lemma 5.2 implies that

$$W_T^\epsilon(\cdot) \to_d W^\epsilon(\cdot)$$

in $l_\infty([0, 1], \mathbb{R})$ and also

$$W^\epsilon(\cdot) \to_d W_1(\cdot)$$

for rational $\epsilon \to 0$ in $l_\infty([\epsilon, 1], \mathbb{R})$.

The convergence of $W_T(\cdot)$ in $l_\infty([0, 1], \mathbb{R})$ follows with Theorem 4.2 in Billingsley (1968) if we can show that

$$\lim_{\epsilon \to 0} \limsup_{T \to \infty} \mathbb{P}(\sup_{z \in [0,1]} |W_T^\epsilon(z) - W_T(z)| \geq \eta) = \lim_{\epsilon \to 0} \limsup_{T \to \infty} \mathbb{P}(\sup_{z \in [0,\epsilon]} |W_T(z)| \geq \eta) = 0$$

for all $\eta > 0$.

For this, note that

$$\begin{aligned} W_T(z) &= \hat{D}\frac{\tau(z)}{\sqrt{T}}\left(\overline{X^2_{\tau(z)}} - \mathsf{E}(X_1^2)\right) - \hat{D}\frac{\tau(z)}{\sqrt{T}}\left(\left(\overline{X_{\tau(z)}}\right)^2 - (\mathsf{E}(X_1))^2\right) \\ &= \hat{D}\frac{\tau(z)}{\sqrt{T}}\left(\overline{X^2_{\tau(z)}} - \mathsf{E}(X_1^2)\right) - \hat{D}\frac{\tau(z)}{\sqrt{T}}\left(\overline{X_{\tau(z)}} - \mathsf{E}(X_1)\right)\left(\overline{X_{\tau(z)}} + \mathsf{E}(X_1)\right). \end{aligned}$$

We can deduce that $\sup_{z\in[0,\epsilon]} |W_T(z)|$ converges in distribution to a random variable that is smaller than

$$C_1 \sup_{z\in[0,\epsilon]} |W_1^*(z)| + C_2 \sup_{z\in[0,\epsilon]} |W_1^{**}(z)|,$$

where C_1 and C_2 are two constants and $W_1^*(z)$ and $W_1^{**}(z)$ are two Brownian motions, respectively. This sum becomes arbitrarily small for $\epsilon \to 0$ and so the lemma is proved. □

Proof of Theorem 2.1

We have

$$\begin{aligned} &\hat{D}\frac{\tau(z)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(z)} - [\mathsf{Var}X]_T\right) \\ &= \hat{D}\frac{\tau(z)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(z)} - \sigma^2\right) + \hat{D}\frac{\tau(z)}{\sqrt{T}}\left(\sigma^2 - [\mathsf{Var}X]_T\right) \\ &= \hat{D}\frac{\tau(z)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(z)} - \sigma^2\right) - \frac{\tau(z)}{T}\hat{D}\frac{\tau(1)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(1)} - \sigma^2\right) \end{aligned}$$

and thus get

$$\hat{D}\frac{\tau(\cdot)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(\cdot)} - [\mathsf{Var}X]_T\right) \to_d A(\cdot)$$

with $A(z) = W_1(z) - zW_1(1)$. This is a representation of a one-dimensional Brownian Bridge. Now, the theorem follows with the continuous mapping theorem. □

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S&P	DAX	REX	CRB
03.02.1988	29.01.1988	12.02.1990	15.03.1988
03.02.1989	28.10.1988	06.06.1994	03.06.1988
12.02.1993	28.11.1988	03.04.1995	07.09.1988
30.09.1993	01.02.1989	01.05.1995	02.02.1990
18.07.1996	19.04.1989	17.11.1995	09.11.1992
10.03.1997	07.06.1989	19.02.1996	05.08.1994
02.02.2005	20.05.1993	28.07.1998	02.12.1996
19.10.2007	17.06.1993	12.12.2000	07.01.1998
28.10.2008	14.03.1994	14.05.2001	19.05.1998
09.10.2009	19.08.1997	01.08.2003	18.06.2001
	27.11.2002	18.10.2004	19.10.2001
	07.04.2003	29.02.2008	23.05.2003
	17.09.2003	27.01.2010	23.06.2003
	06.02.2004		21.07.2003
	07.03.2005		10.05.2004
	14.07.2006		02.09.2008
	06.10.2006		
	14.03.2007		
	24.07.2007		
	06.11.2007		
	24.11.2008		
	28.08.2009		

Table 1: Rejection Dates ($\alpha = 5\%$)

S&P	DAX	REX	CRB
02.12.1993	29.01.1988	10.10.1994	17.11.1998
27.03.1997	12.07.1989	18.03.2009	29.05.2009
15.08.2005	04.10.1994		26.06.2009
11.12.2007	21.10.1997		
01.12.2008	24.03.2003		
10.09.2009	23.12.2004		
	06.10.2008		

Table 2: Rejection Dates ($\alpha = 1\%$)

DAX	Returns	Volatilities
29.01.1988 - 28.10.1988	44,09%	16,07%
28.10.1988 - 28.11.1988	-34,9%	12,63%
28.11.1988 - 01.02.1989	15,34%	13,29%
01.02.1989 - 19.04.1989	27,35%	12,84%
19.04.1989 - 07.06.1989	9,99%	8,96%
07.06.1989 - 20.05.1993	3,26%	20,04%
20.05.1993 - 17.06.1993	54,55%	7,87%
17.06.1993 - 14.03.1994	28,53%	16,41%
14.03.1994 - 19.08.1997	18,48%	14,59%
19.08.1997 - 27.11.2002	-4,46%	28,87%
27.11.2002 - 07.04.2003	-49,58%	43,45%
07.04.2003 - 17.09.2003	63,03%	25,92%
17.09.2003 - 06.02.2004	29,14%	19,17%
06.02.2004 - 07.03.2005	8,63%	15,08%
07.03.2005 - 14.07.2006	15,73%	14,47%
14.07.2006 - 06.10.2006	39,39%	14,67%
06.10.2006 - 14.03.2007	19,13%	11,66%
14.03.2007 - 24.07.2007	48,34%	15,69%
24.07.2007 - 06.11.2007	-5,78%	16,18%
06.11.2007 - 24.11.2008	-58,16%	33,42%
24.11.2008 - 28.08.2009	35,39%	33,66%
28.08.2009 - 01.04.2010	21,25%	18,55%

Table 3: Rejection Dates and annualized market parameters ($\alpha = 5\%$)

	Strategy	CRB	REX	DAX	S&P	Average
Return p.a.	6,78%	2,23%	5,76%	8,09%	6,62%	5,68%
Volatility p.a.	9,09%	6,51%	3,30%	22,65%	17,84%	12,57%

Table 4: Summary statistics for all indices and our strategy

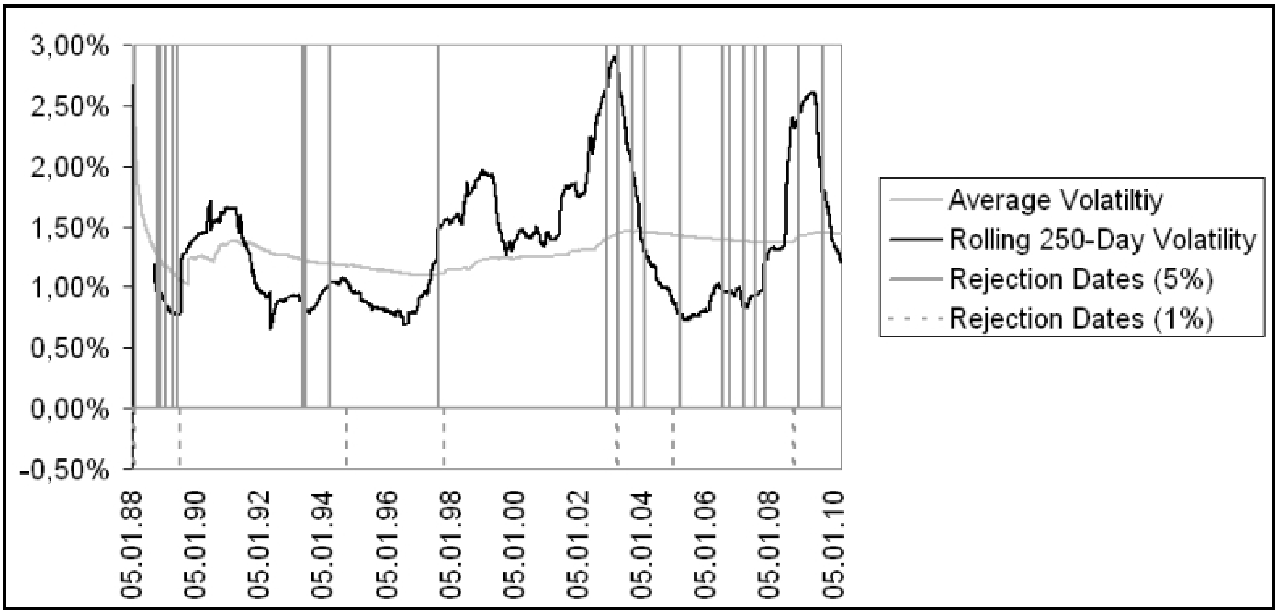

Figure 1: DAX, Volatility and Structural Breaks

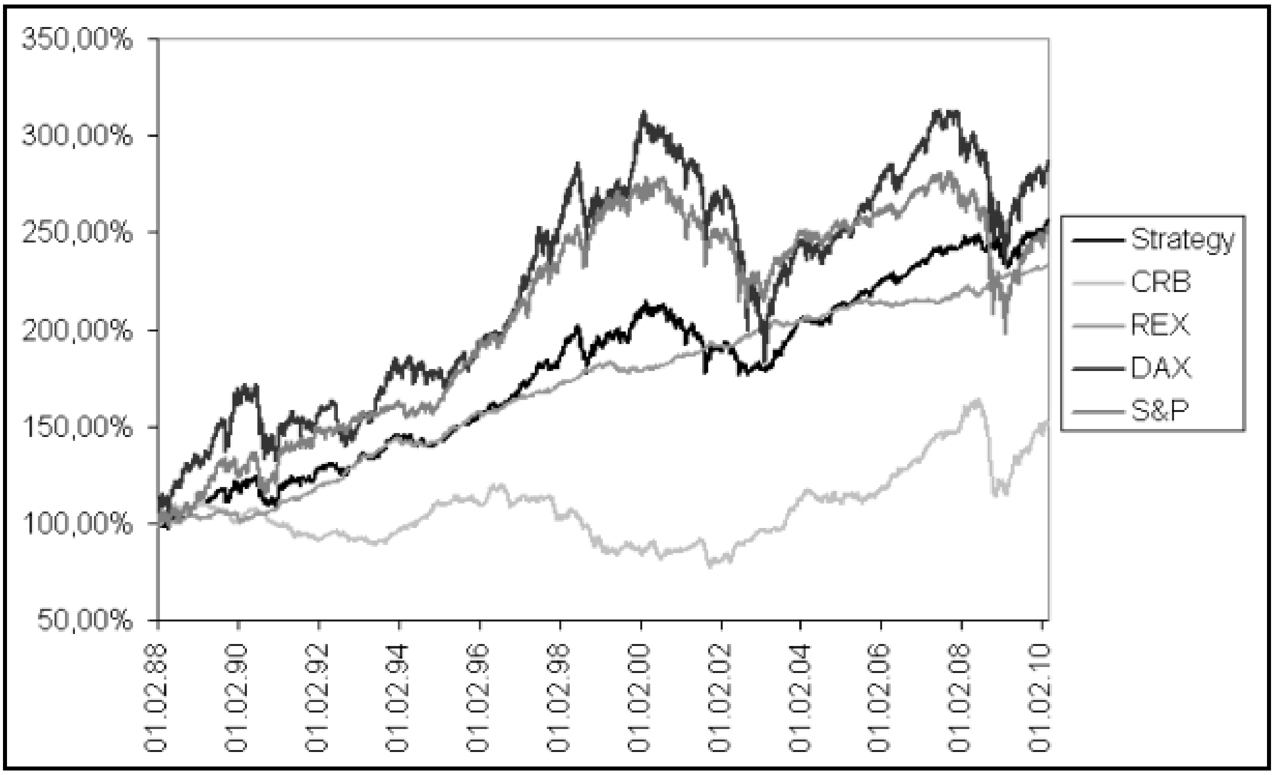

Figure 2: Strategy and underlying assets

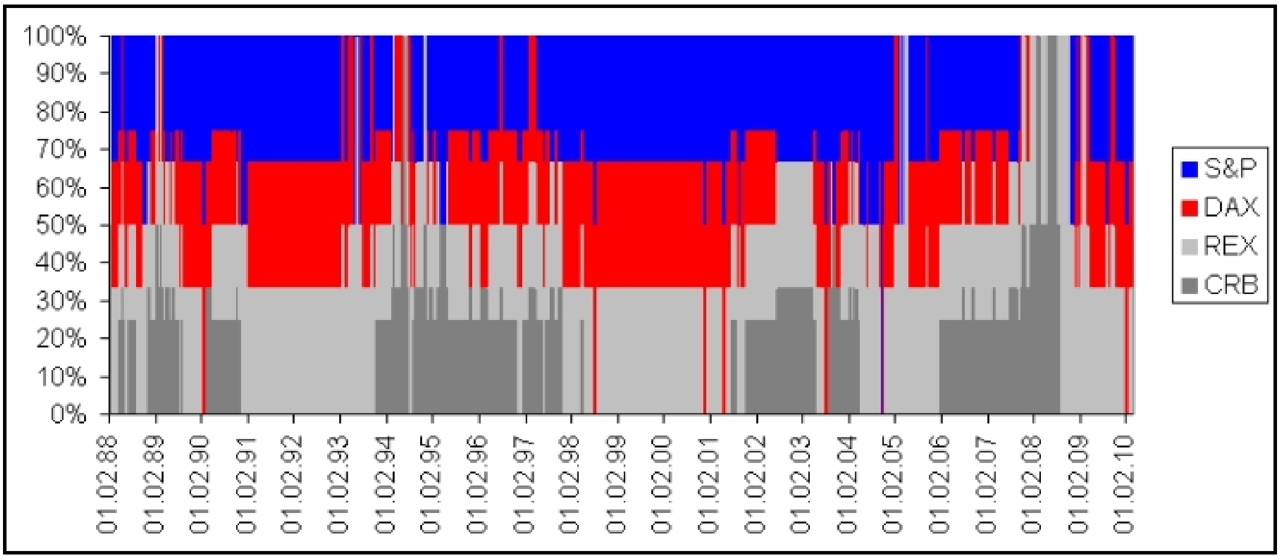

Figure 3: Portfolio Weights