A note on testing hypotheses for stationary processes in the frequency domain

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Abstract

In a recent paper Eichler (2008) considered a class of non- and semiparametric hypotheses in multivariate stationary processes, which are characterized by a functional of the spectral density matrix. The corresponding statistics are obtained using kernel estimates for the spectral distribution and are asymptotically normal distributed under the null hypothesis and local alternatives. In this paper we derive the asymptotic properties of these test statistics under fixed alternatives. In particular we show also weak convergence but with a different rate compared to the null hypothesis.

Keywords: stationary process, goodness-of-fit tests, kernel estimate, smoothed periodogram, weak convergence under the alternative

1 Introduction

In this paper we investigate hypotheses about the second order properties of a multivariate d-dimensional stationary time series $\{\mathbf{X}_t\}_{t\in\mathbb{Z}}$ which can be expressed in terms of functionals of its spectral density matrix $f = (f_{ij})_{i,j=1,\dots,d}$. This problem has been investigated by numerous authors replacing the unknown density in the functional by a corresponding nonparametric estimate [see Taniguchi and Kondo (1993), Taniguchi et al. (1996), Paparoditis (2000) or Dette and Spreckelsen (2003), Dette and Paparoditis

(2009) among others]. Recently Eichler (2008) proposed a test for a very large class of hypotheses of the form

$$(1.1)\qquad H_0 : \int_{-\pi}^{\pi} \parallel \Psi(f(\lambda),\lambda) \parallel^2 d\lambda \;=\; 0 \quad \text{vs.} \quad H_1 : \int_{-\pi}^{\pi} \parallel \Psi(f(\lambda),\lambda) \parallel^2 d\lambda \neq 0$$

where $\Psi : \mathbb{C}^{d\times d} \times [-\pi,\pi] \to \mathbb{C}^r$ is functional characterizing the null hypothesis by the property $\Psi(f(\lambda),\lambda)) \equiv 0$ a.e. on $\Pi = [-\pi,\pi]$ and $\parallel \cdot \parallel$ denotes the euclidian norm on $\mathbb{C}^r$. Typical examples include

$$(1.2)\qquad \Psi(f(\lambda),\lambda) = \Big(\frac{f_{11}(\lambda)}{\frac{1}{d}\sum_{i=1}^m f_{ii}(\lambda)} - 1, \ldots \frac{f_{dd}(\lambda)}{\frac{1}{d}\sum_{i=1}^m f_{ii}(\lambda)} - 1\Big)$$

corresponding to the comparison of the diagonal elements or the null hypothesis $H_0 : f_{11} = \ldots = f_{dd}$, and the problem of testing if the components $\mathbf{X}_t^A$ and $\mathbf{X}_t^B$ of the series $\{\mathbf{X}_t\}_{t\in\mathbb{Z}} = \{(\mathbf{X}_t^A, \mathbf{X}_t^B)\}_{t\in\mathbb{Z}}$ are uncorrelated, which corresponds to the spectral coherence

$$(1.3)\qquad \Psi(f(\lambda),\lambda) = \left(\frac{f_{ij}(\lambda)}{\sqrt{f_{ii}(\lambda)f_{jj}(\lambda)}}\right)_{\substack{i\in\{1,\ldots,d_1\}\\ j\in\{d_1+1,\ldots,d\}}}.$$

Eichler (2008) proposed to estimate the spectral density matrix f by a kernel estimate, say $\hat{f}$ and showed weak convergence of an appropriately standardized version of the statistic

$$(1.4)\qquad S_T(\Psi) = \int_{-\pi}^{\pi} \parallel \Psi(\hat{f}(\lambda),\lambda) \parallel^2 d\lambda$$

under the null hypothesis and local alternatives.
The purpose of the present paper is to provide some more insight in the asymptotic properties of the statistic $S_T(\Psi)$. In particular we consider the case of fixed alternatives and show weak convergence of a of a standardized version of $S_T(\Psi)$ to a normal distribution with a different rate of convergence compared to the null hypothesis. In Section 2 we briefly review the approach of Eichler (2008) and state the necessary assumptions for our results, which are presented in Section 3. Finally, in Section 4 we present two examples illustrating our approach.

2 Preliminaries

Let $\{\mathbf{X}_t\}_{t\in\mathbb{Z}}$ denote a centered d-dimensional stationary process which has a linear representation of the form

$$(2.1)\qquad \mathbf{X}_t = (X_{t1}, X_{t2}, \ldots, X_{td})^T = \sum_{j=-\infty}^{\infty} A_j \varepsilon_{t-j} \qquad t \in \mathbb{Z},$$

where $\{A_j = (a_j(r,s))_{r,s=1,2,\ldots,d}, j \in \mathbb{Z}\}$ is a sequence of matrices with entries satisfying

$$\sum_{j\in\mathbb{Z}} |j|^{1/2}|a_j(r,s)| < \infty, \quad r,s = 1,2,\ldots,d \tag{2.2}$$

and $\{\varepsilon_t, t \in \mathbb{Z}\}$ is a d-dimensional centered white noise process. We further assume that $\{\mathbf{X}_t\}_{t\in\mathbb{Z}}$ satisfies for all $j = 1,\ldots,k-1$ and $a_1,\ldots,a_k \in \{1,\ldots,d\}$ the condition

$$\sum_{u_1,\ldots,u_{k-1}\in\mathbb{Z}} (1+|u_j|^2)|c_{a_1,\ldots,a_k}(u_1,\ldots,u_{k-1})| < \infty \tag{2.3}$$

where

$$c_{a_1,\ldots,a_k}(t_1,\ldots,t_{k-1}) = \mathrm{cum}(X_{t_1a_1},\ldots,X_{t_{k-1}a_{k-1}},X_{0a_k}) \tag{2.4}$$

denotes the cumulant of the random variables $X_{t_1a_1},\ldots,X_{t_{k-1}a_{k-1}},X_{0a_k}$. For a matrix A define $A^* = \overline{A}^T$ as the complex conjugated and transposed matrix A. Let

$$I(\lambda) = (2\pi T)^{-1} d(\lambda)d^*(\lambda) \tag{2.5}$$

$$d(\lambda) = \sum_{t=1}^{T} \mathbf{X_t} e^{-i\lambda t} \tag{2.6}$$

denote the periodogram. Then the spectral density matrix f can be estimated by

$$\hat{f}(\lambda) = \frac{2\pi}{T}\sum_k K_b(\lambda-\lambda_k)I(\lambda_k), \tag{2.7}$$

where K denotes a kernel function, $K_b(\lambda) = K(\lambda/b)/b$, b is a bandwidth converging to 0 with increasing sample size and $\lambda_k = 2\pi k/T$ $(k = -\lfloor (n-1)/2\rfloor,\ldots,\lfloor n/2\rfloor)$ denote the Fourier frequencies. Following Eichler (2008) we assume that the bandwidth satisfies $b \sim T^{-\nu}$ for some $2/9 < \nu < 1/2$ and that the kernel K is a symmetric, bounded and Lipschitz continuous density. Finally, we assume that the function Ψ is defined on $D \times [-\pi,\pi]$, where $D \subset \mathbb{C}^{d\times d}$ is an open set containing the set $\{f(\lambda) \mid \lambda \in [-\pi,\pi]\}$. Throughout this paper we use the notation $\Pi = [-\pi,\pi]$ and require the following assumptions for our asymptotic results

(i) $\Psi(Z,\lambda)$ is holomorphic with respect to the variable Z.

(ii) $\Psi(Z,\lambda)$ and its first derivative with respect $z = \mathrm{vec}(Z)$

$$D_z(\Psi(Z,\lambda) = \frac{\partial\Psi(Z,\lambda)}{\partial z^T}$$

are piecewise Lipschitz continuous with respect to $\lambda \in \Pi = [-\pi,\pi]$.

(iii) There exists a constant ρ, such that the closed ball

$$B_{\rho,\lambda} = \{Z \in \mathbb{C}^{d\times d} \ : \| f(\lambda) - Z \| \leq \rho\}$$

is contained in D for all $\lambda \in \Pi$, and such that

$$\sup_{\lambda\in\Pi} \sup_{Z\in B_{\rho,\lambda}} \| \Psi(Z, \lambda) \| < \infty.$$

(iv) $0 < \int_\Pi \| D_z(\Psi(f(\lambda), \lambda) \| d\lambda < \infty$

Under these assumptions Eichler (2008) showed that under the null hypothesis a centered and standardized version of the statistic $S_T(\Psi)$ is asymptotically normal distributed, that is

$$b^{1/2}TS_T(\Psi) - b^{-1/2}\mu \Rightarrow N(0, \sigma^2),$$

where the terms μ and σ^2 are given by

$$\begin{aligned}
\mu &= \int K^2(u)du \int_\Pi \operatorname{tr}[\Gamma_\Psi(\lambda)(f(\lambda)^T \otimes f(\lambda))]d\lambda \\
\sigma^2 &= (2\pi)^3 \int_\Pi (K * K)^2(\lambda)d\lambda \int_\Pi \operatorname{tr}[\Gamma_\Psi(\lambda)(f^T(\lambda) \otimes f(\lambda))\{\Gamma_\Psi(\lambda) + \tilde{\Gamma_\Psi}(-\lambda) \\
&\quad +\Gamma_\Psi^T(-\lambda) + \tilde{\Gamma_\Psi}^T(\lambda)\}(f(\lambda)^T \otimes f(\lambda))]d\lambda,
\end{aligned}$$

respectively. Here $\Gamma_\Psi(\lambda) = D_Z(\Psi(f(\lambda), \lambda))^* D_Z(\Psi(f(\lambda), \lambda))$, $K * K$ denotes the convolution of the kernel K with itself and the matrix $\tilde{\Gamma}$ is given by

$$\tilde{\Gamma_\Psi}(\lambda) = K_{dd}\Gamma_\Psi(\lambda)K_{dd}, \tag{2.8}$$

K_{dd} denotes the commutation matrix, i.e.

$$K_{dd} = \sum_{i,j=1}^{d} (e_i e_j^T \otimes e_j e_i^T), \tag{2.9}$$

and $e_i \in \mathbb{C}^d$ is the ith unit vector ($i = 1, \ldots, d$). Note that Eichler (2008) considered also the case of a tapered periodogram, but for the sake of a transparent notation we restrict ourselves to the periodogram of the form (2.5) and (2.6). From this result a simple test for the hypothesis (1.1) can be derived by rejecting the null hypothesis whenever

$$S_T(\Psi) > (bT)^{-1}\hat{\mu} + \hat{\sigma}u_{1-\alpha}(b^{1/2}T)^{-1} \tag{2.10}$$

where $\hat{\mu}$ and $\hat{\sigma}^2$ are appropriate estimates of the asymptotic bias and variance, respectively, and $u_{1-\alpha}$ is the $(1-\alpha)$ quantile of the standard normal distribution [see Eichler (2008) for details and examples]. In the following section we investigate the weak convergence of the statistic $S_T(\Psi)$ under fixed alternatives.

3 Weak convergence under the alternative

Note that the statistic $S_T(\Psi)$ converges to

$$M^2 = \int_{-\pi}^{\pi} \| \Psi(f(\lambda), \lambda) \|^2 \, d\lambda \geq 0 \tag{3.1}$$

which is positive if and only if the null hypothesis is not satisfied. Consequently it follows that the test defined by (2.10) is consistent. We now investigate the asymptotic distribution of $S_T(\Psi)$ under the alternative. Throughout this paper we define for non-negative definite matrix $F \in \mathbb{C}^{k\times k}$ a semi-norm on $\mathbb{C}^k$ by $\|x\|_F = (x^* F x)^{1/2}$, the symbol $\Longrightarrow$ denotes weak convergence and $\langle x, y\rangle = x^T \overline{y}$ is the common inner product of the vectors $x, y \in \mathbb{C}^k$.

Theorem 3.1 *If the assumptions stated in Section 2 are satisfied and $M^2 > 0$, then*

$$\sqrt{T}(S_T(\Psi) - M^2 - b_b) \Rightarrow N(0, \tau^2), \tag{3.2}$$

where

$$\begin{aligned}
b_b \;&=\; 2\int_\Pi\int_\Pi Re\big\langle \Psi(f(\lambda),\lambda); D_Z\Psi(f(\lambda),\lambda)(K_b(\lambda-x)vec(f(x)) - vec(f(\lambda)))\big\rangle dx d\lambda \\
\tau^2 \;&=\; \Big[4\pi \int_\Pi \| D_Z\Psi(f(x),x)^T\overline{\Psi(f(x),x)} \|^2_{f(x)\otimes\overline{f(x)}} \\
&\qquad + \| Re(D_Z\Psi(f(x),x)^T\overline{\Psi(f(x),x)}) \|^2_{K_{dd}(\overline{f(x)}\otimes f(x))} \\
&\qquad - \| Im(D_Z\Psi(f(x),x)^T\overline{\Psi(f(x),x)}) \|^2_{K_{dd}(\overline{f(x)}\otimes f(x))} \, dx\Big] + 4 \| A^T \|^2_\kappa \\
A \;&=\; \int_\Pi Re(\Psi(f(x),x)^T \overline{D_Z\Psi(f(x),x)})dx \in \mathbb{R}^{1\times d^2}
\end{aligned} \tag{3.3}$$

and $\kappa \in \mathbb{C}^{d^2\times d^2}$ denotes the matrix containing the fourth cumulants $cum_4(\varepsilon_t^p, \varepsilon_t^q, \varepsilon_t^r, \varepsilon_t^s)$, where ε_t^p denotes the p-th entry of the vector ε_t.

Remark 3.2 If the random variables ε_t in the linear representation (2.1) are normally distributed and the function $D_Z\Psi(f(x),x)^T\overline{\Psi(f(x),x)}$ is real valued, then the asymptotic variance in (3.1) simplifies to

$$\tau^2 \;=\; 4\pi \int_\Pi \| D_Z\Psi(f(x),x)^T\Psi(f(x),x) \|^2_{(f(x)\otimes\overline{f(x)})(I_{d^2\times d^2}+K_{dd})}, \tag{3.4}$$

where $I_{d^2\times d^2}$ denotes the $d^2 \times d^2$ unit matrix.

Remark 3.3 A detailed discussion about the use of Theorem 3.1 can be found in Dette and Paparoditis (2009) and we only briefly mention the potential applications here.

(1) It follows from Theorem 3.1 that the power of the test (2.10)can be approximated by

$$P(H_0 \text{ rejected} \mid H_1 \text{ is true}) \;\approx\; 1 - \Phi\left(-\sqrt{T}(M^2+b_b)/\tau + (b^{-1/2}\mu + \sigma u_{1-\alpha})/\tau\sqrt{Tb}\right).$$

(2) From Theorem 3.1 we obtain an upper (asymptotic) $(1-\alpha)$ confidence bound for the parameter M^2, that is

$$S_T(\Psi) - b_b + \frac{\hat{\tau} u_\alpha}{\sqrt{T}}$$

where $\hat{\tau}^2$ is an appropriate (consistent) estimator of the asymptotic variance given in Theorem 3.1. Such an estimator is obtained, for instance, if f is replaced by its kernel estimator $\hat{f}$.

(3) The results of Theorem 3.1 can be used to the so called *precise hypotheses* [see Berger and Delampady (1987)]

$$H_0 : M^2 > \varepsilon \quad \text{versus} \quad H_1 : M^2 \leq \varepsilon \,, \tag{3.5}$$

where M^2 is the measure defined by (3.1) and $\varepsilon > 0$ is a prespecified constant for which the statistician agrees to analyze the data under the null hypothesis. This formulation of the hypothesis reflects the fact that in applications second order behavior of the d time series will usually never be precisely specified by the identity $M^2 = 0$ and the more realistic question in this context is, if the the null hypothesis is approximately satisfied. An asymptotic α-level test for the hypothesis (3.5) is obtained by rejecting the null hypothesis, whenever $\sqrt{T}(S_T(\Psi) - \varepsilon - b_b) < \hat{\tau} u_{1-\alpha}$.

Proof of Theorem 3.1. We will show at the end of this proof that the stochastic expansion

$$\begin{aligned} \tilde{S}_T &= S_T(\Psi) - M^2 \\ &= 2\int_\Pi \mathrm{Re}\Big\langle \Psi(f(\lambda),\lambda) \ ; \ D_Z\Psi(f(\lambda),\lambda)\mathrm{vec}(\hat{f}(\lambda) - f(\lambda))\Big\rangle \, d\lambda + O_P((bT)^{-1}) \end{aligned} \tag{3.6}$$

is valid. Next we use a decomposition of the dominating term in (3.6)

$$2\sqrt{T}\int_\Pi \mathrm{Re}\Big\langle \Psi(f(\lambda),\lambda); (D_Z\Psi(f(\lambda),\lambda)\mathrm{vec}(\hat{f}(\lambda) - f(\lambda))\Big\rangle d\lambda = B_{1T} + B_{2T} \ ,$$

where the terms B_{1T} and B_{2T} are defined by

$$B_{1T} = 2\sqrt{T}\int_\Pi \mathrm{Re}\Big\langle \Psi(f(\lambda),\lambda); D_Z\Psi(f(\lambda),\lambda)\mathrm{vec}(\sum_k \frac{2\pi}{T} K_b(\lambda-\lambda_k)(I_T(\lambda_k) - f(\lambda_k)))\Big\rangle d\lambda, \tag{3.7}$$

$$B_{2T} = 2\sqrt{T}\int_\Pi \mathrm{Re}\Big\langle \Psi(f(\lambda),\lambda); D_Z\Psi(f(\lambda),\lambda)\mathrm{vec}(\sum_k \frac{2\pi}{T} K_b(\lambda-\lambda_k)f(\lambda_k) - f(\lambda)))\Big\rangle d\lambda, \tag{3.8}$$

respectively. A standard calculation shows

$$\begin{aligned} B_{2T} &= \frac{4\pi}{\sqrt{T}}\sum_k \int_\Pi \mathrm{Re}\Big\langle \Psi(f(\lambda),\lambda); D_Z\Psi(f(\lambda),\lambda)(K_b(\lambda-\lambda_k)\mathrm{vec}(f(\lambda_k)) - \mathrm{vec}(f(\lambda)))\Big\rangle d\lambda \\ &= 2\sqrt{T}\int_\Pi\int_\Pi \mathrm{Re}\Big\langle \Psi(f(\lambda),\lambda); D_Z\Psi(f(\lambda),\lambda)(K_b(\lambda-x)\mathrm{vec}(f(x)) - \mathrm{vec}(f(\lambda))\Big\rangle dx d\lambda (1+o(1)) \\ &= \sqrt{T} b_b (1+o(1)). \end{aligned}$$

From these estimates we have

$$\sqrt{T}(S_T(\Psi) - M^2 - b_b) = B_{1T} + o_p(1) \tag{3.9}$$

and the assertion follows if the weak convergence

$$B_{1T} \Rightarrow N(0;\tau^2) \tag{3.10}$$

can be established. For this purpose we note that it follows from the results of Hannan (1970), p. 249, and a straightforward but tedious calculation that

$$\begin{aligned}
\operatorname{Cov}(\operatorname{vec}(I(\lambda_k)); \overline{\operatorname{vec}(I(\lambda_k))}) &= (K_{dd} \cdot f(\lambda_k) \otimes \overline{f(\lambda_k)}) \cdot (1 + o(1)) & (3.11)\\
\operatorname{Cov}(\operatorname{vec}(I(\lambda_k)); \operatorname{vec}(I(\lambda_k))) &= (\overline{f(\lambda_k)} \otimes f(\lambda_k)) \cdot (1 + o(1)) & (3.12)\\
\operatorname{Cov}(\overline{\operatorname{vec}(I(\lambda_k))}; \overline{\operatorname{vec}(I(\lambda_k))}) &= (f(\lambda_k) \otimes \overline{f(\lambda_k)}) \cdot (1 + o(1)) & (3.13)\\
\operatorname{Cov}(\overline{\operatorname{vec}(I(\lambda_k))}; \operatorname{vec}(I(\lambda_k))) &= (K_{dd} \cdot \overline{f(\lambda_k)} \otimes f(\lambda_k)) \cdot (1 + o(1)) & (3.14)
\end{aligned}$$

uniformly with respect to $\lambda_k \neq 0, \pi$. Obviously, we have $E[I(\lambda_k)] = f(\lambda_k)(1 + o(1))$, which yields

$$E[B_{1T}] = o(1), \tag{3.15}$$

and for the calculation of the second moment we use the decomposition

$$E(B_{1,T}^2) = R_1 + R_2 \tag{3.16}$$

with

$$\begin{aligned}
R_1 = \frac{16\pi^2}{T} E\Big(\sum_k \int_\Pi \int_\Pi & K_b(\lambda - \lambda_k) K_b(\mu - \lambda_k) \\
\times \operatorname{Re}\Big\langle & \Psi(f(\lambda), \lambda); D_Z\Psi(f(\lambda), \lambda)\operatorname{vec}(I_n(\lambda_k) - f(\lambda_k))\Big\rangle \\
\times \operatorname{Re}\Big\langle & \Psi(f(\mu), \mu); D_Z\Psi(f(\mu), \mu)\operatorname{vec}(I_n(\lambda_k) - f(\lambda_k))\Big\rangle d\lambda d\mu\Big)
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
R_2 = \frac{16\pi^2}{T} E\Big(\sum_{k \neq l} \int_\Pi \int_\Pi & K_b(\lambda - \lambda_k) K_b(\mu - \lambda_l) \\
\times \operatorname{Re}\Big\langle & \Psi(f(\lambda), \lambda); D_Z\Psi(f(\lambda), \lambda)\operatorname{vec}(I_n(\lambda_k) - f(\lambda_k))\Big\rangle \\
\times \operatorname{Re}\Big\langle & \Psi(f(\mu), \mu); D_Z\Psi(f(\mu), \mu)\operatorname{vec}(I_n(\lambda_l) - f(\lambda_l))\Big\rangle d\lambda d\mu\Big)
\end{aligned} \tag{3.18}$$

Observing (3.11) - (3.14) we obtain by standard calculations for the first term

$$
\begin{aligned}
R_1 \;=\; & \Big[2\pi \int_\Pi \int_\Pi \int_\Pi K_b(\lambda - x) K_b(\mu - x) (\Psi(f(\lambda),\lambda)^T \overline{D_Z \Psi(f(\lambda),\lambda)} \\
\times\; & (K_{dd} \cdot (\overline{f(x)} \otimes f(x)))\; \overline{D_Z \Psi(f(\mu),\mu)}^T \Psi(f(\mu),\mu) d\lambda d\mu dx \\
+\; & 2\pi \int_\Pi \int_\Pi \int_\Pi K_b(\lambda - x) K_b(\mu - x) (\Psi(f(\lambda),\lambda)^T \overline{D_Z \Psi(f(\lambda),\lambda)} \\
\times\; & f(x) \otimes \overline{f(x)}\; (D_Z \Psi(f(\mu),\mu))^T \overline{\Psi(f(\mu),\mu)} d\lambda d\mu dx \\
+\; & 2\pi \int_\Pi \int_\Pi \int_\Pi K_b(\lambda - x) K_b(\mu - x) \overline{\Psi(f(\lambda),\lambda)^T} D_Z \Psi(f(\lambda),\lambda) \\
\times\; & \overline{f(x)} \otimes f(x)\; \overline{D_Z \Psi(f(\mu),\mu)}^T \Psi(f(\mu),\mu) d\lambda d\mu dx \\
+\; & 2\pi \int_\Pi \int_\Pi \int_\Pi K_b(\lambda - x) K_b(\mu - x) \overline{\Psi(f(\lambda),\lambda)^T} D_Z \Psi(f(\lambda),\lambda) \\
\times\; & (K_{dd} \cdot (f(x) \otimes \overline{f(x)}))\; (D_Z \Psi(f(\mu),\mu))^T \overline{\Psi(f(\mu),\mu)} d\lambda d\mu dx\Big] \cdot (1 + o(1)) \\
=\; & \Big[4\pi \int_\Pi \Psi(f(x),x)^T \overline{D_Z \Psi(f(x),x)} (f(x) \otimes \overline{f(x)})\; D_Z \Psi(f(x),x)^T \overline{\Psi(f(x),x)} dx \\
+\; & 4\pi \int_\Pi \mathrm{Re}\Big(\Psi(f(x),x)^T \overline{D_Z \Psi(f(x),x)} \\
\times\; & (K_{dd} \cdot \overline{f(x)} \otimes f(x)\; \overline{D_Z \Psi(f(x),x)}^T \Psi(f(x),x)\Big) dx\Big] \cdot (1 + o(1)) \\
=\; & \Big[4\pi \int_\Pi \| D_Z \Psi(f(x),x)^T \overline{\Psi(f(x),x)} \|^2_{f(x) \otimes \overline{f(x)}} \\
+\; & \| \mathrm{Re}(D_Z \Psi(f(x),x)^T \overline{\Psi(f(x),x)}) \|^2_{K_{dd} \cdot (\overline{f(x)} \otimes f(x))} \\
-\; & \| \mathrm{Im}(D_Z \Psi(f(x),x)^T \overline{\Psi(f(x),x)}) \|^2_{K_{dd} \cdot (\overline{f(x)} \otimes f(x))}\; dx\Big] \cdot (1 + o(1))
\end{aligned} \tag{3.19}
$$

Next we investigate the term R_2 in (3.16) for which we obtain with the estimates ($k \neq l$)

$$
\begin{aligned}
E(\mathrm{vec}(I_n(\lambda_k) - f(\lambda_k)) \mathrm{vec}(I_n(\lambda_l) - f(\lambda_l))^T) \;&=\; E(\mathrm{vec}(I_n(\lambda_k) - f(\lambda_k))\; \mathrm{vec}(I_n(\lambda_l) - f(\lambda_l))^*) = \frac{\kappa}{T} \\
E(\overline{\mathrm{vec}(I_n(\lambda_k) - f(\lambda_k))}\; \mathrm{vec}(I_n(\lambda_l) - f(\lambda_l))^T) \;&=\; E(\overline{\mathrm{vec}(I_n(\lambda_k) - f(\lambda_k))}\; \mathrm{vec}(I_n(\lambda_l) - f(\lambda_l))^*) = \frac{\kappa}{T}
\end{aligned}
$$

[see Hannan (1970), p. 249]

$$\begin{aligned}
R_2 \;=\; & \frac{4\pi^2}{T^2}\sum_{k\neq l}\int_{\Pi^2} K_b(\lambda-\lambda_k)K_b(\mu-\lambda_l)\Psi(f(\lambda),\lambda)^T\overline{D_Z\Psi(f(\lambda),\lambda)}\cdot\kappa\cdot D_Z\Psi(f(\mu),\mu)^*\Psi(f(\mu),\mu)d\lambda d\mu \\
+ & \frac{4\pi^2}{T^2}\sum_{k\neq l}\int_{\Pi^2} K_b(\lambda-\lambda_k)K_b(\mu-\lambda_l)\Psi(f(\lambda),\lambda)^T\overline{D_Z\Psi(f(\lambda),\lambda)}\cdot\kappa\cdot (D_Z\Psi(f(\mu),\mu))^t\overline{\Psi(f(\mu),\mu)}d\lambda d\mu \\
+ & \frac{4\pi^2}{T^2}\sum_{k\neq l}\int_{\Pi^2} K_b(\lambda-\lambda_k)K_b(\mu-\lambda_l)\Psi(f(\lambda),\lambda)^* D_Z\Psi(f(\lambda),\lambda)\cdot\kappa\cdot D_Z\Psi(f(\mu),\mu)^*\Psi(f(\mu),\mu)d\lambda d\mu \\
+ & \frac{4\pi^2}{T^2}\sum_{k\neq l}\int_{\Pi^2} K_b(\lambda-\lambda_k)K_b(\mu-\lambda_l)\Psi(f(\lambda),\lambda)^* D_Z\Psi(f(\lambda),\lambda\cdot\kappa\cdot (D_Z\Psi(f(\mu),\mu))^T\overline{\Psi(f(\mu),\mu)}d\lambda d\mu \\
= & \frac{4\pi^2}{T^2}\sum_{k,l}\int_{\Pi^2} K_b(\lambda-\lambda_k)K_b(\mu-\lambda_l)\Big(\Psi(f(\lambda),\lambda)^T\overline{D_Z\Psi(f(\lambda),\lambda)}+\Psi(f(\lambda),\lambda)^* D_Z\Psi(f(\lambda),\lambda)\Big) \\
& \times\kappa\cdot\Big(D_Z\Psi(f(\mu),\mu)^*\Psi(f(\mu),\mu)+D_Z\Psi(f(\mu),\mu))^T\overline{\Psi(f(\mu),\mu)}\Big)d\lambda d\mu\cdot(1+o(1)) \\
= & \; 4\int_{\Pi^4} K_b(\lambda-x)K_b(\mu-y)a(\lambda)\cdot\kappa\cdot a(\mu)^T\; d\lambda\; d\mu\; dx\; dy\cdot(1+o(1))
\end{aligned}$$

where we have used the notation $a(x)=Re(\Psi(f(x),x)^T\overline{D_Z\Psi(f(x),x)})$. Finally standard calculations and a combination of this result (3.16) and (3.19) show

$$\begin{aligned}
\text{(3.20)}\quad E(B_{1,T}^2) \;=\; & \Big[4\pi\int_\Pi \| D_Z\Psi(f(x),x)^T\overline{\Psi(f(x),x)} \|^2_{f(x)\otimes\overline{f(x)}} \\
& + \| \mathrm{Re}(D_Z\Psi(f(x),x)^T\overline{\Psi(f(x),x))} \|^2_{K_{dd}\cdot(\overline{f(x)}\otimes f(x))} \\
& - \| \mathrm{Im}(D_Z\Psi(f(x),x)^T\overline{\Psi(f(x),x))} \|^2_{K_{dd}\cdot(\overline{f(x)}\otimes f(x))}\; dx + 4\,\| A^T \|^2_\kappa\Big]\cdot(1+o(1))
\end{aligned}$$

The results (3.15) and (3.20) show that the first two moments of B_{1T} converge to the first two moments of the normal distribution specified in Theorem 3.1. Next we show that all cumulants of order $r\geq 3$ vanish asymptotically, that is

$$\text{(3.21)}\qquad \mathrm{cum}_r(B_{1T})=o(1)\quad\text{for all}\quad r\geq 3,$$

which implies the desired weak convergence. For this purpose we introduce the notation

$$\text{(3.22)}\qquad \Omega_i(\lambda)=(\Psi(f(\lambda),\lambda)^T\overline{D_Z\Psi(f(\lambda),\lambda)})_i$$

for the ith component of the vector $(\Psi(f(\lambda),\lambda)^T\overline{D_Z\Psi(f(\lambda),\lambda)})$ and note that

$$\begin{aligned}
\text{cum}_r(B_{1T}) &= \text{cum}_r\Big(2\sqrt{T}\int_\Pi\int_\Pi K_b(\lambda-\alpha) \\
&\quad\times \text{Re}\Big\langle \Psi(f(\lambda),\lambda); D_Z\psi(f(\lambda),\lambda)\text{vec}(I_n(\alpha)-f(\alpha))\Big\rangle d\lambda d\alpha\Big)\cdot(1+o(1)) \\
&= \text{cum}_r\Big(\sqrt{T}\int_{\Pi^2} K_b(\lambda-\alpha) \\
&\quad\times \sum_{k,l=1}^{d}(\Omega_{(l-1)\cdot d+k}(\lambda)+\overline{\Omega_{(k-1)\cdot d+l}(\lambda)})(I_{kl}(\alpha)-f_{kl}(\alpha))d\lambda d\alpha\Big)\cdot(1+o(1)) \\
&= \Big(\frac{1}{2\pi\sqrt{T}}\Big)^r\Big(\sum_{a_{11}a_{12}\ldots a_{r2}=1}^{d}\int_{\Pi^{2r}}\prod_{i=1}^{r}K_b(\lambda_i-\alpha_i)(\Omega_{(a_{i2}-1)\cdot d+a_{i1}}(\lambda_i)+\overline{\Omega_{(a_{i1}-1)\cdot d+a_{i2}})(\lambda_i)}) \\
&\quad\times \text{cum}\Big(d_{a_{11}}(\alpha_1)d_{a_{12}}(-\alpha_1),\ldots,d_{a_{r1}}(\alpha_r)d_{a_{r2}}(-\alpha_r)\Big)d\lambda_1\ldots d\lambda_r d\alpha_1\ldots d\alpha_r\Big)\cdot(1+o(1)).
\end{aligned}$$

Now the product theorem for cumulants [see Brillinger (1981)] yields

$$\begin{aligned}
\text{cum}_r(B_{1T}) &= \Big(\frac{1}{2\pi\sqrt{T}}\Big)^r\Big(\sum_{a_{11}a_{12}\ldots a_{r2}=1}^{d}\int_{\Pi^{2r}}\prod_{i=1}^{r}K_b(\lambda_i-\alpha_i)\big(\Omega_{(a_{i2}-1)\cdot d+a_{i1}}(\lambda_i)+\overline{\Omega_{(a_{i1}-1)\cdot d+a_{i2}}(\lambda_i)}\big) \\
&\quad\times \sum_Q\prod_{k=1}^{p}\text{cum}\Big(\{d_{a_{ij}}(\gamma_i);(i,j)\in Q_k\}\Big)d\lambda_1\ldots d\lambda_r d\alpha_1\ldots d\alpha_r\Big)\cdot(1+o(1))
\end{aligned}$$

where $\gamma_i=(-1)^{j-1}\alpha_i$ and the summation is performed with respect to all indecomposable partitions $Q=\{Q_1,\ldots,Q_p\}$ of the table

$$\begin{matrix} (1,1) & (1,2) \\ \vdots & \vdots \\ (r,1) & (r,2) \end{matrix} \tag{3.23}$$

Using the fact

$$\text{cum}\{d_{a_1}(\alpha_1),\ldots d_{a_k}(\alpha_k)\} = (2\pi)^{k-1}H(\alpha_1+\ldots+\alpha_k)f_{a_1\ldots a_k}(\alpha_1,\ldots,\alpha_{k-1})+O(1)$$

uniformly with respect to $\alpha_1,\ldots,\alpha_k$ with $H(\lambda)=\sum_{t=1}^T e^{-i\lambda t}$ and

$$f_{a_1\ldots a_k}(\alpha_1,\ldots,\alpha_{k-1}) = (2\pi)^{1-k}\sum_{u_1=-\infty}^{\infty}\cdots\sum_{u_{k-1}=-\infty}^{\infty}\exp\{-i(\sum_{j=1}^{k-1}\alpha_j u_j)\}c_{a_1\ldots a_k}(u_1,\ldots,u_{k-1})$$

[see Theorem 4.3.2 in Brillinger (1981)] we have

$$\text{cum}_r(B_{1T}) = (G_1+G_2+G_3)(1+o(1)), \tag{3.24}$$

where the terms G_1, G_2, G_3 in (3.24) are defined as follows:

$$\begin{aligned} G_1 &= \Big(\frac{1}{2\pi\sqrt{T}}\Big)^r\Big(\sum_{Q^1}\sum_{a_{11}\ldots a_{r2}=1}^{d}\int_{\Pi^{2r}}\prod_{k=1}^{r}K_b(\lambda_i-\alpha_i)(\Omega_{(a_{i2}-1)\cdot d+a_{i1}}(\lambda_i)+\overline{\Omega_{(a_{i1}-1)\cdot d+a_{i2}}(\lambda_i)}) \\ &\times \text{ cum}\Big(\{d_{a_{ij}}(\gamma_i);(i,j)\in Q^1_k\}\Big)d\lambda_1\ldots d\lambda_r d\alpha_1\ldots d\alpha_r\Big) \end{aligned}$$

denotes the sum over all indecomposable partitions $Q^1 = \{Q_1 \ldots Q_p, Q^1_1 \ldots Q^1_y\}$ with $p < r$ and at least one set Q^1_i with only one element. Similarly, we define

$$\begin{aligned} G_2 &= \Big(\frac{1}{2\pi\sqrt{T}}\Big)^r\Big(\sum_{Q^2}\sum_{a_{11}\ldots a_{r2}=1}^{d}\int_{\Pi^{2r}}\prod_{i=1}^{r}K_b(\lambda_i-\alpha_i)(\Omega_{(a_{i2}-1)\cdot d+a_{i1}}(\lambda_i)+\overline{\Omega_{(a_{i1}-1)\cdot d+a_{i2})}(\lambda_i)}) \\ &\times \prod_{k=1}^{p}(2\pi)^{p_k-1}H(\overline{\gamma_k})f_{a_{i_{k1}},j_{k1},\ldots,a_{i_{kp_k}}j_{kp_k}}(\gamma_{i_{k1}},\ldots,\gamma_{i_{kp_k}})d\lambda_1\ldots d\lambda_r d\alpha_1\ldots d\alpha_r\Big) \end{aligned}$$

as the sum over all indecomposable partitions whose sets contain at least 3 elements ($\overline{\gamma}_k = \gamma_{i_{k1}} + \ldots + \gamma_{i_{kp_k}}$) and

$$\begin{aligned} G_3 &= \Big(\frac{1}{2\pi\sqrt{T})}\Big)^r\Big(\sum_{Q^3}\sum_{a_{11}\ldots a_{r2}=1}^{d}\int_{\Pi^{2r}}\prod_{i=1}^{r}K_b(\lambda_i-\alpha_i)(\Omega_{(a_{i2}-1)\cdot d+a_{i1}}(\lambda_i)+\overline{\Omega_{(a_{i1}-1)\cdot d+a_{i2}}(\lambda_i)}) \\ &\times \prod_{k=1}^{r}\text{cum}\Big(d_{a_{i_{k1}j_{k1}}}(\gamma_{i_{k1}}),d_{a_{i_{k2}j_{k2}}}(\gamma_{i_{k2}})\Big)d\lambda_1\ldots d\lambda_r d\alpha_1\ldots d\alpha_r\Big) \end{aligned}$$

is the sum over all indecomposable partitions whose sets contain exactly 2 elements. Obviously we have $G_1 = 0$ because each summand contains at least one term of the form $\text{cum}(d_{a_{ij}}(\gamma_i))$. For the term G_2 we obtain

$$\begin{aligned} |G_2| &= \Bigg|\Big(\frac{1}{2\pi\sqrt{T}}\Big)^r\Big(\sum_{Q^2}\sum_{a_{11}\ldots a_{r2}=1}^{d}\int_{\Pi^{2r}}\prod_{i=1}^{r}K_b(\lambda_i-\alpha_i)(\Omega_{(la_{i2}-1)\cdot d+a_{i1}}(\lambda_i)+\overline{\Omega_{(a_{i1}-1)\cdot d+a_{i2}}(\lambda_i)}) \\ &\times \Big(\prod_{k=1}^{p}(2\pi)^{p_k-1}H(\overline{\gamma_k})f_{a_{i_{k1}},j_{k1},\ldots,a_{i_{kp_k}}j_{kp_k}}(\gamma_{i_{k1}},\ldots,\gamma_{i_{kp_k}})\Big)d\lambda_1\ldots d\lambda_r d\alpha_1\ldots d\alpha_r\Big)\Bigg| \\ &\le \frac{b^r C}{T^{r/2}}\sum_{Q^2}\sum_{a_{11}\ldots a_{r2}=1}^{d}\int_{\Pi^{2r}}\prod_{i=1}^{r}L^{1/b}(\lambda_i-\alpha_i)^2\prod_{k=1}^{p}L^T(\overline{\gamma_k})\ d\lambda_1\ldots d\lambda_r d\alpha_1\ldots d\alpha_r \\ &\le \frac{C}{T^{r/2}}\sum_{Q^2}\sum_{a_{11}\ldots a_{r2}=1}^{d}\int_{\Pi^r}\prod_{k=1}^{p}L^T(\overline{\gamma_k})\ d\alpha_1\ldots d\alpha_r \end{aligned}$$

where we have used the inequality $|H(\lambda)| \le c \cdot L^T(\lambda)$ where L^T denotes the 2π-periodic function defined by

$$L^T(\lambda) = \begin{cases} T & \text{if } |\lambda| \le 1/T \\ \frac{1}{|\lambda|} & \text{if } 1/T < |\lambda| \le \pi \end{cases}$$

and the inequality

$$K_b(\lambda) \le c \cdot b(L^{1/b}(\lambda))^2$$

for some constant $c \in \mathbb{R}^+$. Because G_2 contains no sets with one element and at least one set with 3 elements we have $p < r$ and obtain with Lemma 2 in Eichler (2008) the estimate

$$(3.25) \qquad |G_2| \le \frac{C}{T^{r/2}} \sum_{Q^2} \sum_{a_{11}\dots a_{r2}=1}^{d} \int_{\Pi^2} T \cdot log(T)^{r-2} d\alpha_{i_{n-1}} d\alpha_{i_n} \le \frac{C \cdot log(T)^{r-2}}{T^{r/2-1}}$$

Finally we use again Lemma 2 in Eichler (2008) and obtain for the term G_3

$$\begin{aligned}
|G_3| &= \Bigg| \Big(\frac{1}{2\pi\sqrt{T}}\Big)^r \Big(\sum_{Q^3} \sum_{a_{11}\dots a_{r2}=1}^{d} \int_{\Pi^{2r}} \prod_{i=1}^{r} K_b(\lambda_i - \alpha_i)(\Omega_{(la_{i2}-1)\cdot d + a_{i1}}(\lambda_i) + \overline{\Omega_{(a_{i1}-1)\cdot d + a_{i2})}(\lambda_i)}) \\
&\times \prod_{k=1}^{r} \text{cum}\Big(d_{a_{i_{k1}j_{k1}}}(\gamma_{i_{k1}}), d_{a_{i_{k2}j_{k2}}}(\gamma_{i_{k2}})\Big) d\lambda_1 \dots d\lambda_r d\alpha_1 \dots d\alpha_r \Big) \Bigg| \\
&= \Bigg| \Big(\frac{1}{2\pi\sqrt{T}}\Big)^r \Big(\sum_{Q^3} \sum_{a_{11}\dots a_{r2}=1}^{d} \int_{\Pi^{2r}} \prod_{i=1}^{r} K_b(\lambda_i - \alpha_i)(\Omega_{(la_{i2}-1)\cdot d + a_{i1}}(\lambda_i) + \overline{\Omega_{(a_{i1}-1)\cdot d + a_{i2})}(\lambda_i)}) \\
&\times \Big(\prod_{k=1}^{r} (2\pi)^{2-1} H(\overline{\gamma_k}) f_{a_{i_{k1},j_{k1}}, a_{i_{k2}j_{k2}}}(\gamma_{i_{k1}}, \gamma_{i_{k2}}) \Big) d\lambda_1 \dots d\lambda_r d\alpha_1 \dots d\alpha_r \Big) \Bigg| \\
&\le \frac{C}{T^{r/2}} \Big(\sum_{Q^3} \sum_{a_{11}\dots a_{r2}=1}^{d} \int_{\Pi^r} \prod_{k=1}^{r} L^T(\overline{\gamma_k}) d\alpha_1 \dots d\alpha_r \Big) \\
&\le \frac{C}{T^{r/2}} \Big(\sum_{Q^3} \sum_{a_{11}\dots a_{r2}=1}^{d} \int_{\Pi^2} L^T(\alpha_{i_{r-1}} \pm \alpha_{i_r})^2 log(T)^{r-2} d\alpha_{i_{r-1}} d\alpha_{i_r} \Big) \\
&\le \frac{C \cdot log(T)^{r-2}}{T^{r/2}} \Big(\sum_{Q^3} \sum_{a_{11}\dots a_{r2}=1}^{d} \int_{\Pi} \int_{\Pi \pm \alpha_{i_r}} L^T(\alpha_{i_{r-1}})^2 d\alpha_{i_{r-1}} d\alpha_{i_r} \Big) = O\Big(\frac{log(T)^{r-2}}{T^{r/2-1}}\Big).
\end{aligned}$$

A combination of (3.24), (3.25) and (3.26) yields $\text{cum}_r(B_{1T}) = o(1)$, whenever $r \ge 3$. This shows (3.9) and the assertion of Theorem 3 follows from (3.8). The proof will now be completed by a proof of the stochastic approximation (3.6).

Proof of (3.6)*:* By means of a Taylor expansion we have

$$(3.26) \qquad \tilde{S}_T = \int_{\Pi} \| \text{vec}(\hat{f}(\lambda) - f(\lambda) \|_{\Gamma_\Psi(\lambda)} d\lambda + \sum_{j=0}^{3} C_j,$$

where the matrix $\Gamma_\Psi(\lambda)$ is defined by $\Gamma_\Psi(\lambda) = D_Z\Psi(f(\lambda),\lambda)^* D_Z\Psi(f(\lambda),\lambda)$, we have used the notation

$$\begin{aligned} C_0 &= 2\int_\Pi \mathrm{Re}\Big\langle \Psi(f(\lambda),\lambda); D_Z\Psi(f(\lambda),\lambda)\mathrm{vec}(\hat f(\lambda)-f(\lambda))\Big\rangle d\lambda \\ C_1 &= \int_\Pi \parallel R(\lambda)\parallel^2 d\lambda \\ C_2 &= 2\int_\Pi \mathrm{Re}\Big\langle \Psi(f(\lambda),\lambda); R\Big\rangle\, d\lambda \\ C_3 &= 2\int_\Pi \mathrm{Re}\Big\langle (D_Z\Psi(f(\lambda),\lambda)\cdot \mathrm{vec}(\hat f(\lambda)-f(\lambda))); R(\lambda)\Big\rangle\, d\lambda, \end{aligned}$$

and the remainder is given by

$$R(\lambda) = \Psi(\hat f(\lambda),\lambda) - \Psi(f(\lambda),\lambda) - D_Z\Psi(f(\lambda),\lambda)\mathrm{vec}(\hat f(\lambda)-f(\lambda)). \tag{3.27}$$

We will show exemplarily that $C_2 = O_P((bT)^{-1})$ a corresponding result for C_1 and C_3 can be obtained by similar arguments. For this purpose note for any $\eta > 0, \delta > 0$

$$\begin{aligned} P_2 &= \mathbb{P}\Big(\int_\Pi \mathrm{Re}\Big\langle \Psi(f(\lambda),\lambda); R(\lambda)\Big\rangle\, d\lambda > \eta(bT)^{-1}\Big) \le \mathbb{P}\Big(\int_\Pi |\langle \Psi(f(\lambda),\lambda); R(\lambda)\rangle|\, d\lambda > \eta(bT)^{-1}\Big) \\ &= \mathbb{P}\Big(\int_\Pi \|\Psi(f(\lambda),\lambda)\|\cdot\|R(\lambda)\|\, d\lambda > \eta(bT)^{-1},\ A_\delta\Big) + \mathbb{P}\Big(\int_\Pi \|\Psi(f(\lambda),\lambda)\|\cdot\|R(\lambda)\|\, d\lambda > \eta(bT)^{-1},\ A_\delta^c\Big) \end{aligned}$$

where $A_\delta = \{\omega |\max_{\lambda\in\Pi} \parallel \hat f(\lambda) - f(\lambda)\parallel \le \delta\}$. Observing that $\parallel \Psi(f(\lambda),\lambda)\parallel$ is bounded by a constant, say K (see Assumption (iii) on page 5) it follows

$$\begin{aligned} P_2 &\le \mathbb{P}\Big(K\int_\Pi \|R(\lambda)\|\, d\lambda > \eta(bT)^{-1}\ ,\ A_\delta\Big) + \mathbb{P}(\ A_\delta^c) \\ &\le \mathbb{P}(K\int_\Pi C_\delta \|\hat f(\lambda) - f(\lambda)\| l^2\, d\lambda > \eta(bT)^{-1},\ A_\delta) + \mathbb{P}(\ A_\delta^c) \end{aligned}$$

where we have used the fact that for any $\delta > 0$ there exists a constant $C_\delta > 0$, such that $\|R(\lambda)\| \le C_\delta\|\hat f(\lambda) - f(\lambda)\|^2$ uniformly with respect to $\lambda \in \Pi$, whenever $\max_{\lambda\in\Pi}|\hat f(\lambda) - f(\lambda)| \le \delta$. Therefore an application of the estimate (5) in Eichler (2008) and equation (6.1.17) in Taniguchi and Kakizawa (2000) shows that for every ε there exists a constant η_ε such that

$$P_2 = \mathbb{P}\Big(\int_\Pi Re\Big\langle \Psi(f(\lambda),\lambda); R(\lambda)\Big\rangle\, d\lambda > \eta_\varepsilon(bT)^{-1}\Big) < \varepsilon$$

for sufficiently large T, which yields the estimate $C_2 = 2\int_\Pi \mathrm{Re}\langle \Psi(f(\lambda),\lambda); R(\lambda)\rangle d\lambda = O_P((bT)^{-1})$. Similar arguments for the first term in (3.26) and the terms C_1 and C_3 give

$$C_i = O_P((bT)^{-1})\ ,\ i = 1,2,3 \tag{3.28}$$

$$\int_\Pi \parallel \mathrm{vec}(\hat f(\lambda) - f(\lambda)\parallel_{\Gamma_\Psi}(\lambda)(\lambda) d\lambda = O_P((bT)^{-1}) \tag{3.29}$$

Combining (3.26), (3.28) and (3.29) yields the assertion (3.6), which completes the proof of Theorem 3.1. □

4 Examples

In this Section we discuss several examples to illustrate Theorem 3.1. In particular we consider the problem of testing for "no-correlation" and the problem of comparing the spectral densities of the components of the d-dimensional time series. Throughout this section we assume that the random variables ε_t in (2.1) are normally distributed and real valued. Otherwise a corresponding term reflecting the dependence on cumulants or order 4 has to be added (see Theorem 3.1).

4.1 Comparing spectral densities

The problem of comparing spectral densities has also found considerable attention in the literature. [see e.g. Carmona and Wang (1996), Coates and Diggle (1986), Swanepoel and van Wyk (1986) or Diggle and Fisher (1991) among others. Recently Dette and Paparoditis (2009) considered the case $d = 2$ and proposed to base a test for the hypothesis $H_0 : \ f_{11} = \ldots = f_{dd}$ on the statistic $S_T(\Psi)$ with the functional (1.2). This yields

$$D_Z\Psi(f(\lambda), \lambda) = \frac{d}{(\sum_{i=1}^d f_{ii}(\lambda))^2} \begin{pmatrix} \sum_{i=2}^d f_{ii}(\lambda) & 0 \ldots 0 & -f_{11}(\lambda) & 0 \ldots & -f_{11}(\lambda) \\ -f_{22}(\lambda) & 0 \ldots 0 & \sum_{\substack{i=1 \\ i\neq 2}}^d f_{ii}(\lambda) & 0 \ldots & -f_{22}(\lambda) \\ \vdots & & & & \vdots \\ -f_{dd}(\lambda) & 0 \ldots 0 & -f_{dd}(\lambda) & 0 \ldots & \sum_{i=1}^{d-1} f_{ii}(\lambda) \end{pmatrix},$$

and by a straightforward but tedious calculation we obtain for the quantities M^2, b_b and τ^2 in Theorem 3.1

$$\begin{aligned} M^2 &= \int_\Pi \frac{d^2 \sum_{i=1}^d f_{ii}^2(\lambda) - d(\sum_{i=1}^d f_{ii}(\lambda))^2}{(\sum_{i=1}^d f_{ii}(\lambda))^2} \, dx \\ b_b &= 2d \int_{\Pi^2} K_b(\lambda - x) \frac{\sum_{i=1}^d \left(df_{ii}(\lambda) - \sum_{j=1}^d f_{jj}(\lambda)\right)\left(f_{ii}(x) \sum_{j=1}^d f_{jj}(\lambda) - f_{ii}(\lambda) \sum_{j=1}^d f_{jj}(x)\right)}{(\sum_{i=1}^d f_{ii}(\lambda))^3} \, d\lambda dx \\ \tau^2 &= 8\pi d^4 \int_\Pi \frac{\sum_{b,c=1}^d |f_{bc}(x)|^2 \left(f_{bb}(x) - \frac{\sum_{i=1}^d f_{ii}^2(x)}{\sum_{i=1}^d f_{ii}(x)}\right)\left(f_{cc}(x) - \frac{\sum_{i=1}^d f_{ii}^2(x)}{\sum_{i=1}^d f_{ii}(x)}\right)}{(\sum_{i=1}^d f_{ii}(x))^4} \, dx \end{aligned}$$

Note that in the case $d = 2$ this result does not coincide with the corresponding statement in Dette and Paparoditis (2009) and that there is minor error in this reference.

4.2 Testing for no correlation

The problem of testing for no correlation between $\mathbf{X}_t^A = (X_{t1}, ..., X_{td_1})$ and $\mathbf{X}_t^B = (X_{td_1+1}, ..., X_{td})$ of the real valued d-dimensional stationary process $\{\mathbf{X}_t\}_{t\in\mathbb{Z}}$ with $d = d_1 + d_2$ has been considered in the context of ARMA processes by El Himdi and Roy (1997), Hallin and Saidi (2005), Bouhaddioui and

Roy (2006) and Saidi (2007) and by Eichler (2008) for general multivariate stationary processes using the functional (1.3). In this case we have

$$\left(D_Z\Psi(f(\lambda),\lambda)^T\right)_{ab} = \begin{cases} -\frac{f_{ij}(\lambda)}{2\sqrt{f_{ii}(\lambda)^3 f_{jj}(\lambda)}} & \text{if} \quad a=(i-1)d+i \quad \text{and} \quad b=(i-1)d_2+j \\ \frac{1}{\sqrt{f_{ii}(\lambda)f_{jj}(\lambda)}} & \text{if} \quad a=(i-1)d+j \quad \text{and} \quad b=(i-1)d_2+j \\ -\frac{f_{ij}(\lambda)}{2\sqrt{f_{ii}(\lambda)f_{jj}(\lambda)^3}} & \text{if} \quad a=(j-1)d+j \quad \text{and} \quad b=(i-1)d_2+j \\ 0 & \text{else} \end{cases}$$

which yields for the constants in Theorem 3.1

$$M^2 = \sum_{i=1}^{d_1}\sum_{j=d_1+1}^{d}\int_\Pi \frac{|f_{ij}|^2(\lambda)}{f_{ii}(\lambda)f_{jj}(\lambda)}\, d\lambda$$

$$b_b = \sum_{i=1}^{d_1}\sum_{j=d_1+1}^{d}\int_\Pi\int_\Pi \frac{K_b(\lambda-x)}{f_{ii}(\lambda)f_{jj}(\lambda)}\Big(2\mathrm{Re}(f_{ij}(\lambda)\overline{f_{ij}(x)}) - |f_{ij}(\lambda)|^2\Big(\frac{f_{ii}(x)}{f_{ii}(\lambda)}+\frac{f_{jj}(x)}{f_{jj}(\lambda)}\Big)\Big)dx\, d\lambda$$

$$\tau^2 = 4\pi\sum_{i,j,k,l=1}^{d}\int_\Pi f_{ik}(\lambda)\overline{f_{jl}(\lambda)}g_{(i-1)d+j}(\lambda)\overline{g}_{(k-1)d+l}(\lambda) + \mathrm{Re}(f_{il}(\lambda)\overline{f_{jk}(\lambda)}g_{(i-1)d+j}(\lambda)\overline{g}_{(k-1)d+l}(\lambda)),$$

where $g_i(\lambda)$ is defined as

$$g_i = \begin{cases} -\sum_{c=d_1+1}^{d}\frac{|f_{ac}(\lambda)|^2}{2f_{aa}^2(\lambda)f_{cc}(\lambda)} & \text{if} \quad i=(a-1)d+a \quad a\in\{1,...,d_1\} \\ -\sum_{c=1}^{d_1}\frac{|f_{ca}(\lambda)|^2}{2f_{aa}^2(\lambda)f_{cc}(\lambda)} & \text{if} \quad i=(a-1)d+a \quad a\in\{d_1+1,...,d\} \\ \frac{\overline{f_{ba}(\lambda)}}{f_{aa}(\lambda)f_{bb}(\lambda)} & \text{if} \quad i=(a-1)d+b \quad a\in\{d_1+1,...,d\} \quad b\in\{1,...,d_1\} \\ 0 & \text{else} \end{cases}$$

If we are interested whether the i and jth component of $\mathbf{X}_t$ are uncorrelated the function Ψ is given by

$$\Psi(f(\lambda),\lambda) = \frac{f_{ij}(\lambda)}{\sqrt{f_{ii}(\lambda)f_{jj}(\lambda)}}.$$

and the terms b_b and τ^2 reduce to

$$b_b = \int_\Pi\int_\Pi \frac{K_b(\lambda-x)}{f_{ii}(\lambda)f_{jj}(\lambda)}\Big(2\mathrm{Re}(f_{ij}(\lambda)\overline{f_{ij}(x)}) - |f_{ij}(\lambda)|^2\Big(\frac{f_{ii}(x)}{f_{ii}(\lambda)}+\frac{f_{jj}(x)}{f_{jj}(\lambda)}\Big)\Big)dx\, d\lambda$$

and

$$\tau^2 = 4\pi\int_\Pi \frac{|f_{ij}(\lambda)|^2}{f_{ii}^3(\lambda)f_{jj}^3(\lambda)}\Big(|f_{ij}(\lambda)|^4 + f_{ii}^2(\lambda)f_{jj}^2(\lambda) - 2f_{ii}(\lambda)f_{jj}(\lambda)\mathrm{Re}(f_{ij}(\lambda))^2\Big)d\lambda$$

respectively.

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