Characteristic numbers from 2-cocycles on formal groups

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Abstract. We give explicit polynomial generators for the homology rings of $BSU$ and $BSpin$ for complex oriented theories. Using these we are able to provide an alternative proof of the result of Hopkins, Ando and Strickland for symmetric 2-cocycles on formal group laws.

1. Introduction and statement of results

Hopkins, Ando and Strickland have recently shown (see [AHS98][Hop95][HMM98]) that for any complex oriented theory $E$ the ring $E_\ast BSU$ carries the universal symmetric 2-cocycle on the formal group of $E$. In this paper we give an alternative proof of their result which is based on the following choice of polynomial generators for $E_\ast BSU$: Let $L$ be the canonical line bundle over $\mathbb{C}P_\infty$ and let $\beta_i \in E_2 \mathbb{C}P_\infty$ be dual to $c_1(L^i)$. Let $f : \mathbb{C}P_\infty \times \mathbb{C}P_\infty \to BSU$ be the map which classifies the product $(1 - L_1)(1 - L_2)$. For each natural number $k$ and $1 \leq i \leq k - 1$ choose integers $n_{ik}$ such that
\[ \sum_{i=1}^{k-1} n_{ik}^i \binom{k}{i} = \text{g.c.d.} \{ \binom{k}{1}, \ldots, \binom{k}{k-1} \} . \]

Then our first result is

**Theorem 1.1.** Define elements
\[ d_k = \sum_{i=1}^{k-1} n_{ik}^i f_\ast (\beta_i \otimes \beta_{k-i}) \in E_{2k} BSU. \]

Then for any complex oriented $E$ we have
\[ E_\ast BSU \cong E[c_2, d_3, d_4, \ldots]. \]

It must be emphasized that the conceptual basis and the proof of the above theorem owes many ideas to the work of [AHS98]. However, the present approach is more elementary and does not use the language of schemes. We also show how the generators relate to the map from $E_\ast BSpin$.

Next we investigate the homology ring of $BSpin$ for mod 2 $K$-theory. Our main result is

**Theorem 1.2.** Let $\omega$ be the canonical quaternion line bundle over $\mathbb{H}P_\infty$ and let $z_k \in K_\ast(\mathbb{H}P_\infty; \mathbb{F}_2)$ be dual to $c_2(\omega)^i$. Setting $d_{2k}' = d_{2k} + z_k \in K_\ast(BSpin; \mathbb{F}_2)$ for all $k$ we have
\[ K_0(BSpin; \mathbb{F}_2) \cong \mathbb{F}_2[d_{2k} | k \neq 2^s] \otimes \mathbb{F}_2[d_4', d_8', d_16', \ldots]. \]

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Moreover, each \( z_k \) is decomposable in \( K_0(\text{BSpin}; F_2) \).

As a consequence, we are able to give a new proof of the result of [HAS99] that the ring \( K_*(\text{BSpin}; F/2) \) carries the universal real symmetric 2-cocycle for the multiplicative formal group.

2. The homology of \( \mathbb{C}P^\infty \) and binomial coefficients of formal group laws

Recall from [Ada74] that for any complex oriented ring theory \( E \) we are given a class \( x \in \tilde{E}^2\mathbb{C}P^\infty \) such that

\[
E^*\mathbb{C}P^\infty \cong \pi^*E[x].
\]

The \( H \)-space structure of \( BS^1 \cong \mathbb{C}P^\infty \) induces a comultiplication

\[
\mu^*: E^*\mathbb{C}P^\infty \rightarrow E^*\mathbb{C}P^\infty \hat{\otimes} E^*\mathbb{C}P^\infty; \; x \mapsto x + F y
\]

and a ring structure map

\[
E_*\mathbb{C}P^\infty \otimes E_*\mathbb{C}P^\infty \rightarrow E_*\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\mu_*} E_*\mathbb{C}P^\infty.
\]

Let \( \beta_0 = 1, \beta_1, \beta_2, \ldots \) be the additive basis of \( E_*\mathbb{C}P^\infty \) dual to \( x^0, x^1, x^2, \ldots \).

**Definition 2.1.** The binomial coefficients of the formal group law \( F \)

\[
\binom{k}{i,j}_F \in \pi_{2(i+j-k)}E
\]

are defined by the equation

\[
(x + F y)^k = \sum_{i,j} \binom{k}{i,j}_F x^i y^j.
\]

With this notation we easily see

**Lemma 2.2.**

\[
\beta_i \beta_j = \sum_{k=0}^{i+j} \binom{k}{i,j}_F \beta_k.
\]

**Example 2.3.** Let \( E \) be integral singular homology. Then \( F \) is the additive formal group law \( \tilde{G}_a \) and

\[
\binom{k}{i,j}_{\tilde{G}_a} = \begin{cases} \binom{k}{i} & \text{if } i + j = k \\ 0 & \text{else} \end{cases}.
\]

Hence \( \mathbb{H}Z_*\mathbb{C}P^\infty \) is the divided power algebra \( \Gamma[\beta_1] \).

Next let \( E \) be \( K \)-theory with its standard orientation \( F = \tilde{G}_m \). Then

\[
(x + \tilde{G}_m y)^k = (x + y - v^{-1}xy)^k = \sum_{s=0}^{k} \sum_{t=0}^{k} \binom{k}{s,t}(-v)^{s-k}x^{k-s+t}y^{k-t}
\]

and hence

\[
\binom{k}{i,j}_{\tilde{G}_m} = \binom{k}{2k-i-j}_F \binom{k}{k-i-j}(-v)^{k-i-j} = \frac{k!}{(i+j-k)! (k-j)! (k-i)!} (-v)^{k-i-j}
\]

Finally, let \( E \) be complex bordism \( MU \). The coefficients of the universal formal group law \( FGL \) are the Milnor hypersurfaces \( H_{i,j} \) of type \((1,1)\) in \( \mathbb{C}P^i \times \mathbb{C}P^j \) and hence

\[
\binom{k}{i,j}_{FGL} = \sum_{i_1 + \ldots + i_k = i} \prod_{j=1}^{k} H_{i,j}.
\]
We close this section with an observation which will help finding the binomial coefficients in the situation of positive characteristic. Let $F$ be a formal group law with coefficients in an $F_p$-algebra. Then we define

\[ x + pF y = \sum_{i,j} \binom{k}{i,j}_F x^i y^j. \]

Since $x^p + pF y^p = (x + F y)^p$ we obtain a new formal group law $pF$, i.e. it satisfies the associativity condition. The group law $pF$ carries the name Frobenius of $F$. Be aware that the grading has changed

\[ \binom{k}{i,j}_F = 2p(i + j - k) \]

**Example 2.4.** Let $H_n$ be the Honda formal group law of height $n$. Then the $[p]$-series of $pH_n$ read

\[ [p](x^p) = x^p + pH_n \cdots + pH_n x^p = (x + H_n \cdots + H_n x)^p = (v_n x^{p^n})^p = v_n^p x^{p^{n+1}}. \]

Hence $pH_n$ again is the Honda group law of height $n$ over the ring $F_p[v_n^{\pm 1}]$.

**Lemma 2.5.** Let $k = \sum_{i=0}^n k_ip^i$ be the $p$-adic expansion of $k$. Then mod $p$

\[ \binom{k}{i,j}_F = \sum_{i_0, j_0} \prod_{s=0}^{n} \binom{k_s}{i_s, j_s}_F p^{i_s} p^{j_s}. \]

In particular

\[ \binom{pk}{p^i, p^j}_F = \binom{k}{i,j}_F. \]

**Proof.**

\[ (x + F y)^k = \prod_{s=0}^{n} (x^{p^s} + p^sF y^{p^s})^{k_s} = \prod_{s=0}^{n} \sum_{i_s, j_s} \binom{k_s}{i_s, j_s}_F x^{i_s p^s} y^{j_s p^s} \]

\[ = \sum_{i_0, j_0} \prod_{s=0}^{n} \binom{k_s}{i_s, j_s}_F x^{i_0 p^s} y^{j_0 p^s}. \]

\[ \square \]

### 3. The homology ring of $BSU$ and symmetric 2-cocycles on formal groups

In the following let $f : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \rightarrow BSU$ be the map which classifies $(1-L_1)(1-L_2)$. Even though $f$ is not a map of $H$-spaces we may use it to produce interesting classes in $E_1BSU$. Let $a_{ij} \in E_2(i+j)BSU$ be the image of $\beta_i \otimes \beta_j$ under the induced map $f_*$.  

**Lemma 3.1.** The following relations hold for all $i, j, k$

\[ (4) \quad a_{0,0} = 1 \quad ; \quad a_{0i} = a_{i0} = 0 \quad \text{for all } i \neq 0 \]

\[ (5) \quad a_{ij} = a_{ji} \]

\[ (6) \quad \sum_{l,s,t} \binom{l}{s,t}_F a_{j-s, k-t} a_{i} = \sum_{l,s,t} \binom{l}{s,t}_F a_{lk} a_{i-s, j-t} \]
The first relation is obvious. Let $\tau$ be the self map of $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$ which switches the two factors. Then the second relation immediately follows from the fact that $f\tau$ is homotopic to $f$. To do the last consider the two maps $g, h$ from $(\mathbb{CP}^\infty)^3$ to $(\mathbb{CP}^\infty)^4$ given by

$$
g(x, y, z) = (y, z, x, yz)$$

$$
h(x, y, z) = (xy, z, x, y)$$

Their effect on our generators is

$$
g_*(\beta_i \otimes \beta_j \otimes \beta_k) = \sum_{l,s,t} \left( \begin{array}{c} l \end{array} \right)_F \beta_{j-s} \otimes \beta_{k-t} \otimes \beta_i \otimes \beta_l$$

(7)

$$
h_*(\beta_i \otimes \beta_j \otimes \beta_k) = \sum_{l,s,t} \left( \begin{array}{c} l \end{array} \right)_F \beta_i \otimes \beta_k \otimes \beta_{-s} \otimes \beta_{-t}$$

(8)

This can be verified by pairing the left hand side with the cohomological monomials in the the $x_i$'s. The maps $g$ and $h$ become homotopic in $BSU$ when composed with $\mu(f \times f)$ since

$$
(\mu(f \times f)g)_*\xi_{univ} = (1 - L_2)(1 - L_3) + (1 - L_1)(1 - L_2) = (\mu(f \times f)h)_*\xi_{univ}
$$

The desired relation now follows from the above by applying $\mu(f \times f)_*$ to the right hand side of (7) and (8).

There is another way to look at the classes $a_{ij}$ and the relations of 3.1. First note that $E \wedge BSU_+$ is itself a complex oriented ring theory with

$$
x_{E \wedge BSU_+} = (1 \wedge \eta)_*x_E
$$

In abuse of the notation we will simply denote this orientation by $x$ in the following. Hence, we may view

$$
(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)_+ \xrightarrow{f_+} BSU_+ \xrightarrow{\eta \wedge 1} E \wedge BSU_+
$$

as a power series

$$
f(x, y) = 1 + \sum_{i,j \geq 1} b_{ij} x^i y^j \in (E \wedge BSU_+)^0(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)
$$

for some $b_{ij} \in E_{2(i+j)}BSU$. Of course, we have

$$
b_{ij} = \sum_{k,l} b_{ij}(1 \wedge \eta)_* (\beta_i \otimes \beta_j, x^k y^l)
$$

$$
= \langle (1 \wedge \eta)_* \beta_i \otimes (1 \wedge \eta)_* \beta_j, 1 + \sum_{k,l} b_{kl} x^k y^l \rangle
$$

$$
= \langle (1 \wedge \eta)_*(\beta_i \otimes \beta_j), f^*(\eta \wedge 1) \rangle = (\mu f(1 \wedge \eta))_*(\beta_i \otimes \beta_j) = a_{ij}.
$$

The power series $f$ is a symmetric 2-cocycle or 2-structure on the formal group law $F$ in the sense of [AHS98][3.1][HAS99][1.2]: This means that the relations

$$
f(x, 0) = f(0, y) = 1$$

(9)

$$
f(x, y) = f(y, x)$$

(10)

$$
f(y, z)f(x, y + F z) = f(x + F y, z)f(x, y).
$$

(11)

hold. In fact, one easily checks that these are equivalent to 3.1.
Remark 3.2. There is another way to look at symmetric 2-cocycles: any such \( f \) defines a commutative central extension
\[
G_m \rightarrow E \rightarrow F
\]
of \( F \) by the multiplicative formal group. Here, \( E \) is the product \( F \times G_m \) and the group structure is given by the formula
\[
(a, \lambda) \cdot (b, \mu) = (a + b, f(a, b)\lambda \mu).
\]
The equation 11 is then equivalent to the associativity of the multiplication. These objects have been extensively studied; for example Lazard’s symmetric 2-cocycle lemma classifies central extensions of the additive formal group by itself
\[
G_a \rightarrow E \rightarrow G_a.
\]

Definition 3.3. For any complex oriented \( E \) let \( C_2(E) \) be the graded ring freely generated by symbols \( a_{ij} \)'s subject to the relations of 3.1. We write \( \alpha \) for the canonical map from \( C_2(E) \) to \( E^* \text{BSU} \).

Here, we denoted the generators of \( C_2(E) \) and \( E^* \text{BSU} \) by the same letters to simplify the notation. We will see below that the map \( \alpha \) is an isomorphism.

Equivalently, given any 2-cocycle \( f' \) on the formal group \( F \) over an \( \pi^* E \)-algebra \( S \) then there is a unique algebra homomorphism \( \varphi : E^* \text{BSU} \rightarrow S \) with \( \varphi f = f' \). Hence, \( E^* \text{BSU} \) carries the universal symmetric 2-cocycle on \( F \) which is the result of [AHS98].

Consider the canonical map of \( H \)-spaces \( \iota : \text{BSU} \rightarrow \text{BU} \). Writing \( g \) for the map from \( CP^\infty \) to \( \text{BU} \) which classifies \( 1 - L \) we see from the homotopy equivalence \( \iota + g : \text{BSU} \times CP^\infty \rightarrow \text{BU} \) that \( \iota \) is an inclusion in homology. It is well known [Ada74] that \( E_* \text{BU} \) is a polynomial algebra with generators \( b_i = g_* \beta_i \). Alternatively, let \( g' \) be the map which classifies \( L - 1 \). Then the classes \( g' \beta_i = b'_i \) again give polynomial generators of \( E_* \text{BU} \). Since the map \( \mu_{\text{BU}}(g \times g') \Delta \) is null the generators \( b_i \) and \( b'_i \) are related by the equation
\[
\sum_{i=0}^\infty b'_i x^i = (\sum_{i=0}^\infty b_i x^i)^{-1}.
\]

Proposition 3.4. We have the formula
\[
\iota_* a_{ij} = \sum_{s=0, \ldots, i \geq 0, \ldots, j \geq t} \binom{k}{s, t}_F b'_{k i - s} b_{j - t}
\]
In particular, modulo decomposables in \( \tilde{E}_* \text{BU} \)
\[
\iota_* a_{ij} = \sum_{k=0}^{i+j} \binom{k}{i, j}_F b'_k.
\]

Proof. Decompose \( f \) by writing
\[
f^* \xi_{\text{univ}} = 1 - L_1 - L_2 + L_1 L_2 = (L_1 L_2 - 1) + (1 - L_1) + (1 - L_2)
\]
\[
= (g' \mu_{CP} + g p_1 + g p_2)^* \xi_{\text{univ}}
\]
and calculate
\[ \iota_* a_{ij} = \iota_* f_*(\beta_i \otimes \beta_j) \]
\[ = \mu_*^{BU}(g\mu_*^{pp} \otimes g p_1 \otimes g p_2) \sum_{i_1 + j_2 + j_3 = i} \beta_{i_1} \otimes \beta_{j_2} \otimes \beta_{j_3} \otimes \beta_{j_3} \]
\[ = \sum_{i_1 + j_2 + j_3 = i} \sum_{k=0}^{i_1 + j_1} \binom{k}{i_1, j_2} F b_k b_i b_j. \]

Now choose \( n_i^k \) and \( d_k \) as in 1.1 and set
\[ \epsilon(k) \coln \sum \binom{n_i^k}{i} = \gcd \{ \binom{k}{1}, \ldots, \binom{k}{k-1} \} = \{ p \text{ for } k = p^2 \} \]

For a graded ring \( R \) we write \( R^+ \) for the elements in positive degrees and \( Q(R) = R/R^2 \).

**Corollary 3.5.** In \( Q(E_*(BU)) \) we have \( \iota_* d_k = \epsilon(k)b'_k \).

**Proof.** Using the identity \( \binom{s+t}{s} \coln \binom{s+t}{t} \) we compute with the proposition
\[ \iota_* d_k = \sum_{i+j=k} n_k^i \iota_* a_{ij} = \sum_{i=1}^{k-1} \binom{k}{i} b_k = \epsilon(k)b'_k. \]

Before proving 1.1 we need two more lemmas.

**Lemma 3.6.** For all \( s, t \) we have
\[ a_{st} = \binom{s+t}{s} / \epsilon(s+t) \coln Q_2(s+t)(C_2(E)). \]

**Proof.** The third relation of 3.1 reads modulo \( C_2(E)^2 \)
\[ \binom{n}{n-t} a_{mn} = \binom{s}{m} a_{st} \]
for all \( m + n = s + t, m \leq s \). We conclude
\[ \binom{s+t}{s} / \epsilon(s+t) \coln \sum_{m+n=s+t} n_{s+t}^{m+n} \binom{s+t}{m} = a_{st} \]

**Lemma 3.7.** (i) For all \( s \geq 0 \) and prime numbers \( p \) the map
\[ Q_2 p^{s+t} : Q_2 p^s(H_*(BSU; F_p)) \rightarrow Q_2 p^t(H_*(BU; F_p)) \]
is null.
(ii) Let \( \rho_t \) denote the Poincaré series of a graded vector space. Then we have
\[ \rho_t(Q(H_*(BSU; F_p)) = (1 - t^3)^{-1} - t^2. \]
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PROOF. (i) For a Hopf algebra $A$ let us write $P(A)$ for the group of primitives. It is enough to show the dual statement that the map

$$P_{2p^r} : P_{2p^r}(H^*(BU; F_p)) \to P_{2p^r}(H^*(BSU; F_p))$$

vanishes. The $p^r$-power of the first Chern class generates the source since

$$c_1^{p^r}(\xi \oplus \eta) = c_1^{p^r}(\xi) + c_1^{p^r}(\eta)$$

and the dual is one dimensional. This class obviously vanishes in $H^*(BSU; F_p)$.

(ii) As in [Sin67] 1.4 and 1.5 one sees

$$\rho_t(Q(H_*(BSU; F_p))) = \rho_t(P(H_*(BSU; F_p))) = \rho_t(Q(H^*(BSU; F_p)))$$

$$= \rho_t(Q(F_2[c_2, c_3, \ldots])) = (1 - t^2)^{-1} - t^2$$

□

PROOF OF 1.1. The proof will fall into several steps: First consider the case when $E$ is rational ordinary homology. Then by 3.5 the composite

$$\mathbb{Q}[d_2, d_3, \ldots] \to H_*(BSU; \mathbb{Q}) \xrightarrow{\sim} H_*(BU; \mathbb{Q}) \to \mathbb{Q}[b_1', b_2', \ldots]/b_1'$$

is a surjection and consequently is an isomorphism. Thus there cannot be any relation between the monomials in the $d_i$’s and the statement follows from the homotopy equivalence

$$e + g : BSU \times \mathbb{C}P^\infty \cong BU.$$  

by counting dimensions in each degree.

Next observe that the class $d_k$ must generate $Q_{2k}(H_*(BSU; F_p))$ for all $k \geq 2$: by 3.7(ii) this vector space is one dimensional for any prime $p$. Pick a generator $e$ of the latter. Then a multiple $n$ of $e$ coincides with $d_k$. If $k$ is not a prime power the integer $n$ is invertible since the element $d_k$ is sent to the generator $b_k$ under the map to $BU$. For prime power degrees the integer $n$ again can not be a multiple of $p$ since else a multiple of $e$ is mapped to $b_k$ by 3.5 which contradicts 3.7 (i). In particular, we have shown that the canonical map

$$F_p[d_2, d_3, \ldots] \to H_*(BSU; F_p)$$

is a surjection which in turn means that it is an isomorphism. The theorem now holds for integral singular homology.

Next let $E$ be complex bordism $MU$. Since the Atiyah Hirzebruch spectral sequence collapses we may choose an isomorphism of $\pi_*MU$-modules

$$MU_*, BSU \cong E_\infty = E_2 = H_*(BSU; \pi_*MU).$$

It is enough to show that a monomial in the $d_k$’s reduces to the corresponding monomial on the 0-line of the $E_2$-term $H_*(BSU; \mathbb{Z})$ since then the canonical map

$$\pi_*MU[d_2, d_3, \ldots] \xrightarrow{\cong} MU_*, BSU$$

is an isomorphism. This follows from the fact that the map induced from the complex orientation from $MU, BSU$ to $H_*(BSU; \mathbb{Z})$ respects the $d_k$’s and is the projection onto the 0 line of the spectral sequence.

Finally, for arbitrary complex oriented $E$ we may simply tensor the isomorphism

$$\pi_*MU[d_2, d_3, \ldots] \xrightarrow{\cong} MU_*, BSU$$

with $\pi_*E$ and the result follows from the universal coefficients spectral sequence [Ada69]. □

COROLLARY 3.8. The map $\alpha : C_2(E) \to E_*, BSU$ is an isomorphism.
Proof. Consider the obvious map $\varphi : \pi_* E[d_2, d_3, \ldots] \to C_0(E)$. Since its composite with $\alpha$ is an isomorphism $\varphi$ must be injective. Moreover, 3.6 tells us that the $a_{ij}$ can be written as polynomials in the $d_i$’s. Consequently $\varphi$ is an isomorphism and so is $\alpha$.

It is hard to give an explicit formula for the $n_{ij}$. However, in the situation of positive characteristic we are better off.

Lemma 3.9. (i) It is possible to choose $n_{ij} = 0 \bmod p$ for all $i$ not equal to $p^{r+1}$.
(ii) If $n$ is not a prime power it is possible to choose $n_i = 0 \bmod p$ for all $i$ not equal to $p^{e(n)}$. Here, $\nu_p(n)$ is the exponent of $p$ in the prime decomposition of $n$.

Proof. $p^s!/(p^s - p^{s-1})!$ is once more divisible than $p^{s-1}$. Hence $(p^{s-1})/p$ is not divisible by $p$ and we find $a, b \in \mathbb{Z}$ such that

$$a \left( \frac{p^s}{1} \right) + b \left( \frac{p^{s-1}}{p} \right) = 1.$$ 

Hence we take $n_{ij} = pa$ and $n_{p^{s-1}}^p = b$.

A similar argument works for the second statement: Since $(n_{p^{e(n)}}) \bmod p$ is not divisible by $p$ we find natural numbers $a_0, a_1, \ldots, a_{n-1}$ such that

$$a_0 \left( \frac{n}{p^{e(n)}} \right) + \sum_{i=1, \ldots, n-1; i \neq p^{e(n)}} a_i p \left( \frac{n}{i} \right) = 1$$

which gives the result. 

4. The homology ring of $BSp$

In this section we are going to determine the canonical map from $BSp$ to $BSU$ in $E$-homology for complex oriented theories. The calculations will prove useful in things to come.

The (trivial) fibration $det : U(n) \to S^1$ with fibre $SU(n)$ allows an identification of $E^*BU(n)$ with $E^*BU(2) = \pi^* E[c_1, \ldots, c_n]$ divided by the ideal generated by the first $E$-Chern class $c_1$ of the determinant bundle. The determinant restricted to the standard maximal torus of $U(n)$ is just the multiplication map. Hence in formal Chern roots we compute

$$c_1(det) = x_1 + x_2 + \ldots + x_n.$$ 

In particular, for the $E$-cohomology of $\mathbb{H}P^\infty \cong BSU(2)$ we have

$$c_1(det) = x_1 + x_2 = c_1(\omega) + \text{terms of higher order}$$

Here, $\omega$ is the canonical quaternionic line bundle over $\mathbb{H}P^\infty$. This gives the

Proposition 4.1. $E^* \mathbb{H}P^\infty \cong \pi^* E[c_2(\omega)]$.

Observe that in general $c_1(\omega)$ does not vanish: for $K$-theory we get

$$x_1 + x_2 = x_1 + x_2 - x_1x_2 = c_1(\omega) - c_2(\omega)$$

and the first two Chern classes hence coincide.

Let $z_i \in E_1[\mathbb{H}P^\infty]$ be dual to $c_2(\omega)^i$. Abusing the notation denote the image of $z_i$ under the canonical map

$$E_* \mathbb{H}P^\infty = E_* BSp(1) \to E_* BSp.$$ 

by the same letter. Then one easily verifies

Proposition 4.2. $E_* BSp \cong \pi_* E[z_1, z_2, \ldots]$. 


We next consider the standard fibration \( p : \mathbb{C}P^{2k+1} \to \mathbb{H}P^k \) with fibre \( \mathbb{C}P^1 \).

The quaternion line bundle \( \omega \) splits on the total space \( p^* \omega \cong L \oplus jL \).

Here, \( L \) is the canonical complex line bundle. Moreover, since \( i \) anti-commutes with \( j \) we see that \( jL \cong \bar{L} \).

When passing to infinity the fibration fits into the commutative diagram

\[
\begin{array}{ccc}
\mathbb{H}P^\infty & \xrightarrow{h} & \text{BSU} \\
p \downarrow & & p^* \downarrow \\
\mathbb{C}P^\infty & \xrightarrow{(1 \land 1) \Delta} & \mathbb{C}P^\infty \times \mathbb{C}P^\infty
\end{array}
\]

because \( (1 \land 1)^* \Delta^*(1 - L_1)(1 - L_2) = 1 - L - \bar{L} + L\bar{L} = (1 - L) + (1 - \bar{L}) \).

Hence, in homology the map \( E_* \mathbb{C}P^\infty \to E_* \mathbb{H}P^\infty \to E_* \text{BSU} \) sends \( \beta_k \) to the \( k \)th coefficient of the power series

\[
f(x, -Fx) = \sum_{i,j} a_{ij} x^i (-Fx)^j.\]

Here, \( f(x, y) \) is the universal symmetric 2-cocycle on the formal group law \( F \). It is not hard to see that \( p \) is a surjection in homology. Thus we should be able to lift each \( z_k \) to \( E_* \mathbb{C}P^\infty \) and compute its image from there. In fact, we have the following nice formula:

**Theorem 4.3.** The map \( p : \mathbb{C}P^+ \to \mathbb{H}P^+ \land E \) is given by the power series

\[
p(x) = \sum_{j=0}^{\infty} z_j x^j (-Fx)^j.
\]

As a consequence, the map \( h \) is determined by the equality of power series

\[
\sum_{i,j} a_{ij} x^i (-Fx)^j = f(x, -Fx) = hp(x) = \sum_j z_j x^j (-Fx)^j.
\]

**Proof.** It remains to compute the image of \( \beta_i \) under \( p_* \):

\[
\left\langle p_* \beta_i, c_2 \right\rangle = \left\langle \beta_i, p^* c_2 \right\rangle = \left\langle \beta_i, c_2(L + \bar{L})^j \right\rangle = \left\langle \beta_i, (x(-Fx))^j \right\rangle.
\]

Hence, \( \beta_i \) is sent to \( \sum_j \left\langle \beta_i, x^j (-Fx)^j \right\rangle z_j \) and the claim follows. \( \square \)

Let us see how this formula works for \( K \)-theory. Setting \( y = -Fx \) the left hand side becomes the symmetric polynomial

\[
\sum_{i,j} a_{ij} x^i y^j = \sum_i a_{ii} (xy)^i + \sum_{i<j} a_{ij} (x^i y^j + x^j y^i)
\]

Let \( Q_k \) be the Newton polynomial expressing the power sum in terms of the elementary symmetric functions \( e_1, e_2 \). That is,

\[
t_1^k + t_2^k = Q_k(e_1, e_2)
\]

and set \( q_k(a) = Q_k(a, a) \). Then since

\[
x + (-Fx) = c_1(\omega) = c_2(\omega) = x(-Fx)
\]
we have
\[ x^k + y^k = q_k(e_2(\omega)). \]
The polynomials \( q_k \) satisfy the Newton identities
\[ q_k = a(q_{k-1} - q_{k-2}) \]
and a simple induction shows
\[ q_k = \sum a_{k,i}x^i y^{k-i}. \]

Hence equation (12) reads
\[ \sum_{i,j} a_{ij} x^i y^j = \sum_k a_{kk} c_k^2 + \sum_{i<j} (-1)^{j-i} \left( \binom{r-i-1}{j-r-1} + \binom{r-i}{j-r} \right) a_{ij} c_j'. \]

We have shown

**Corollary 4.4.** The map \( h_* : K_* BSp \to K_* BSU \) is given by the formula
\[ z_r \mapsto a_{rr} + \sum_{i<j} (-1)^{j-i} \left( \binom{r-i-1}{j-r-1} + \binom{r-i}{j-r} \right) a_{ij}. \]

**5. The \( K \)-homology ring of \( BSpin \) and real symmetric 2-cocycles**

In this section we are going to prove 1.2. First we need

**Theorem 5.1 (Sna75 8.4 8.11).** 
(i) The canonical map
\[ K_* (BSpin; \mathbb{F}_2) \to K_* (BSO; \mathbb{F}_2) \]
is an algebra isomorphism.

(ii) The composite of
\[ K_* (pt; \mathbb{F}_2)[b_2, b_4, b_6, \ldots] \to K_* (BU; \mathbb{F}_2) \xrightarrow{\rho_*} K_* (BSO; \mathbb{F}_2) \]
is an algebra isomorphism. Moreover, each \( \rho_* b_{2k+1} \) lies in the image of
\[ K_* (pt; \mathbb{F}_2)[b_2, b_4, \ldots, b_{2k}]. \]

**Lemma 5.2.** The composite
\[ K_* (H\mathbb{P}^\infty; \mathbb{F}_2) \xrightarrow{h} K_* (BSU; \mathbb{F}_2) \xrightarrow{i_*} K_* (BSO; \mathbb{F}_2) \]
sends \( z_j \) to \( b_j^2 \) modulo the ideal generated by \( b_1^2, b_2^2, \ldots, b_{j-1}^2 \). Hence \( z_j \) is decomposable in \( K_* (BSpin; \mathbb{F}_2) \).

**Proof.** Since the diagram
\[
\begin{array}{ccc}
CP^\infty & \xrightarrow{1} & CP^\infty \\
\downarrow & & \downarrow \\
BU & \longrightarrow & BSO
\end{array}
\]
commutes we get
\[ g(x) = g(-\hat{G}_m x) \in K_* (BSO; \mathbb{F}_2)[x]. \]

Hence using 3.4 we compute
\[ f(x, -\hat{G}_m x) = g'(x + (-\hat{G}_m x)) g(x) g(-\hat{G}_m x) = g(x)^2 = \sum_i b_i^2 x^{2i}. \]
and with 5.2
\[ f(x, -G_m x) = \sum_i z_i x^i (-G_m x)^i. \]

Hence the assertion follows by induction since \( x^i (-G_m x)^j = x^{2j} + o(x^{2j+1}). \) The last claim is a consequence of 5.1(ii).

**Definition 5.3.** For any natural number \( i \) let \( 1_i \) be the set of indices of the 1-digits of \( i \) in its binary decomposition. Declare a new product \( n \star m \) of natural numbers \( n, m \) by
\[ 1_{n \star m} = 1_n \cup 1_m. \]

**Example 5.4.** Let \( i = \sum_j i_j 2^j \) be the 2-adic expansion of \( i \). Then by its definition \( \nu_2(i) \) is the minimum of the set \( 1_i \) and hence
\[ i = (i - 2^{\nu_2(i)}) \star 2^{\nu_2(i)}. \]

The importance of the \( \star \)-product comes from the

**Lemma 5.5.** For the multiplicative formal group law we have modulo 2
\[ (k_{i,j}) = 1 \text{ iff } k = i \star j. \]

**Proof.** Since
\[ (1,1) = (1,0) = (0,0) = 1 \]
and
\[ (0,1) = (0,0) = (0,1) = (1,0) = 0 \]
the lemma holds for \( k = 0,1 \). Hence for arbitrary \( k \) we see with 2.5 that the binomial coefficient is non zero iff \( k_s = i_s \star j_s \) for all \( s \) and the assertion follows.

It is interesting to note

**Corollary 5.6.** Modulo decomposables we have
\[ a_{ij} = \begin{cases} d_{2^{s+1}} & \text{for } i = j = 2^s \smallsetminus \text{else} \\ \end{cases} \]

**Proof.** Modulo decomposables (6) gives
\[ a_{i,j,k} = \sum_i \binom{l}{j,k} a_{il} = \sum_i \binom{l}{i,j} a_{lk} = a_{i \star j,k}. \]

In particular if \( i \neq j \neq 2^s \)
\[ a_{ij} = a_{2^{\nu_2(i) \star j}, i \star j} = a_{2^{\nu_2(i) \star j}, i \star j - 2^{\nu_2(i) \star j}} = d_{i \star j}. \]

Here we used 3.9.

**Proof of 1.2.** We may assume that we have chosen the \( n_k^t \) as in 3.9. By 5.1 it remains to show that the map
\[ \mathbb{F}_2[d_k | k \neq 2^s] \otimes \mathbb{F}_2[d'_s, d'_8, \ldots] \rightarrow \mathbb{F}_2[b_2, b_4, \ldots]; \quad d_k \mapsto \iota_*(d_k), \quad d'_k \mapsto \iota_*(d_{2k} + h_{2^s k}) \]
is an isomorphism. By 3.4 and 5.5 we have
\[ \iota_*(d_{2^{s+1}}) = \iota_* a_{2^{s+2}} = \sum_{s,t=0}^{2^r} b_{s,t} b_{2^r - s} b_{2^r-t} = \sum_{s=0}^{2^r} b'_s b'_{2^r-s} \]
and hence with 5.2
\[ \iota_*(d'_{2^{s+1}}) = b'_{2^r} \mod (b_2, b_4, \ldots, b_{2^r-2}). \]
The claim follows since a similar relation holds for the other $d_k$'s by 3.5:
$\iota_* d_k = \iota_* a_{2\nu(2^i), k-2\nu(2^i)} = b'_k \text{ mod } (b_1 b_2, \ldots, b_{k-1})$ for $k \neq 2^s$.

Since the composite
\[ \mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{f} BSU \xrightarrow{\nu} BSpin \]
is null we have for any complex oriented $E$ the relation
\[ f(x, y) = f(-Fx, -Fy) \in E_* BSpin[x, y] \quad (13) \]

**Example 5.7.** Explicitly, the relations (13) read for $F = \hat{G}_m$
\[ \sum_{s,t} \left( \begin{array}{c} i \\ s-1 \end{array} \right) \left( \begin{array}{c} j \\ t-1 \end{array} \right) a_{st} = a_{ij} \text{ for all } i, j \]
as one checks easily.

Recall from [HAS99] the

**Definition 5.8.** For any complex oriented $E$ let $C_1^r(E)$ be the ring which carries the universal real symmetric 2-cocycles on $F$. That is, $C_1^r(E)$ is the quotient of the graded ring $C_2^r(E)$ by the relations implied by the equation 13. We write $\beta$ for the canonical map from $C_2^r(E)$ to $E_* BSpin$.

We are going to show that $\beta$ is an isomorphism for mod 2 $K$-theory. Note that this statement is wrong for mod 2 singular homology. However, for general $E$ we have

**Lemma 5.9.** The map
\[ (\iota + g)_* : E_* BSU \otimes_{E_* \mathbb{C}P^\infty} E_* BSU \times \mathbb{C}P^\infty \rightarrow E_* BSU \times \mathbb{C}P^\infty \rightarrow E_* BU \]
is an isomorphism of $E_* BSU$-modules.

**Proof.** The diagram
\[ \begin{array}{ccc}
BSU \times BSU \times \mathbb{C}P^\infty & \xrightarrow{\iota \times 1 \times g} & BU \times BU \times BU \\
\downarrow{\rho \times 1} & & \downarrow{\rho \times 1} \\
BSU \times \mathbb{C}P^\infty & \xrightarrow{\iota \times g} & BU \times BU \end{array} \]
commutes.

A complex 1-structure on $F$ simply is a power series $g(x)$ with leading term 1. The universal ring $C_1^r(E)$ of these objects can be identified with $E_* BU$ in the obvious way. A real 1-structure is such a power series $g$ which satisfies the real relation
\[ g(x) = g(-Fx). \]

Let us write $C_1^r(E)$ for the universal ring of real 1-structures. That is $C_1^r(E)$ is $C_1(E)$ subject to the real relation. It is clear that the map $\iota : C_2 \rightarrow C_1$ for which
\[ \iota(f(x, y)) = g(x + Fy)g(x)^{-1}g(y)^{-1} \]
induces a map on the real universal rings which we denote with the same letter. For mod 2 $K$-theory we have

**Proposition 5.10.** The obvious map from $C_1^r$ to $K_*(BSO, F_2)$ is an isomorphism.
Proof. This is an immediate consequence from 5.1 as the composite
\[ F_2[b_2, b_1, \ldots] \longrightarrow C_1(KF_2) \longrightarrow C'_1(KF_2) \]
is easily checked to be surjective with the real relations. □

There is a ring inbetween \( C'_1 \) and \( C'_2 \) which will be useful in the sequel: let \( T(x) \) be the power series \( g(x)g(-\hat{c}_m x)^{-1} \) and let \( C'_{1} \) be the quotient ring of \( C_1 \) subject to the relation generated by the set \( I' \) consisting of the coefficients of (14)
\[ T(x + \hat{c}_m y) = T(x)T(y). \]
Then we have

Lemma 5.11. (i) The canonical map \( \iota' : C'_2 \longrightarrow C'_{1} \) is an injection.
(ii) \( T \) is an even power series.

Proof. For (i) observe that we have
\[ \iota' \frac{f(x + y)}{f(-x, -y)} = \frac{g(x + y)g(x)g(y)}{g(-x - y)g(-x)g(-y)} = \frac{T(x + y)}{T(x)T(y)} \]
when omitting the formal addition from the notation. Hence it suffices to check that the ideal \( I'C_2 \) generated by \( f(x, y)f(-x, -y)^{-1} \) is the intersection of the ideal \( I'C_1 \) with \( C_2 \). By 5.9 there exists a retraction homomorphism \( \rho : C_1 \cong K_*(BU; F_2) \longrightarrow K_*(BSU; F_2) \cong C_2 \) of \( C_2 \)-modules. Hence any
\[ a = \sum_k i_k s_k \in C_2 \]
with \( s_k \in C'_{1} \) and \( i_k \in I' \) satisfies
\[ a = \rho(a) = \sum_k i_k \rho(s_k) \in IC_2 \]
and the first part of the lemma follows.

For the second we have with \( T(x) = \sum t_i x^i \) and 5.5
\[ T(x + y) = \sum_{i,j,k} t_i^j \binom{i}{j,k} x^j y^k = \sum_{j,k} t_{j^*k} x^j y^k \]
and hence with \( T(x + y) = T(x)T(y) \) for each odd \( n \)
\[ t_n = t_{1^*n} = t_1 t_n = 0 \]
since \( t_1 = 0 \). □

Lemma 5.12. The power series \( S(x) = f(x, -x)f(x, x)^{-1} \) with coefficients in \( C'_2 \) satisfies the relation
\[ f(x^2, y^2) = \frac{S(x)S(y)}{S(x + y)}. \]

Proof. The cocycle relation (11) gives
\[
\frac{S(x)S(y)}{S(x + y)} = \frac{f(x + y, x + y)f(x, -x)f(y, -y)f(-x, -y)}{f(-x, x + y)f(y, -y)f(x, x)f(y, y)} = \frac{f(x + y, x + y)f(x, y)f(-x, -y)}{f(x, x)f(y, y)}
\]
and 

\[ f(x^2, y^2) = \frac{f(x^2 + y, y)f(x^2, y)}{f(y, y)} = \frac{f(x, y)f(x + y, x + y)f(x, y)f(x, x + y)}{f(y, y)f(x + y, x)f(x, x)} \]

\[ = \frac{f(x, y)f(x + y, x + y)f(x, y)}{f(y, y)f(x, x)} \]

Hence the claim follows from the real relation \( f(x, y) = f(-x, -y) \). 

**Corollary 5.13.** \( \beta : C_2^0(K\mathbb{F}_2) \longrightarrow K_*(BS\text{Spin}; \mathbb{F}_2) \) is an isomorphism.

**Proof.** The composite of

\[ \pi_1 K\mathbb{F}_2[d_k | k \neq 2^r] \otimes_{\pi_1 K\mathbb{F}_2} \pi_1 K\mathbb{F}_2[d_4, d_8, d_16, \ldots] \longrightarrow C_2^0(K\mathbb{F}_2) \]

with \( \beta \) is an isomorphism. Hence, \( \beta \) must be surjective. It remains to check that the map \( \iota \) from \( C_2^0 \) to \( C_1^0 \) is an injection. First we claim that the power series \( S(x) \) of 5.12 is even. Since \( f(x, x) \) is even we only need to investigate \( f(x, -x) \). Using the injection \( \iota' \) of 5.11 we get

\[ \iota' f(x, -x) = g(x)g(-x)^{-1} = T(x)g(-x)^{-2}. \]

Since \( T(x) \) was even by 5.11(ii) the assertion follows. Next define the ring homomorphism \( \kappa \) from \( C_1^0 \) to \( C_2^0 \) by demanding

\[ \kappa g(x^2) = S(x)^{-1}. \]

Then we see with 5.12

\[ \kappa \iota f(x^2, y^2) = \frac{\kappa g(x^2 + y^2)}{\kappa (g(y^2))} = S(x)S(y) \frac{S(x + y)}{S(x)S(y)} = f(x^2, y^2). \]

Thus the universal property of \( C_2^0 \) shows that we have constructed a left inverse to the map \( \iota \).

**References**


