

# Matrix measures on the unit circle, moment spaces, orthogonal polynomials and the Geronimus relations

Holger Dette

Ruhr-Universität Bochum

Fakultät für Mathematik

44780 Bochum, Germany

e-mail: holger.dette@rub.de

Jens Wagoner

Ruhr-Universität Bochum

Fakultät für Mathematik

44780 Bochum, Germany

e-mail: jens.wagoner@rub.de

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## Abstract

We study the moment space corresponding to matrix measures on the unit circle. Moment points are characterized by non-negative definiteness of block Toeplitz matrices. This characterization is used to derive an explicit representation of orthogonal polynomials with respect to matrix measures on the unit circle and to present a geometric definition of canonical moments. It is demonstrated that these geometrically defined quantities coincide with the Verblunsky coefficients, which appear in the Szegő recursions for the matrix orthogonal polynomials. Finally, we provide an alternative proof of the Geronimus relations which is based on a simple relation between canonical moments of matrix measures on the interval  $[-1,1]$  and the Verblunsky coefficients corresponding to matrix measures on the unit circle.

Keyword and Phrases: Matrix measures on the unit circle, orthogonal polynomials, canonical moments, Verblunsky coefficients, Geronimus relations.

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## 1 Introduction

In recent years considerable interest has been shown in moment problems, orthogonal polynomials, continued fractions and quadrature formulas corresponding to matrix measures on the real line

or on the unit circle. Early work dates back to Krein (1949), while more recent results on matrix measures on the real line can be found in the papers of Rodman (1990), Duran (1995, 1996) and Defez et al. (2000) among many others. Additionally, several authors have discussed matrix measures on the unit circle [see Delsarte et al. (1978), Geronimo (1981), Marcellán and Rodríguez (1989), Sinap and Van Assche (1994, 1996), Yakhlef and Marcellán (2001, 2002), Cantero et al. (2003)].

The purpose of the present paper is to investigate some geometric properties of the moment space corresponding to matrix measures on the unit circle. In Section 2 we present a characterization of the moment space in terms of nonnegative definiteness of block Toeplitz matrices. We also provide a geometric definition of canonical moments of matrix measures on the unit circle, which generalizes the scalar case discussed by Dette and Studden (1997) in a nontrivial way. In Section 3 an explicit determinantal representation of orthogonal matrix polynomials with respect to matrix measures on the unit circle is presented, which generalizes the classical representation in the one-dimensional case [see e.g. Geronimus (1962)]. These results are used to identify the canonical moments as Verblunsky coefficients, which appear in the Szegő relations for the corresponding orthonormal and reversed matrix polynomials [see Delsarte et al. (1978), Sinap and Van Assche (1996) or Damanik et al. (2008)]. In particular our results provide a geometric definition of Verblunsky coefficients corresponding to matrix measures on the unit circle. Roughly speaking, the Verblunsky coefficient of order  $m$  can be characterized as the distance of the  $m$ th trigonometric moment to a center of a matrix disc relative to the diameter of this disc (see Section 3 for more details). Finally, in Section 4 these results are used to present an alternative proof of the Geronimus relations for monic orthogonal polynomials, which describe the relation between the coefficients in the three-term recursive relation of orthogonal polynomials with respect to a matrix measure on a compact interval and the coefficients in the Szegő recursion of an associated matrix measure on the unit circle.

## 2 The moment space of matrix measure on the unit circle

A matrix measure  $\mu$  on the unit circle is defined as a  $p \times p$  matrix of complex valued Borel measures  $\mu = (\mu_{ij})_{i,j=1,\dots,p}$  on the unit circle  $\partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$  such that for each Borel set  $A \subset \partial\mathbb{D}$  the matrix  $\mu(A)$  is nonnegative definite, i.e.  $\mu(A) \geq 0$ . Throughout this paper we use the usual parametrization  $z = e^{i\theta}$ ,  $\theta \in [-\pi, \pi)$  and the notation  $\mu(\theta)$  for the sake of simplicity.

The  $k$ th moment of a matrix measure  $\mu$  on the unit circle is defined by

$$(2.1) \quad \Gamma_k = \Gamma_k(\mu) = \int_{-\pi}^{\pi} e^{ik\theta} d\mu(\theta) = \alpha_k + i\beta_k \quad k \in \mathbb{Z}$$

where  $\alpha_k = \alpha_k(\mu) = \int_{-\pi}^{\pi} \cos(k\theta) d\mu(\theta)$ ,  $\beta_k = \beta_k(\mu) = \int_{-\pi}^{\pi} \sin(k\theta) d\mu(\theta)$  ( $k = 0, 1, \dots$ ) are the trigonometric moments and the dependence on the given measure  $\mu$  is omitted in the notation, whenever it is clear from the context. Throughout this paper let  $m \in \mathbb{N}_0$   $\lambda(\mu) = (\alpha_0, \alpha_1, \beta_1, \dots, \alpha_m, \beta_m) \in (\mathbb{C}^{p \times p})^{2m+1}$  denote the vector of trigonometric moments of order  $m$  and define

$$(2.2) \quad \mathcal{M}_{2m+1} = \{\lambda(\mu) \mid \mu \text{ is a matrix measure on } \partial\mathbb{D}\} \subset (\mathbb{C}^{p \times p})^{2m+1}$$

as the  $(2m+1)$ th moment space of matrix measures on the unit circle. The set  $\mathcal{M}_{2m+1}$  and its interior  $\text{Int}(\mathcal{M}_{2m+1})$  can be characterized as follows.

**Theorem 2.1**  $\lambda = (\alpha_0, \alpha_1, \beta_1, \dots, \alpha_m, \beta_m) \in \mathcal{M}_{2m+1}$  if and only if

$$(2.3) \quad \sum_{i=0}^m \sum_{j=0}^m \text{trace}(B_i B_j^* \Gamma_{i-j}) \geq 0 \quad \forall B_0, \dots, B_m \in \mathbb{C}^{p \times p},$$

where the matrices  $\Gamma_{-m}, \Gamma_{-m+1}, \dots, \Gamma_m$  are defined in (2.1).

$\lambda = (\alpha_0, \alpha_1, \beta_1, \dots, \alpha_m, \beta_m) \in \text{Int}(\mathcal{M}_{2m+1})$  if and only if there is strict inequality in (2.3) except if  $B_0 = \dots = B_m = 0$ .

**Proof:** We start with a proof of the first part. Assume that  $\lambda \in \mathcal{M}_{2m+1}$  and consider matrices  $B_0, \dots, B_m \in \mathbb{C}^{p \times p}$ . With the notation

$$(2.4) \quad B(\theta) = \sum_{k=0}^m B_k e^{ik\theta} \quad (\theta \in [-\pi, \pi))$$

it follows that the polynomial  $P(\theta) = B(\theta)(B(\theta))^*$  is obviously nonnegative definite, i.e.

$$(2.5) \quad P(\theta) = B(\theta)(B(\theta))^* \geq 0 \quad \forall \theta \in [-\pi, \pi).$$

A straightforward calculation shows that the polynomial  $P$  can be represented as

$$(2.6) \quad P(\theta) = D_0 + \sum_{k=1}^m D_k \cos(k\theta) + E_k \sin(k\theta),$$

where the hermitian  $p \times p$  matrices  $D_0, \dots, D_m, E_1, \dots, E_m$  are defined by  $D_0 = A_0$ , and for  $k = 1, \dots, m$

$$D_k = A_k + A_{-k}, \quad E_k = i(A_k - A_{-k})$$

and

$$A_k = \sum_{l=0}^{m-k} B_{k+l} B_l^* \quad \text{and} \quad A_{-k} = A_k^*.$$

Because it is easy to see that the moment space  $\mathcal{M}_{2m+1}$  is the convex hull of the set

$$\left\{ (aa^*, \cos(\theta)aa^*, \sin(\theta)aa^*, \dots, \cos(m\theta)aa^*, \sin(m\theta)aa^*) \mid a \in \mathbb{C}^p, \theta \in [-\pi, \pi] \right\},$$

a similar argument as in Corollary 2.2 of Dette and Studden (2002) now shows that (2.5) and (2.6) imply

$$\begin{aligned} 0 &\leq \text{trace}(D_0\alpha_0) + \sum_{k=1}^m \text{trace}(D_k\alpha_k) + \text{trace}(E_k\beta_k) \\ &= \text{trace} \left( \int_{-\pi}^{\pi} d(D_0\mu(\theta)) + \sum_{k=1}^m \int_{-\pi}^{\pi} \cos(k\theta) d(D_k\mu(\theta)) + \int_{-\pi}^{\pi} \sin(k\theta) d(E_k\mu(\theta)) \right) \\ &= \text{trace} \left( \int_{-\pi}^{\pi} \sum_{k=-m}^m e^{ik\theta} d(A_k\mu(\theta)) \right) \\ &= \text{trace} \left( \int_{-\pi}^{\pi} \sum_{k=0}^m e^{ik\theta} d \left( \sum_{l=0}^{m-k} B_{k+l} B_l^* \mu(\theta) \right) + \int_{-\pi}^{\pi} \sum_{k=1}^m e^{-ik\theta} d \left( \sum_{l=0}^{m-k} B_l B_{k+l}^* \mu(\theta) \right) \right) \\ &= \text{trace} \left( \sum_{k=0}^m \sum_{l=0}^m \int_{-\pi}^{\pi} e^{i(k-l)\theta} d(B_k B_l^* \mu(\theta)) \right) \\ &= \sum_{k=0}^m \sum_{l=0}^m \text{trace}(B_k B_l^* \Gamma_{k-l}), \end{aligned}$$

which proves (2.3). On the other hand assume that the inequality (2.3) is satisfied for all matrices  $B_0, \dots, B_m \in \mathbb{C}^{p \times p}$  and consider a nonnegative definite matrix polynomial

$$(2.7) \quad P(\theta) = D_0 + \sum_{k=1}^m D_k \cos(k\theta) + E_k \sin(k\theta) \geq 0 \quad \forall \theta \in [-\pi, \pi].$$

with hermitian matrices  $D_0, \dots, D_m, E_1, \dots, E_m \in \mathbb{C}^{p \times p}$ . It now follows from Malyshev (1982) that there exists a matrix polynomial

$$B(\theta) = \sum_{k=0}^m B_k e^{ik\theta}$$

such that  $P(\theta) = B(\theta)(B(\theta))^*$ , and the same calculation as in the first part of the proof yields

$$\text{trace}(D_0\alpha_0) + \sum_{k=1}^m \text{trace}(D_k\alpha_k) + \text{trace}(E_k\beta_k) = \sum_{i=0}^m \sum_{j=0}^m \text{trace}(B_i B_j^* \Gamma_{i-j}) \geq 0.$$

By similar arguments as in Lemma 2.3 of Dette and Studden (2002) it follows that this is sufficient for  $\lambda \in \mathcal{M}_{2m+1}$ .

Finally, the second part of the Theorem is shown similarly observing the fact that  $(\alpha_0, \alpha_1, \beta_1, \dots, \alpha_m, \beta_m) \in \text{Int}(\mathcal{M}_{2m+1})$  if and only if

$$\text{trace}(D_0\alpha_0) + \sum_{k=1}^m \text{trace}(D_k\alpha_k) + \text{trace}(E_k\beta_k) > 0$$

for any nonnegative definite polynomial  $P(\theta)$  of the form (2.6) with  $P(\theta) \neq 0 \forall \theta \in [-\pi, \pi]$ . This characterization can be shown by the same arguments as presented in Dette and Studden (2002) who proved a corresponding statement for the moment space of matrix measures on the interval  $[0, 1]$ .  $\square$ .

Throughout this paper let

$$(2.8) \quad T_m = T_m(\mu) = \begin{pmatrix} \Gamma_0 & \cdots & \Gamma_m \\ \vdots & \ddots & \vdots \\ \Gamma_{-m} & \cdots & \Gamma_0 \end{pmatrix} \in \mathbb{C}^{p(m+1) \times p(m+1)}$$

denote the Block Toeplitz matrix, where the blocks  $\Gamma_i = \Gamma_i(\mu)$  ( $i = -m, \dots, m$ ) are the moments of a matrix measure  $\mu$  on the unit circle defined by (2.1) (note that  $T_m$  is hermitian). The following characterization of the moment space  $\mathcal{M}_{2m+1}$  by nonnegative definiteness of Toeplitz matrices is now easily obtained.

**Corollary 2.2** *Assume that  $\lambda = (\alpha_0, \alpha_1, \beta_1, \dots, \alpha_m, \beta_m) \in (\mathbb{R}^{p \times p})^{2m+1}$  and that  $T_m$  is defined by (2.8) with  $\Gamma_k = \alpha_k + i\beta_k$  and  $\Gamma_{-k} = \alpha_k - i\beta_k$ .*

(a)  $\lambda \in \mathcal{M}_{2m+1}$  if and only if  $T_m \geq 0$ .

(b)  $\lambda \in \text{Int}(\mathcal{M}_{2m+1})$  if and only if  $T_m > 0$ .

**Proof:** We only proof part (a); part (b) is shown by similar arguments. First assume that  $\lambda \in \mathcal{M}_{2m+1}$ , then we obtain from Theorem 2.1 for all matrices  $B_0, \dots, B_m \in \mathbb{C}^{p \times p}$

$$\sum_{i=0}^m \sum_{j=0}^m \text{trace}(B_i B_j^* \Gamma_{j-i}) \geq 0.$$

Consequently, if  $a_0, \dots, a_m \in \mathbb{C}^p$ ,  $a = (a_0^T, \dots, a_m^T)^T \in \mathbb{C}^{p(m+1)}$  we put  $B_i = (a_i, 0, \dots, 0) \in \mathbb{C}^{p \times p}$  ( $i = 0, \dots, m$ ) and it follows

$$a^* T_m a = \text{trace}(a a^* T_m) = \sum_{i=0}^m \sum_{j=0}^m \text{trace}(a_i a_j^* \Gamma_{j-i}) = \sum_{i=0}^m \sum_{j=0}^m \text{trace}(B_i B_j^* \Gamma_{j-i}) \geq 0,$$

which shows that the matrix  $T_m$  is nonnegative definite. To prove the converse assume that  $T_m \geq 0$ , i.e.

$$(2.9) \quad 0 \leq a^* T_m a = \sum_{i=0}^m \sum_{j=0}^m \text{trace}(a_i a_j^* \Gamma_{j-i}).$$

for all  $a = (a_0^T, \dots, a_m^T)^T \in \mathbb{C}^{p(m+1)}$ . If  $B_0, \dots, B_m \in \mathbb{C}^{p \times p}$ , and  $a_j^{(i)}$  denotes the  $i$ th column of the matrix  $B_j$  ( $j = 0, \dots, m$ ,  $i = 1, \dots, p$ ), then

$$B_j B_k^* = \sum_{i=1}^p a_j^{(i)} \left( a_k^{(i)} \right)^*$$

and we obtain from (2.9)

$$\sum_{i=0}^m \sum_{j=0}^m \text{trace}(B_i B_j^* \Gamma_{j-i}) = \sum_{k=1}^p \sum_{i=0}^m \sum_{j=0}^m \text{trace} \left( a_i^{(k)} \left( a_j^{(k)} \right)^* \Gamma_{j-i} \right) \geq 0.$$

By Theorem 2.1 it follows that  $\lambda \in \mathcal{M}_{2m+1}$ , which completes the proof of the Corollary.  $\square$

With the aid of Theorem 2.1 and Corollary 2.2 we are now able to define geometrically canonical moments for matrix measures on the unit circle. It turns out that these geometrically defined quantities are exactly the Verblunsky coefficients of matrix measures on the unit circle as introduced by Damanik et al. (2008) (see Section 3 where we prove this identity). For this purpose let  $W$  denote a  $p \times p$  matrix and define

$$(2.10) \quad A = A(W) = \begin{pmatrix} \Gamma_0 & \Gamma_1 & \cdots & \Gamma_m & W \\ \Gamma_{-1} & \Gamma_0 & \cdots & \Gamma_{m-1} & \Gamma_m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Gamma_{-m} & \Gamma_{-m+1} & \cdots & \Gamma_0 & \Gamma_1 \\ W^* & \Gamma_{-m} & \cdots & \Gamma_{-1} & \Gamma_0 \end{pmatrix} \in \mathbb{C}^{p(m+2) \times p(m+2)}.$$

Let  $\Gamma^{(m)} = (\Gamma_{-m}, \Gamma_{-m+1}, \dots, \Gamma_{m-1}, \Gamma_m) \in (\mathbb{C}^{p \times p})^{2m+1}$  denote a vector of moments of a matrix measure on the unit circle, that is  $(\alpha_0, \alpha_1, \beta_1, \dots, \alpha_m, \beta_m) \in \mathcal{M}_{2m+1}$ , where  $\Gamma_k = \alpha_k + i\beta_k$ . Define  $\mathcal{P}_{\Gamma^{(m)}}$  as the set of all matrix measures  $\mu$  on the unit circle with moments of order  $m$  given by  $\Gamma^{(m)}$ ,

that is  $\Gamma_j = \int_{-\pi}^{\pi} e^{ik\theta} d\mu(\theta)$  ( $j = -m, \dots, m$ ). By Corollary 2.2 it follows that the matrix  $W$  is the  $(m+2)$ th moment of a matrix measure  $\mu \in \mathcal{P}_{\Gamma(m)}$  if and only if  $A(W) \geq 0$ . We assume without loss of generality that  $(\alpha_0, \alpha_1, \beta_1, \dots, \alpha_m, \beta_m) \in \text{Int}(\mathcal{M}_{2m+1})$  which is equivalent to  $T_m > 0$  by Corollary 2.2. From Theorem 1 in Fritzsche and Kirstein (1987) it follows that

$$A(W) \geq 0$$

if and only if there exists a  $p \times p$  matrix  $U$  with  $UU^* \leq I_p$  such that the matrix  $W$  can be represented as

$$(2.11) \quad W = (\Gamma_1 \dots \Gamma_m) T_{m-1}^{-1} (\Gamma_{-m} \dots \Gamma_{-1})^* + L_m^{1/2} U R_m^{1/2},$$

where the matrices  $L_m$  and  $R_m$  are defined by

$$(2.12) \quad L_m = \Gamma_0 - (\Gamma_1 \dots \Gamma_m) T_{m-1}^{-1} (\Gamma_1 \dots \Gamma_m)^*,$$

$$(2.13) \quad R_m = \Gamma_0 - (\Gamma_{-m} \dots \Gamma_{-1}) T_{m-1}^{-1} (\Gamma_{-m} \dots \Gamma_{-1})^*,$$

respectively. Note that the matrices  $L_m$  and  $R_m$  are Schur complements of the positive definite matrix  $T_m$  and as a consequence are also positive definite [see Horn and Johnson (1985)]. This means that the matrix  $W$  is the  $(m+2)$ th moment of the matrix measure  $\mu \in \mathcal{P}_{\Gamma(m)}$ , if and only if it is an element of the “ball”

$$(2.14) \quad K_m := \{W \in \mathbb{C}^{p \times p} \mid L_m^{-1/2} (W - M_m) R_m^{-1/2} = U, UU^* \leq I_p\},$$

where the “center” of the ball is given by the matrix

$$(2.15) \quad M_m = (\Gamma_1 \dots \Gamma_m) T_{m-1}^{-1} (\Gamma_{-m} \dots \Gamma_{-1})^*.$$

We are now in a position to define the canonical moments of a matrix measure on the unit circle (or Verblunsky coefficients as shown in Section 3).

**Definition 2.3** *Let  $\mu$  denote a matrix measure on the unit circle with moments  $\Gamma_k = \alpha_k + i\beta_k$  ( $k \geq 0$ ),  $\lambda_{2m+1}(\mu) = (\alpha_0, \alpha_1, \beta_1, \dots, \alpha_m, \beta_m) \in (\mathbb{R}^{p \times p})^{m+1}$  ( $m \geq 0$ ) and define*

$$(2.16) \quad N(\mu) = \min \{m \in \mathbb{N} \mid \lambda_{2m+1}(\mu) \in \partial \mathcal{M}_{2m+1}\},$$

*as the minimum number  $m \in \mathbb{N}$  such that  $\lambda_{2m+1}$  is a boundary point of the moment space  $\mathcal{M}_{2m+1}$  (if  $\lambda_{2m+1} \in \text{Int}(\mathcal{M}_{2m+1})$  for all  $m \in \mathbb{N}$  we put  $N(\mu) = \infty$ ). For each  $m = 0, \dots, N(\mu) - 1$  the*

quantity

$$\begin{aligned}
(2.17) \quad A_{m+1} = A_{m+1}(\mu) &= L_m^{-1/2} (\Gamma_{m+1} - M_m) R_m^{-1/2} \\
&= [\Gamma_0 - (\Gamma_1, \dots, \Gamma_m) T_{m-1}^{-1} (\Gamma_1, \dots, \Gamma_m)^*]^{-1/2} \\
&\quad \times (\Gamma_{m+1} - (\Gamma_1, \dots, \Gamma_m) T_{m-1}^{-1} (\Gamma_{-m}, \dots, \Gamma_{-1})^*) \\
&\quad \times [\Gamma_0 - (\Gamma_{-m}, \dots, \Gamma_{-1}) T_{m-1}^{-1} (\Gamma_{-m}, \dots, \Gamma_{-1})^*]^{-1/2}
\end{aligned}$$

is called the  $(m + 1)$ th canonical moment of the matrix measure  $\mu$ .

Definition 2.3 is a generalization of the definition of canonical moments of scalar measures on the unit circle in Dette and Studden (1997). In general the explicit representation of the canonical moments in terms of the moments  $\Gamma_0, \Gamma_1, \dots$  is very difficult. For example if  $m = 0$  we have

$$(2.18) \quad A_1 = \Gamma_0^{-1/2} \Gamma_1 \Gamma_0^{-1/2}$$

and in the case  $m = 1$  we obtain from Definition 2.3

$$(2.19) \quad A_2 = (\Gamma_0 - \Gamma_1 \Gamma_0^{-1} \Gamma_{-1})^{-1/2} (\Gamma_2 - \Gamma_1 \Gamma_0^{-1} \Gamma_1) (\Gamma_0 - \Gamma_{-1} \Gamma_0^{-1} \Gamma_{-1})^{-1/2}$$

In the following section we will demonstrate that the quantities defined by Definition 2.3 are the well known Verblunsky coefficients, which are usually obtained from the recursive relations of the orthonormal polynomials with respect to matrix measures on the unit circle [see for example Delsarte et al. (1978) where these matrices do not have any special name, Sinap and Van Assche (1996) where they are called reflection coefficients or Damanik et al. (2008)]. For this purpose we use an explicit determinant representation of the matrix orthogonal polynomials, which is of interest by itself and given in the following section.

### 3 Orthogonal matrix polynomials

A  $p \times p$  matrix polynomial is a  $p \times p$  matrix with polynomial entries. It is of degree  $n$  if all the polynomial entries are of degree less than or equal to  $n$  and is usually written in the form

$$(3.1) \quad P(z) = \sum_{i=0}^n A_i z^i.$$

with coefficients  $A_i \in \mathbb{C}^{p \times p}$  and  $z \in \mathbb{C}$ . Recall that for matrix polynomials  $P$  and  $Q$  the right and left inner product are defined by

$$(3.2) \quad \langle P, Q \rangle_R = \int_{-\pi}^{\pi} P(e^{i\theta})^* d\mu(\theta) Q(e^{i\theta}),$$

$$(3.3) \quad \langle P, Q \rangle_L = \int_{-\pi}^{\pi} P(e^{i\theta}) d\mu(\theta) Q(e^{i\theta})^*,$$

respectively [see for example Sinap and Van Assche (1996)]. The matrix polynomials  $P$  and  $Q$  are called orthogonal with respect to the right inner product  $\langle \cdot, \cdot \rangle_R$  if

$$(3.4) \quad \langle P, Q \rangle_R = 0$$

and orthogonality with respect to the left inner product  $\langle \cdot, \cdot \rangle_L$  is defined analogously. The matrix polynomials  $P_0(z), P_1(z), P_2(z), \dots$  are called orthonormal with respect to the right inner product if for each  $m \in \mathbb{N}_0$   $P_m(z)$  is of degree  $m$ ,  $P_m(z)$  and  $P_{m'}(z)$  are orthogonal with respect to  $\langle \cdot, \cdot \rangle_R$  whenever  $m \neq m'$  and

$$(3.5) \quad \langle P_m, P_m \rangle_R = I_p,$$

where  $I_p$  denotes the  $p \times p$  identity matrix. Orthonormal polynomials with respect to the left inner product  $\langle \cdot, \cdot \rangle_L$  are defined analogously. Orthonormal polynomials with respect to the inner products  $\langle \cdot, \cdot \rangle_R$  and  $\langle \cdot, \cdot \rangle_L$  are determined uniquely up to multiplication by unitary matrices. In the following discussion we will derive an explicit representation of these polynomials in terms of the moments of matrix measure  $\mu$ . A representation very similar to the well known determinant representation in the scalar case [see for example Geronimus (1946)] was given by Miranian (2005, 2009) in the matrix case on the real line and on the circle. Here we develop another explicit representation using determinants.

For this purpose consider a matrix measure  $\mu$  on the unit circle with moments  $\Gamma_{-m}, \dots, \Gamma_m$  and recall the definition of the corresponding block Toeplitz matrix  $T_m$  in (2.8). We define for  $m \in \mathbb{N}$  matrix polynomials by

$$(3.6) \quad \Psi_m^R(z) = (T_{ij}^R(z))_{i,j=1,\dots,p},$$

$$(3.7) \quad \Psi_m^L(z) = (T_{ij}^L(z))_{i,j=1,\dots,p},$$

where the elements  $T_{ij}^R(z)$  and  $T_{ij}^L(z)$  in these matrices are given by the determinants

$$(3.8) \quad T_{ij}^R(z) = \begin{vmatrix} \Gamma_0 & \Gamma_1 & \dots & \Gamma_m \\ \Gamma_{-1} & \Gamma_0 & \dots & \Gamma_{m-1} \\ \vdots & \vdots & & \vdots \\ \Gamma_{-m+1} & \Gamma_{-m+2} & \dots & \Gamma_1 \\ \Gamma_{-m}^{ij}(z) & \Gamma_{-m+1}^{ij}(z) & \dots & \Gamma_0^{ij}(z) \end{vmatrix}; \quad i, j = 1, \dots, p$$

and

$$(3.9) \quad T_{ij}^L(z) = \begin{vmatrix} \tilde{\Gamma}_0^{ij}(z) & \Gamma_1 & \dots & \Gamma_m \\ \tilde{\Gamma}_{-1}^{ij}(z) & \Gamma_0 & \dots & \Gamma_{m-1} \\ \vdots & \vdots & & \vdots \\ \tilde{\Gamma}_{-m}^{ij}(z) & \Gamma_{-m+1} & \dots & \Gamma_0 \end{vmatrix}; \quad i, j = 1, \dots, p,$$

respectively, and the matrices  $\Gamma_{-m+k}^{ij}$  (and  $\tilde{\Gamma}_{-m+k}^{ij}$ ) are obtained replacing the  $j$ th row (and the  $i$ th column) in the matrix  $\Gamma_{-m+k}$  by  $e_i^T z^k$  (and  $e_j z^{m-k}$ ). The following result shows that these polynomials are orthogonal with respect to the given matrix measure  $\mu$ .

**Theorem 3.1** *For a given matrix measure  $\mu$  on the unit circle let  $\Psi_m^R(z)$  and  $\Psi_m^L(z)$  ( $m \in \mathbb{N}$ ) denote the matrix polynomials defined by (3.6) and (3.7), respectively, then we have*

$$(3.10) \quad \langle z^k I_p, \Psi_m^R \rangle_R = 0, \quad (k = 0, \dots, m-1); \quad \langle z^m I_p, \Psi_m^R \rangle_R = |T_m| I_p$$

$$\langle \Psi_m^L, z^k I_p \rangle_L = 0, \quad (k = 0, \dots, m-1); \quad \langle \Psi_m^L, z^m I_p \rangle_L = |T_m| I_p.$$

**Proof:** We will only give a proof for the polynomials  $\Psi_m^R(z)$ , the remaining part of Theorem 3.1 is shown similarly. The element  $B_{ij}^R$  in the position  $(i, j)$  of the matrix

$$B^R := \langle z^k I, \Psi_m^R \rangle_R = \int_{-\pi}^{\pi} e^{-ik\theta} d\mu(\theta) (T_{ij}^R(e^{i\theta}))_{i,j=1,\dots,p} \quad (k = 0, \dots, m),$$

is given by

$$(3.11) \quad B_{ij}^R = \sum_{l=1}^p \int_{-\pi}^{\pi} e^{-ik\theta} T_{lj}^R(e^{i\theta}) d\mu_{il}(\theta).$$

An expansion of the determinant  $T_{lj}^R(e^{i\theta})$  with respect to the  $(mp+j)$ th row yields

$$(3.12) \quad T_{lj}^R(e^{i\theta}) = \sum_{n=0}^m (-1)^{(m+n)p+j+l} e^{in\theta} |T_m^{(mp+j),(np+l)}|,$$

where the matrix  $T_m^{(mp+j),(np+l)}$  is obtained from  $T_m$  by deleting the  $(mp+j)$ th row and  $(np+l)$ th column. If  $\gamma_{n,ij} = \int_{-\pi}^{\pi} e^{in\theta} d\mu_{ij}$  denotes the element of the matrix  $\Gamma_n$  in the position  $(i, j)$ , where  $n \in \{-m, \dots, m\}$ , it follows that

$$(3.13) \quad B_{ij}^R = \sum_{n=0}^m \sum_{l=1}^p (-1)^{(m+n)p+j+l} |T_m^{(mp+j),(np+l)}| \gamma_{n-k,il}.$$

Now it is easy to see that the right hand side of (3.13) is the determinant of the matrix  $T_m$ , where the  $(mp+j)$ th row has been replaced by the vector

$$(\gamma_{-k,i1}, \dots, \gamma_{-k,ip}, \gamma_{-k+1,i1}, \dots, \gamma_{-k+1,ip}, \dots, \gamma_{m-1-k,i1}, \dots, \gamma_{m-1-k,ip}, \gamma_{m-k,i1}, \dots, \gamma_{m-k,ip})$$

Consequently, if  $k \in \{0, \dots, m-1\}$  the  $(mp+j)$ th and  $(kp+i)$ th row in this matrix coincide and we have  $B_{ij}^R = 0$ , which proves the first identity in (3.10).

For a proof of the second identity we note that in the case  $k = m$  and  $i \neq j$  the same argument yields  $B_{ij} = 0$ . If  $k = m$  and  $i = j$  it follows that  $B_{ij}$  is exactly the determinant of the matrix  $T_m$ , which completes the proof of the first assertion of Theorem 3.1.  $\square$

In the following discussion we derive several consequences of the representations (3.6) and (3.7), which will be useful to identify the canonical moments as Verblunsky coefficients. In particular we determine the corresponding leading coefficients and identify the orthonormal polynomials with respect to the measure  $\mu$ . For this purpose recall that a matrix polynomial of the form (3.1) is called monic, if the coefficient of the leading term is the identity matrix, that is  $A_n = I_p$ .

**Corollary 3.2** *For a given matrix measure  $\mu$  on the unit circle let  $\Psi_m^R(z)$  and  $\Psi_m^L(z)$  be defined by (3.6) and (3.7) and consider for  $m \leq N(\mu)$  the matrix polynomials*

$$(3.14) \quad \Phi_m^R(z) = \Psi_m^R(z)|T_m|^{-1}R_m,$$

$$(3.15) \quad \Phi_m^L(z) = |T_m|^{-1}L_m\Psi_m^L(z),$$

where the matrices  $R_m$  and  $L_m$  are defined by (2.13) and (2.12), respectively. The polynomials  $\Phi_m^R(z)$  (and  $\Phi_m^L(z)$ ) are monic orthogonal matrix polynomials with respect to the right (and left) inner product  $\langle \cdot, \cdot \rangle_R$  (and  $\langle \cdot, \cdot \rangle_L$ ).

Similarly, define for  $m \leq N(\mu)$

$$(3.16) \quad \phi_m^R(z) = \Psi_m^R(z)|T_m|^{-1}R_m^{1/2},$$

$$(3.17) \quad \phi_m^L(z) = |T_m|^{-1}L_m^{1/2}\Psi_m^L(z),$$

then the matrix polynomial  $\phi_m^R(z)$  (and  $\phi_m^L(z)$ ) are orthonormal polynomials with respect to the right (and left) inner product  $\langle \cdot, \cdot \rangle_R$  (and  $\langle \cdot, \cdot \rangle_L$ ). The leading coefficients of  $\phi_m^R(z)$  and  $\phi_m^L(z)$  are given by  $R_m^{-1/2}$  and  $L_m^{-1/2}$ , respectively.

**Proof:** In the first part we will prove that the leading coefficients of the polynomials  $\Psi_m^R(z)$  and  $\Psi_m^L(z)$  defined by (3.6) and (3.7) are given by

$$(3.18) \quad L_m^R = |T_m|R_m^{-1},$$

$$(3.19) \quad L_m^L = |T_m|L_m^{-1},$$

respectively. With these representations we obtain from Theorem 3.1

$$\langle \Psi_m^R, \Psi_m^R \rangle_R = |T_m|(L_m^R)^* ; \quad \langle \Psi_m^L, \Psi_m^L \rangle_L = |T_m|(L_m^L)^*,$$

and the assertion of the Corollary follows by a straightforward calculation.

In order to prove (3.18) and (3.19) we restrict ourselves to the first case; the second case is shown similarly. Observing the definition of the determinants  $T_{ij}^R(z)$  in (3.8) we obtain for the entry in the position  $(i, j)$  of the leading coefficient of the matrix polynomial  $\Psi_m^R(z)$

$$(L_m^R)_{ij} = (-1)^{2mp+i+j} |T_m^{(mp+j), (mp+i)}|,$$

where we have used an expansion of the determinant with respect to the  $(mp + j)$  row and the matrix  $T_m^{(mp+j), (mp+i)}$  is obtained from  $T_m$  by deleting the  $(mp + j)$ th row and  $(mp + i)$ th column. This means that  $(L_m^R)_{ij}$  is the entry in the position  $(mp + i, mp + j)$  of the adjoint of the matrix  $T_m$  ( $i, j = 1, \dots, p$ ), and consequently  $L_m^R/|T_m|$  is the  $p \times p$  block in the position  $(m + 1, m + 1)$  of the matrix  $T_m^{-1}$ , which is given by

$$(\Gamma_0 - (\Gamma_{-m} \dots \Gamma_{-1}) T_{m-1}^{-1} (\Gamma_{-m} \dots \Gamma_{-1})^*)^{-1} = R_m^{-1}$$

[see e.g. Horn and Johnson (1985)]. This proves the assertion (3.18) and completes the proof of the Corollary.  $\square$ .

We are now in a position to identify the canonical moments introduced in Definition 2.3 as Verblunsky coefficients which are defined as coefficients in the Szegő relation of the matrix orthonormal polynomials  $\phi_n^L(z)$  and  $\phi_n^R(z)$ . For this purpose we introduce for a given matrix polynomial  $P_n$  of degree  $n$  the corresponding reversed polynomial

$$\tilde{P}_n(z) = z^n P_n \left( \frac{1}{\bar{z}} \right)^*,$$

where  $\bar{z}$  denotes the complex conjugation of  $z \in \mathbb{C}$ . Obviously we have for any  $p \times p$  matrix  $A$

$$\widetilde{AP}_n(z) = \tilde{P}_n(z) A^*.$$

In the following discussion let  $\kappa_m^R = R_m^{-1/2}$  and  $\kappa_m^L = L_m^{-1/2}$  ( $m = 1, \dots, N(\mu) - 1$ ) denote the leading coefficients of the orthonormal matrix polynomials  $\phi_m^R(z)$  and  $\phi_m^L(z)$  with respect to the right and left inner product induced by the matrix measure  $\mu$  and define the matrices

$$(3.20) \quad \rho_m^R = (\kappa_{m+1}^R)^{-1} \kappa_m^R \quad \text{and} \quad \rho_m^L = \kappa_m^L (\kappa_{m+1}^L)^{-1} \quad (m = 1, \dots, N(\mu) - 1).$$

Then it follows from Damanik et al. (2008) that there exist  $p \times p$  matrices  $H_m$  such that the orthonormal matrix polynomial with respect to the measure  $\mu$  on the unit circle satisfy the Szegő recursions

$$(3.21) \quad z \phi_m^L(z) - \rho_m^L \phi_{m+1}^L(z) = H_{m+1} \tilde{\phi}_m^R(z),$$

$$(3.22) \quad z \phi_m^R(z) - \phi_{m+1}^R(z) \rho_m^R = \tilde{\phi}_m^L(z) H_{m+1}.$$

The matrices  $H_m$  are uniquely determined and called Verblunsky or reflection coefficients, because they were introduced for the scalar case in two seminal papers by Verblunsky (1935, 1936). The final result of this section shows that the Verblunsky coefficients coincide with the canonical moments introduced in Definition 2.3.

**Theorem 3.3** *Let  $\mu$  denote a matrix measure on the unit circle and assume that  $0 \leq m < N(\mu)$ . If  $A_{m+1}$  is the  $(m+1)$ th canonical moment of  $\mu$  defined in Definition 2.3 and  $H_{m+1}$  is the  $(m+1)$ th Verblunsky coefficient defined by the Szegő recursions (3.21) and (3.22), then*

$$(3.23) \quad A_{m+1} = H_{m+1}.$$

**Proof:** Integrating the recursion (3.22) we obtain

$$\langle I_p, z\phi_m^R - \phi_{m+1}^R \rho_m^R \rangle_R = \langle I_p, \tilde{\phi}_m^L H_{m+1} \rangle_R$$

and

$$\langle I_p, z\Psi_m^R \rangle_R |T_m|^{-1} R_m^{1/2} = \langle I_p, \tilde{\Psi}_m^L \rangle_R |T_m|^{-1} L_m^{1/2} H_{m+1},$$

where we have used the orthogonality of the matrix polynomials  $\Psi_{m+1}^R(z)$  stated in Theorem 3.1 and the representations of the orthonormal polynomials  $\phi_m^R$  and  $\phi_m^L$  in Corollary 3.2. Observing Theorem 3.1 and the identity

$$(3.24) \quad \langle I_p, \tilde{\Psi}_m^L \rangle_R = \int_{-\pi}^{\pi} d\mu(\theta) e^{im\theta} (\Psi_m^L(e^{i\theta}))^* = \langle z^m I_p, \Psi_m^L \rangle_L = |T_m| I_p$$

yields

$$(3.25) \quad \begin{aligned} H_{m+1} &= L_m^{-1/2} \langle I_p, \tilde{\Psi}_m^L \rangle_R^{-1} \langle I_p, z\Psi_m^R \rangle_R R_m^{1/2} \\ &= L_m^{-1/2} |T_m|^{-1} \langle I_p, z\Psi_m^R \rangle_R R_m^{1/2}. \end{aligned}$$

The matrix polynomial  $\Psi_m^R(z)$  has the representation

$$\Psi_m^R(z) = L_m^R z^m + \sum_{k=0}^{m-1} K_k^R z^k,$$

where  $K_0^R, \dots, K_{m-1}^R$  denote  $p \times p$  matrices and the leading coefficient  $L_m^R$  is given by (3.18). Integrating with respect to  $d\mu(\theta)$  gives

$$\langle I_p, z\Psi_m^R \rangle_R = \langle I_p, z^{m+1} + \sum_{k=0}^{m-1} K_k^R (L_m^R)^{-1} z^{k+1} \rangle_R |T_m| R_m^{-1},$$

and it follows from (3.25) that

$$(3.26) \quad H_{m+1} = L_m^{-1/2} \langle I_p, z^{m+1} + \sum_{k=0}^{m-1} K_k^R (L_m^R)^{-1} z^{k+1} \rangle_R R_m^{-1/2}.$$

Observing the definition of the canonical moments in (2.17) and the definition of the center (2.15) the assertion of the Theorem follows if the identity

$$(3.27) \quad \langle I_p, z^{m+1} + \sum_{k=0}^{m-1} K_k^R (L_m^R)^{-1} z^{k+1} \rangle_R = \Gamma_{m+1} - (\Gamma_1 \dots \Gamma_m) T_{m-1}^{-1} (\Gamma_{-m} \dots \Gamma_{-1})^*.$$

can be established. For this purpose we determine the matrices  $K_k^R$  ( $k = 0, \dots, m-1$ ) explicitly using the representation of the orthogonal matrix polynomials  $\Psi_m^R(z)$  in (3.6). From this definition it follows that the element in the position  $(i, j)$  of the matrix  $K_k^R$  is obtained by deleting the  $(mp + j)$ th row and the  $(kp + i)$ th column in the determinant  $T_{ij}^R(z)$  defined by (3.8), that is

$$(K_k^R)_{ij} = (-1)^{(m+k)p+i+j} |T_m^{(mp+j), (kp+i)}|.$$

Here again  $T_m^{(mp+j), (kp+i)}$  denotes the matrix obtained  $T_m$  by deleting the  $(mp + j)$ th row and  $(kp + i)$ th column, which coincides with the entry in the position  $(kp + i, mp + j)$  of the adjoint of the matrix  $T_m$ . Consequently, it follows that

$$(K_k^R)_{ij} = |T_m| (T_m^{-1})_{kp+i, mp+j},$$

and the “vector”

$$\frac{1}{|T_m|} \begin{pmatrix} K_0^R \\ \vdots \\ K_{m-1}^R \end{pmatrix} \in (\mathbb{C}^{p \times p})^m$$

coincides with the right upper block of size  $mp \times p$  of the matrix  $T_m^{-1}$ . By standard result in linear algebra this block is given by

$$-T_{m-1}^{-1} (\Gamma_{-m} \dots \Gamma_{-1})^* R_m^{-1},$$

which yields

$$\begin{aligned} \langle I_p, \sum_{k=0}^{m-1} K_k^R z^{k+1} \rangle_R &= \sum_{k=0}^{m-1} \Gamma_{k+1} K_k^R \\ &= (\Gamma_1 \dots \Gamma_m) ((K_0^R)^* \dots (K_{m-1}^R)^*)^* \\ &= -|T_m| (\Gamma_1 \dots \Gamma_m) T_{m-1}^{-1} (\Gamma_{-m} \dots \Gamma_{-1})^* R_m^{-1}. \end{aligned}$$

Combining this result with the identity  $(L_m^R)^{-1} = R_m |T_m|^{-1}$  finally gives (3.27), which completes the proof Theorem 3.3.  $\square$

## 4 Geronimus relations for monic polynomials

In this section we present a new proof of the Geronimus relations, which provide a representation of the canonical moments (or Verblunsky coefficients) of a symmetric matrix measure on the unit circle in terms of the coefficients in the recurrence relations of a sequence of orthogonal polynomials with respect to an associated matrix measure on the interval  $[-1, 1]$ . There exists several alternative proofs of these relations in the literature [see Yakhlef and Marcellán (2001) and Damanik et al. (2008)], but the one presented here explicitly uses the theory of canonical moments of matrix measures as introduced in Dette and Studden (2002). As a by-product we derive several interesting properties of the Verblunsky coefficients.

To be precise let  $\mu_C$  denote a symmetric (with respect to the point 0) matrix measure on the unit disc (i.e.  $\mu_C$  is invariant with respect to the transformation  $\theta \mapsto -\theta$ ). We associate to  $\mu_C$  a corresponding matrix measure, say  $\mu_I$ , on the the interval  $[-1, 1]$ , which is defined by the property

$$(4.1) \quad \int_{-1}^1 f(x) d\mu_I(x) = \int_{-\pi}^{\pi} f(\cos(\theta)) d\mu_C(\theta)$$

for all integrable functions  $f$  defined on the interval  $[-1, 1]$ . Note that the relation  $Sz : d\mu_C \mapsto d\mu_I$  is called Szegő mapping in the literature, where the matrix measure  $\mu_I$  is usually defined on the interval  $[-2, 2]$ . We will work with the interval  $[-1, 1]$  in this section, because this interval is also used in the classical papers of Szegő (1922) and Geronimus (1946) and in the monograph on canonical moments by Dette and Studden (1997).

Note that the inverse of the Szegő mapping (4.1) is characterized by the property

$$(4.2) \quad \int_{-\pi}^{\pi} g(\theta) d\mu_C(\theta) = \int_{-1}^1 g(\arccos(x)) d\mu_I(x),$$

where  $g$  denotes any integrable function on  $\partial\mathbb{D}$  with  $g(\theta) = g(-\theta)$  for all  $\theta \in [-\pi, \pi)$ . For a proof of the Geronimus relations we need several preparations. Our first results shows that the canonical moments (or Verblunsky coefficients) of a symmetric matrix measure on the unit circle are hermitian matrices. The result was also proved by Damanik et al. (2008). We provide here an alternative proof, because several steps in the proof are used later.

**Lemma 4.1** *For any symmetric matrix measure  $\mu_C$  on the unit circle the corresponding canonical moments  $A_m$  are hermitian .*

**Proof:** By the symmetry of the matrix measure  $\mu_C$  we have  $\Gamma_k = \int_{-\pi}^{\pi} e^{ik\theta} d\mu_C(\theta) = \int_{-\pi}^{\pi} e^{-ik\theta} d\mu_C(\theta) = \Gamma_{-k}$  which yields  $\Gamma_k = \int_{-\pi}^{\pi} \cos(k\theta) d\mu_C(\theta)$ . Consequently, the block Toeplitz

matrix associated with  $\mu_C$  is given by

$$(4.3) \quad T_m = \begin{pmatrix} \Gamma_0 & \dots & \Gamma_m \\ \vdots & \ddots & \vdots \\ \Gamma_m & \dots & \Gamma_0 \end{pmatrix}.$$

We denote by  $[A]_{(k,l)}$  the  $p \times p$  block in the position  $(k,l)$  of the  $mp \times mp$ -block matrix  $A$ . We will show at the end of this proof that

$$(4.4) \quad [T_{m-1}^{-1}]_{(k,l)} = [T_{m-1}^{-1}]_{(m+1-k, m+1-l)}.$$

From this identity and the property  $\Gamma_k = \Gamma_k^*$  we obtain

$$\begin{aligned} (\Gamma_1, \dots, \Gamma_m) T_{m-1}^{-1} (\Gamma_m, \dots, \Gamma_1)^* &= \sum_{k,l=1}^m \Gamma_k [T_{m-1}^{-1}]_{(k,l)} \Gamma_{m+1-l} = \sum_{k,l=1}^m \Gamma_{m-k+1} [T_{m-1}^{-1}]_{(m-k+1, m-l+1)} \Gamma_l \\ &= \sum_{k,l=1}^m \Gamma_{m-k+1} [T_{m-1}^{-1}]_{(k,l)} \Gamma_l = (\Gamma_m, \dots, \Gamma_1) T_{m-1}^{-1} (\Gamma_1, \dots, \Gamma_m)^*, \end{aligned}$$

and by similar arguments

$$(4.5) \quad (\Gamma_1, \dots, \Gamma_m) T_{m-1}^{-1} (\Gamma_1, \dots, \Gamma_m)^* = (\Gamma_m, \dots, \Gamma_1) T_{m-1}^{-1} (\Gamma_m, \dots, \Gamma_1)^*.$$

Observing the definition of the canonical moments  $A_{m+1}$  it now follows that

$$\begin{aligned} A_{m+1}^* &= [\Gamma_0 - (\Gamma_m, \dots, \Gamma_1) T_{m-1}^{-1} (\Gamma_m, \dots, \Gamma_1)^*]^{-1/2} (\Gamma_{m+1} - (\Gamma_1, \dots, \Gamma_m) T_{m-1}^{-1} (\Gamma_m, \dots, \Gamma_1)^*)^* \\ &\quad \times [\Gamma_0 - (\Gamma_1, \dots, \Gamma_m) T_{m-1}^{-1} (\Gamma_1, \dots, \Gamma_m)^*]^{-1/2} \\ &= A_{m+1} \end{aligned}$$

which proves the remaining assertion of Lemma 4.1.

*Proof of the identity (4.4).* The element in the position  $(i,j)$  of the matrix  $[T_{m-1}^{-1}]_{(k,l)}$  and  $[T_{m-1}^{-1}]_{(m+1-k, m+1-l)}$  are given by

$$|T_{m-1}|^{-1} (-1)^{(l+k)p+i+j} \left| T_{m-1}^{((l-1)p+j), ((k-1)p+i)} \right|$$

and

$$|T_{m-1}|^{-1} (-1)^{(2m-l-k)p+i+j} \left| T_{m-1}^{((m-l)p+j), ((m-k)p+i)} \right|,$$

respectively, where  $T_{m-1}^{((m-l)p+j), ((m-k)p+i)}$  denotes the matrix obtained from  $T_{m-1}$  by deleting the  $(m-l)p+j$  row and  $(m-k)p+i$  column (note that both expressions have the same sign). In the

following discussion we denote by  $A^{(\cdot),(i)}$  and  $A^{(j),(\cdot)}$  the matrix obtained from  $A$  by deleting the  $i$ th column or the  $j$ th row, respectively. Then interchanging first columns and then rows yields

$$\begin{aligned}
\left| T_{m-1}^{((l-1)p+j),((k-1)p+i)} \right| &= \begin{vmatrix} \Gamma_0 & \cdots & \Gamma_{k-2} & \Gamma_{k-1}^{(\cdot),(i)} & \Gamma_k & \cdots & \Gamma_{m-1} \\ \vdots & & & \vdots & \vdots & & \vdots \\ \Gamma_{l-2} & \cdots & \Gamma_{|l-k|} & \Gamma_{|l-k-1|}^{(\cdot),(i)} & \Gamma_{|l-k-2|} & \cdots & \Gamma_{m-l+1} \\ \Gamma_{l-1}^{(j),(\cdot)} & \cdots & \Gamma_{|l-k+1|}^{(j),(\cdot)} & \Gamma_{|l-k|}^{(j),(i)} & \Gamma_{|l-k-1|}^{(j),(\cdot)} & \cdots & \Gamma_{m-l}^{(j),(\cdot)} \\ \Gamma_l & \cdots & \Gamma_{|l-k+2|} & \Gamma_{|l-k+1|}^{(\cdot),(i)} & \Gamma_{|l-k|} & \cdots & \Gamma_{m-l-1} \\ \vdots & & & \vdots & \vdots & & \vdots \\ \Gamma_{m-1} & \cdots & \Gamma_{m-k+1} & \Gamma_{m-k}^{(\cdot),(i)} & \Gamma_{m-k-1} & \cdots & \Gamma_0 \end{vmatrix} \\
&= (-1)^\gamma \begin{vmatrix} \Gamma_{m-1} & \cdots & \Gamma_k & \Gamma_{k-1}^{(\cdot),(i)} & \Gamma_{k-2} & \cdots & \Gamma_0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \Gamma_{m-l+1} & \cdots & \Gamma_{|l-k-2|} & \Gamma_{|l-k-1|}^{(\cdot),(i)} & \Gamma_{|l-k|} & \cdots & \Gamma_{l-2} \\ \Gamma_{m-l}^{(j),(\cdot)} & \cdots & \Gamma_{|l-k-1|}^{(j),(\cdot)} & \Gamma_{|l-k|}^{(j),(i)} & \Gamma_{|l-k+1|}^{(j),(\cdot)} & \cdots & \Gamma_{l-1}^{(j),(\cdot)} \\ \Gamma_{m-l-1} & \cdots & \Gamma_{|l-k|} & \Gamma_{|l-k+1|}^{(\cdot),(i)} & \Gamma_{|l-k+2|} & \cdots & \Gamma_l \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \Gamma_0 & \cdots & \Gamma_{m-k-1} & \Gamma_{m-k}^{(\cdot),(i)} & \Gamma_{m-k+1} & \cdots & \Gamma_{m-1} \end{vmatrix} \\
&= (-1)^{2\gamma} \begin{vmatrix} \Gamma_0 & \cdots & \Gamma_{m-k-1} & \Gamma_{m-k}^{(\cdot),(i)} & \Gamma_{m-k+1} & \cdots & \Gamma_{m-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \Gamma_{m-l-1} & \cdots & \Gamma_{|l-k|} & \Gamma_{|l-k+1|}^{(\cdot),(i)} & \Gamma_{|l-k+2|} & \cdots & \Gamma_l \\ \Gamma_{m-l}^{(j),(\cdot)} & \cdots & \Gamma_{|l-k-1|}^{(j),(\cdot)} & \Gamma_{|l-k|}^{(j),(i)} & \Gamma_{|l-k+1|}^{(j),(\cdot)} & \cdots & \Gamma_{l-1}^{(j),(\cdot)} \\ \Gamma_{m-l+1} & \cdots & \Gamma_{|l-k-2|} & \Gamma_{|l-k-1|}^{(\cdot),(i)} & \Gamma_{|l-k|} & \cdots & \Gamma_{l-2} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \Gamma_{m-1} & \cdots & \Gamma_k & \Gamma_{k-1}^{(\cdot),(i)} & \Gamma_{k-2} & \cdots & \Gamma_0 \end{vmatrix} \\
&= \left| T_{m-1}^{((m-l)p+j),((m-k)p+i)} \right|,
\end{aligned}$$

for some  $\gamma \in \mathbb{N}$ , because the number of changed columns coincides with the number of changed rows. This implies (4.4) and completes the proof of Lemma 4.1.  $\square$

For the next step we need to define canonical moments of matrix measures on the interval  $[-1, 1]$ . Because the main arguments here are very similar to the proceeding in Dette and Studden (2002), who considered matrix measures on the interval  $[0, 1]$ , we only state the main differences without proofs. To be precise, define for a matrix measure  $\mu_I$  on the interval  $[-1, 1]$  the moments  $S_k =$

$S_k(\mu_I) = \int_{-1}^1 x^k d\mu_I(x)$  ( $k = 0, 1, \dots$ ) and a vector  $c_n(\mu_I) = (S_0(\mu_I), \dots, S_n(\mu_I)) \in (\mathbb{C}^{p \times p})^{n+1}$ . We consider the moment space

$$(4.6) \quad \mathcal{M}_{n+1}^{(I)} = \{c_n(\mu_I) \mid \mu_I \text{ is a matrix measure on } [-1, 1]\} \subset (\mathbb{C}^{p \times p})^{n+1}$$

corresponding to the first  $n$  moments of matrix measures on the interval  $[-1, 1]$ . For a matrix measure  $\mu_I$  on the interval  $[-1, 1]$  we define the block Hankel matrices  $\overline{H}_j$  and  $\underline{H}_j$

$$\begin{aligned} \underline{H}_{2m} &= \begin{pmatrix} S_0 & \dots & S_m \\ \vdots & \ddots & \vdots \\ S_m & \dots & S_{2m} \end{pmatrix}, \\ \overline{H}_{2m} &= \begin{pmatrix} S_0 - S_2 & \dots & S_{m-1} - S_{m+1} \\ \vdots & \ddots & \vdots \\ S_{m-1} - S_{m+1} & \dots & S_{2m-2} - S_{2m} \end{pmatrix}, \\ \underline{H}_{2m+1} &= \begin{pmatrix} S_0 + S_1 & \dots & S_m + S_{m+1} \\ \vdots & \ddots & \vdots \\ S_m + S_{m+1} & \dots & S_{2m} + S_{2m+1} \end{pmatrix}, \\ \overline{H}_{2m+1} &= \begin{pmatrix} S_0 - S_1 & \dots & S_m - S_{m+1} \\ \vdots & \ddots & \vdots \\ S_m - S_{m+1} & \dots & S_{2m} - S_{2m+1} \end{pmatrix}. \end{aligned}$$

We introduce the notation

$$\begin{aligned} \underline{h}_{2m} &= (S_m, \dots, S_{2m-1})^T, & \overline{h}_{2m} &= (S_{m-1} - S_{m+1}, \dots, S_{2m-3} - S_{2m-1})^T, \\ \underline{h}_{2m+1} &= (S_m + S_{m+1}, \dots, S_{2m-1} + S_{2m})^T, & \overline{h}_{2m+1} &= (S_m - S_{m+1}, \dots, S_{2m-1} - S_{2m})^T, \end{aligned}$$

and define  $S_1^+ = S_0$ ,  $S_2^+ = S_0$ ,

$$(4.7) \quad S_{2m}^+ = S_{2m-2} - \overline{h}_{2m}^T \overline{H}_{2m-2}^{-1} \overline{h}_{2m} \quad (m \geq 2),$$

$$S_{2m+1}^+ = S_{2m} - \overline{h}_{2m+1}^T \overline{H}_{2m-1}^{-1} \overline{h}_{2m+1} \quad (m \geq 1),$$

and  $S_1^- = -S_0$ ,

$$(4.8) \quad S_{2m}^- = \underline{h}_{2m}^T \underline{H}_{2m-2}^{-1} \underline{h}_{2m} \quad (m \geq 1),$$

$$S_{2m+1}^- = \underline{h}_{2m+1}^T \underline{H}_{2m-1}^{-1} \underline{h}_{2m+1} - S_{2m} \quad (m \geq 1).$$

Note that the quantities  $S_n^+$  and  $S_n^-$  are determined by  $S_0, \dots, S_{n-1}$ . It can be shown by the same argument as in Dette and Studden (2002) that for  $(S_0, \dots, S_{n-1}) \in \text{Int}(\mathcal{M}_n)$  and any matrix measure  $\mu_I$  on the interval  $[-1, 1]$  with moments satisfying  $S_j(\mu_I) = S_j$  ( $j = 0, \dots, n-1$ ), the moment of order  $n$   $S_n(\mu_I) = \int_{-1}^1 x^n d\mu_I(x)$  satisfies

$$(4.9) \quad S_n^- \leq S_n(\mu_I) \leq S_n^+,$$

With these preparations we can define the canonical moments of a matrix measure on the interval  $[-1, 1]$  with moments  $S_0, \dots, S_{n-1}$ .

**Definition 4.2** Let  $\mu_I$  denote a matrix measure on the interval  $[-1, 1]$  with moments  $S_k = S_k(\mu_I) = \int_{-1}^1 x^k d\mu_I(x)$  ( $k = 0, 1, \dots$ ) and define

$$(4.10) \quad N(\mu_I) = \min \left\{ k \in \mathbb{N} \mid (S_0, \dots, S_k) \in \partial \mathcal{M}_{k+1}^{(I)} \right\}.$$

For any  $n = 0, \dots, N(\mu_I) - 1$  the (hermitian) canonical moments of the matrix measure  $\mu_I$  are defined by

$$(4.11) \quad U_{n+1} = (S_{n+1}^+ - S_{n+1}^-)^{-1/2} (S_{n+1} - S_{n+1}^-) (S_{n+1}^+ - S_{n+1}^-)^{-1/2},$$

where the quantities  $S_{n+1}^+$  and  $S_{n+1}^-$  are given by (4.7) and (4.8), respectively.

Note that Dette and Studden (2002) use a non hermitian definition of canonical moments of matrix measures on the interval  $[0, 1]$ , that is

$$(4.12) \quad \bar{U}_{n+1} = (S_{n+1}^+ - S_{n+1}^-)^{-1} (S_{n+1} - S_{n+1}^-).$$

This non hermitian definition turns out to be more useful when working with monic orthogonal polynomials but in the present context the hermitian version has advantages. We are now in a position to prove the main result of this section, which relates the canonical moments of a symmetric matrix measure on the unit circle and the canonical moments of the associated matrix measure on the interval  $[-1, 1]$  by the Szegö mapping. For this purpose recall the definition of the matrix ball  $K_m$  in (2.14) and the definition for the matrices  $L_m$ ,  $R_m$  and  $M_m$  (2.12), (2.13) and (2.15), respectively. If the given measure  $\mu_C$  on the unit circle is symmetric, then it follows from (4.5)

$$(4.13) \quad L_m = R_m.$$

The following result is the main step for the proof of the Geronimus relations.

**Theorem 4.3** Let  $\mu_C$  denote a symmetric matrix measure on the unit circle and denote by  $\mu_I = Sz(\mu_C)$  the associated matrix measure on the interval  $[-1, 1]$  defined by the Szegö mapping (4.1). The canonical moments  $A_n$  and  $U_n$  of the matrix measures  $\mu_C$  and  $\mu_I$  satisfy

$$A_n = 2U_n - I_p ; \quad n = 1, \dots, N(\mu_C).$$

Similarly, the non symmetric canonical moments  $\bar{U}_n$  defined in (4.12) satisfy

$$(4.14) \quad 2\bar{U}_n - I_p = \bar{A}_n ; \quad n = 1, \dots, N(\mu_C),$$

where the quantities  $\bar{A}_n$  are given by

$$(4.15) \quad \bar{A}_n = L_{n-1}^{-1/2} A_n L_{n-1}^{1/2}.$$

**Proof:** We only prove the first part of the Theorem. The second part is shown by similar arguments. Assume that  $m < N(\mu_C)$  and let  $\Gamma_0, \Gamma_1, \dots$ , denote moments of the matrix measure on the unit circle  $\mu_C$ . For  $j = 0, 1, \dots$  we define  $T_j(x) = \cos(j \arccos x)$  as the  $j$ th (scalar) Chebychev polynomial of the first kind, then it follows from (4.2) and from Rivlin (1990) that

$$(4.16) \quad \begin{aligned} \Gamma_j &= \int_{-\pi}^{\pi} \cos(j\theta) d\mu_C(\theta) = \int_{-1}^1 T_j(x) d\mu_I(x) \\ &= \sum_{k=0}^{\lfloor j/2 \rfloor} (-1)^k \frac{j\Gamma(j-k)}{\Gamma(k+1)\Gamma(j-2k+1)} 2^{j-2k-1} S_{j-2k}, \end{aligned}$$

where  $S_l = \int_{-1}^1 x^l d\mu_I(x)$  ( $l = 0, 1, \dots$ ) denote the moments of the associated matrix measure  $\mu_I = Sz(\mu_C)$  on the interval. Recall the definition of  $S_{m+1}^+$  and  $S_{m+1}^-$  in (4.7) and (4.8), then there exist matrix measures  $\mu_I^+$  and  $\mu_I^-$  on the interval  $[-1, 1]$  such that  $S_j = S_j(\mu_I^\pm)$  ( $j = 0, \dots, m$ ) and

$$S_{m+1}^+ = \int_{-1}^1 x^{m+1} d\mu_I^+(x) \quad \text{and} \quad S_{m+1}^- = \int_{-1}^1 x^{m+1} d\mu_I^-(x).$$

We define

$$(4.17) \quad \Gamma_{m+1}^+ = 2^m S_{m+1}^+ + \sum_{k=1}^{\lfloor (m+1)/2 \rfloor} (-1)^k \frac{(m+1)\Gamma(m+1-k)}{\Gamma(k+1)\Gamma(m-2k+2)} 2^{m-2k} S_{m+1-2k}$$

$$(4.18) \quad \Gamma_{m+1}^- = 2^m S_{m+1}^- + \sum_{k=1}^{\lfloor (m+1)/2 \rfloor} (-1)^k \frac{(m+1)\Gamma(m+1-k)}{\Gamma(k+1)\Gamma(m-2k+2)} 2^{m-2k} S_{m+1-2k}.$$

With the inverse Szegö mapping we obtain the symmetric measures  $\mu_C^+ = (Sz)^{-1}(\mu_I^+)$  and  $\mu_C^- = (Sz)^{-1}(\mu_I^-)$  on the unit circle and the representation (4.16) yields that the measures  $\mu_C^-$  and  $\mu_C^+$  satisfy

$$\int_{-\pi}^{\pi} \cos((m+1)\theta) d\mu_C^+(\theta) = \Gamma_{m+1}^+ \quad \text{and} \quad \int_{-\pi}^{\pi} \cos((m+1)\theta) d\mu_C^-(\theta) = \Gamma_{m+1}^-.$$

Consequently, recalling the definition of the set  $K_m$  in (2.14) we have  $\Gamma_{m+1}^+, \Gamma_{m+1}^- \in K_m$  and from the extremal property of the moments  $S_{m+1}^+$  and  $S_{m+1}^-$  we obtain that  $\Gamma_{m+1}^+, \Gamma_{m+1}^- \in \partial K_m$ . By the definition of the set  $K_m$  in (2.14) it therefore follows that the canonical moments  $A_{m+1}^+$  and  $A_{m+1}^-$  corresponding to matrix measures  $\mu_C^+$  and  $\mu_C^-$ , respectively, are unitary. Moreover, Lemma 4.1, implies that the matrices  $A_{m+1}^+$  and  $A_{m+1}^-$  are hermitian, which yields

$$(A_{m+1}^+)^2 = I_p \quad \text{and} \quad (A_{m+1}^-)^2 = I_p.$$

Consequently all eigenvalues of the matrices  $A_{m+1}^+$  and  $A_{m+1}^-$  are given by  $-1$  and  $1$ .

We now define the matrices

$$(4.19) \quad \tilde{\Gamma}_{m+1}^+ = M_m + L_m \quad \text{and} \quad \tilde{\Gamma}_{m+1}^- = M_m - L_m,$$

which are obviously elements of the set  $K_m$  because by (4.13) we have  $L_m = R_m$ . Consequently, there exist matrix measures  $\tilde{\mu}_C^+$  and  $\tilde{\mu}_C^-$  such that  $\Gamma_j(\tilde{\mu}_C^\pm) = \Gamma_j$  ( $j = 0, \dots, m$ ) and

$$\begin{aligned} \Gamma_{m+1}(\tilde{\mu}_C^+) &= \tilde{\Gamma}_{m+1}^+ \\ \Gamma_{m+1}(\tilde{\mu}_C^-) &= \tilde{\Gamma}_{m+1}^- \end{aligned}$$

Without loss of generality we assume that  $\tilde{\mu}_C^+$  and  $\tilde{\mu}_C^-$  are symmetric with respect to the point 0 [otherwise use  $\frac{1}{2}(\tilde{\mu}_C^+(\theta) + \tilde{\mu}_C^+(-\theta))$ ] and we define  $\tilde{\mu}_I^+ = Sz(\tilde{\mu}_C^+)$  and  $\tilde{\mu}_I^- = Sz(\tilde{\mu}_C^-)$  as the associated measures on the interval  $[-1, 1]$  with  $(m+1)$ th moments  $\tilde{S}_{m+1}^+$  and  $\tilde{S}_{m+1}^-$ , respectively. These matrices satisfy the identities

$$\begin{aligned} \tilde{\Gamma}_{m+1}^+ &= 2^m \tilde{S}_{m+1}^+ + \sum_{k=1}^{\lfloor (m+1)/2 \rfloor} (-1)^k \frac{(m+1)\Gamma(m+1-k)}{\Gamma(k+1)\Gamma(m-2k+2)} 2^{m-2k} S_{m+1-2k} \\ \tilde{\Gamma}_{m+1}^- &= 2^m \tilde{S}_{m+1}^- + \sum_{k=1}^{\lfloor (m+1)/2 \rfloor} (-1)^k \frac{(m+1)\Gamma(m+1-k)}{\Gamma(k+1)\Gamma(m-2k+2)} 2^{m-2k} S_{m+1-2k} \end{aligned}$$

From the inequalities (4.9) it follows that  $S_{m+1}^+ \geq \tilde{S}_{m+1}^+$  and  $\tilde{S}_{m+1}^- \geq S_{m+1}^-$  (note that  $\tilde{S}_{m+1}^+$  and  $\tilde{S}_{m+1}^-$  are moments of a matrix measure on the interval  $[-1, 1]$  with moments  $S_0, \dots, S_m$ ). On the other hand we have

$$\begin{aligned} 2^m \left( \tilde{S}_{m+1}^+ - S_{m+1}^+ \right) &= \tilde{\Gamma}_{m+1}^+ - \Gamma_{m+1}^+ \\ &= M_m + L_m - (M_m + L_m^{1/2} A_{m+1}^+ L_m^{1/2}) \\ &= L_m^{1/2} (I_p - A_{m+1}^+) L_m^{1/2} \\ &\geq 0, \end{aligned}$$

because the eigenvalues of the matrix  $I_p - A_{m+1}$  are given by 0 and 2. So we obtain

$$\tilde{S}_{m+1}^+ = S_{m+1}^+,$$

while a similar argument shows

$$\tilde{S}_{m+1}^- = S_{m+1}^-.$$

Consequently, it follows that

$$\begin{aligned} A_{m+1}^+ &= I_p ; & A_{m+1}^- &= -I_p ; \\ \tilde{\Gamma}_{m+1}^+ &= \Gamma_{m+1}^+ ; & \tilde{\Gamma}_{m+1}^- &= \Gamma_{m+1}^- ; \end{aligned}$$

and we obtain from the definitions of  $\tilde{\Gamma}_{m+1}^+$ ,  $\tilde{\Gamma}_{m+1}^-$  in (4.19)

$$M_m = \frac{1}{2}(\Gamma_{m+1}^+ + \Gamma_{m+1}^-), \quad L_m = \frac{1}{2}(\Gamma_{m+1}^+ - \Gamma_{m+1}^-).$$

The definition of the  $(m+1)$ th canonical moment  $A_{m+1}$  of the matrix measure  $\mu$  and (4.17)-(4.18) now imply

$$\begin{aligned} A_{m+1} &= L_m^{-1/2}(\Gamma_{m+1} - M_m)L_m^{-1/2} \\ &= \left(\frac{1}{2}(\Gamma_{m+1}^+ - \Gamma_{m+1}^-)\right)^{-1/2} \left(\Gamma_{m+1} - \frac{1}{2}(\Gamma_{m+1}^+ + \Gamma_{m+1}^-)\right) \left(\frac{1}{2}(\Gamma_{m+1}^+ - \Gamma_{m+1}^-)\right)^{-1/2} \\ &= (S_{m+1}^+ - S_{m+1}^-)^{-1/2} (2S_{m+1} - (S_{m+1}^+ + S_{m+1}^-)) (S_{m+1}^+ - S_{m+1}^-)^{-1/2} \\ &= 2(S_{m+1}^+ - S_{m+1}^-)^{-1/2} (S_{m+1} - S_{m+1}^-) (S_{m+1}^+ - S_{m+1}^-)^{-1/2} - I_p \\ &= 2U_{m+1} - I_p, \end{aligned}$$

where the last equality is a consequence of the definition of canonical moments of matrix measures on the interval  $[-1, 1]$ . This proves the assertion of the theorem.  $\square$

Our final result gives the Geronimus relations for monic orthogonal matrix polynomials, which generalize the results obtained by Geronimus (1946) and Faybusovich and Gekhtman (1999) for the scalar case. To be precise note that Corollary 3.2 together with (4.13) yield for the monic orthogonal polynomials  $\Phi_m^R$  and  $\Phi_m^L$  defined in (3.14) and (3.15), respectively

$$\begin{aligned} \rho_m^L \phi_{m+1}^L &= L_m^{-1/2} \Phi_{m+1}^L, & \phi_{m+1}^R \rho_m^R &= \Phi_{m+1}^R L_m^{-1/2} \\ \tilde{\phi}_m^R &= L_m^{-1/2} \tilde{\Phi}_m^R, & \tilde{\phi}_m^L &= \tilde{\Phi}_m^L L_m^{-1/2}. \end{aligned}$$

Using these equations we obtain from (3.21), (3.22) and the second part of Theorem 4.3 the Szegő recursion for the monic orthogonal matrix polynomials with respect to a matrix measure on the

unit circle, that is

$$\begin{aligned} z\Phi_m^L(z) - \Phi_{m+1}^L(z) &= \overline{A_{m+1}}^* \tilde{\Phi}_m^R(z), \\ z\Phi_m^R(z) - \Phi_{m+1}^R(z) &= \tilde{\Phi}_m^R(z) \overline{A_{m+1}} \end{aligned}$$

Consequently, the matrices  $\overline{A_{m+1}}$  defined by (4.15) are the Verblunsky coefficients corresponding to the monic orthogonal polynomials and we obtain the following result.

**Theorem 4.4** *Let  $\mu_C$  denote a symmetric matrix measure on the unit circle and denote by  $\mu_I = Sz(\mu_C)$  the associated matrix measure on the interval  $[-1, 1]$  defined by the Szegö mapping (4.1). If  $P_0, P_1, \dots$  be the monic polynomials orthogonal with respect to the matrix measure  $\mu_I$  satisfying the three term recurrence recursion*

$$(4.20) \quad (1+t)P_{m+1}(t) = P_{m+2}(t) + P_{m+1}(t)C_{m+1} + P_m(t)B_m,$$

( $P_0(t) = I_p, P_{-1}(t) = 0_p$ ), then the matrices  $B_m$  and  $C_{m+1}$  satisfy

$$\begin{aligned} B_m &= \frac{1}{4}(I_p - \overline{A_{2m}})(I_p - \overline{A_{2m+1}}^2)(I_p + \overline{A_{2m+2}}), \\ C_{m+1} &= \frac{1}{2}(I_p - \overline{A_{2m+1}})(I_p + \overline{A_{2m+2}}) + \frac{1}{2}(I_p - \overline{A_{2m+2}})(I_p + \overline{A_{2m+3}}), \end{aligned}$$

where the quantities  $\overline{A_n}$  are defined in (4.15).

**Proof:** It follows analogously to Dette and Studden (2002) that the matrices  $B_m$  and  $C_{m+1}$  are given by

$$\begin{aligned} B_m &= (S_{2m} - S_{2m}^-)^{-1}(S_{2m+2} - S_{2m+2}^-), \\ C_{m+1} &= (S_{2m+2} - S_{2m+2}^-)^{-1}(S_{2m+3} - S_{2m+3}^-) + (S_{2m+1} - S_{2m+1}^-)^{-1}(S_{2m+2} - S_{2m+2}^-). \end{aligned}$$

and that the non hermitian canonical moments defined by (4.12) satisfy

$$2\overline{V_{n-1}}\overline{U}_n = (S_{n-1} - S_{n-1}^-)^{-1}(S_n - S_n^-),$$

whenever  $n \leq N(\mu_I)$ , where  $\overline{V}_n = I_p - \overline{U}_n$ . Consequently, the assertion follows by a direct application of the second part of Theorem 4.3.  $\square$

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## References

- Cantero, M. J., Ferrer, M. P., Moral, L., and Vézquez, L. (2003). A connection between orthogonal polynomials on the unit circle and matrix orthogonal polynomials on the real line. *J. Comput. Appl. Math.*, 154:247–252.
- Damanik, D., Pushnitski, A., and Simon, B. (2008). The analytic theory of matrix orthogonal polynomials. *Surveys in Approximation Theory*, 4:1–85.
- Defez, E., Jódar, L., Law, A., and Ponsoda, E. (2000). Three-term recurrences and matrix orthogonal polynomials. *Utilitas Mathematica*, 57:129–146.
- Delsarte, P., Genin, Y. V., and Kamp, Y. G. (1978). Orthogonal polynomial matrices on the unit circle. *IEEE Trans. Circuits and Systems*, 25:149–160.
- Detle, H. and Studden, W. J. (1997). *Canonical moments with applications in Statistics, Probability and Analysis*. Wiley and Sons, New York.
- Detle, H. and Studden, W. J. (2002). Matrix measures, moment spaces and Favard’s theorem on the interval  $[0; 1]$  and  $[0; \infty)$ . *Lin. Alg. and Appl.*, 345:169–193.
- Duran, A. J. (1995). On orthogonal polynomials with respect to a positive definite matrix of measures. *Can. J. Math.*, 47:88–112.
- Duran, A. J. (1996). Markov’s Theorem for orthogonal matrix polynomials. *Can. J. Math.*, 48:1180–1195.
- Faybusovich, L. and Gekhtman, M. (1999). On Schur flows. *J. Phys. A: Math. Gen.*, 32:4671–4680.
- Fritzsche, B. and Kirstein, B. (1987). An extension problem for non-negative hermitian block toeplitz matrices. *Mathematische Nachrichten*, 130:121–135.
- Geronimo, J. S. (1981). Matrix orthogonal polynomials on the unit circle. *J. Math. Phys.*, 22:1359–1365.
- Geronimus, J. (1946). On the trigonometric moment problem. *The Annals of Mathematics, Second Series*, 47(4):742–761.
- Geronimus, Y. L. (1962). Polynomials orthogonal on a circle and their applications. *American Mathematical Society Translations*, 3(1):1–78.

- Horn, R. A. and Johnson, C. R. (1985). *Matrix analysis*. Cambridge University Press, Cambridge.
- Krein, M. G. (1949). Infinite J-matrices and a matrix moment problem. *Dokl. Akad. Nauk SSSR*, 69:125–128.
- Malyshev, A. N. (1982). Factorization of matrix polynomials. *Sibirsk. Mat. Zh.*, 23:136–146.
- Marcellán, F. and Rodriguez, I. (1989). A class of matrix orthogonal polynomials on the unit circle. *Lin. Alg. and Appl.*, 121:233–241.
- Miranian, L. (2005). Matrix-valued orthogonal polynomials on the real line: some extensions of the classical theory. *J. Phys. A: Math. Gen.*, 38:5731–5749.
- Miranian, L. (2009). Matrix valued orthogonal polynomials on the unit circle: Some extensions of the classical theory. *Canad. Math. Bull.*, 52:95–104.
- Rivlin, T. J. (1990). *Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory*. Wiley, New York.
- Rodman, L. (1990). Orthogonal matrix polynomials. In Nevai, T., editor, *Orthogonal Polynomials: Theory and Practice*, pages 345–362. Kluwer, Dordrecht.
- Sinap, A. and Van Assche, W. (1994). Orthogonal matrix polynomials on the unit circle and applications. In Alfaro, M., García, A., Jagels, C., and Marcellán, F., editors, *Orthogonal polynomials on the unit circle: theory and applications*, pages 159–172. Universidad Carlos III de Madrid, Madrid.
- Sinap, A. and Van Assche, W. (1996). Orthogonal matrix polynomials and applications. *J. Comput. Appl. Math.*, 66:27–52.
- Szegő, G. (1922). Über den asymptotischen Ausdruck von Polynomen, die durch eine Orthogonalitätseigenschaft definiert sind. *Math. Ann.*, 86:114–139.
- Verblunsky, S. (1935). On positive harmonic functions: A contribution to the algebra of Fourier series. *Proc. London Math. Soc.*, 38:125–157.
- Verblunsky, S. (1936). On positive harmonic functions (second paper). *Proc. London Math. Soc.*, 40:290–320.

- Yakhlef, H. O. and Marcellán, F. (2001). Orthogonal matrix polynomials, connection between recurrences on the unit circle and on a finite interval. In *Approximation, Optimization and Mathematical Economics*, pages 369–382. Physica, Heidelberg.
- Yakhlef, H. O. and Marcellán, F. (2002). Relative asymptotics for orthogonal matrix polynomials with respect to a perturbed matrix measure on the unit circle. *Approx. Theory Appl.*, 18:1–19.