The extremes of random walks in random sceneries

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Abstract

In this article we analyse the behaviour of the extremes of a random walk in a random scenery. The random walk is assumed to be in the domain of attraction of a stable law and the scenery is assumed to be in the domain of attraction of an extreme value distribution. The resulting random sequence is stationary and strongly dependent if the underlying random walk is recurrent. We prove a limit theorem for the extremes of the resulting stationary process. However, if the underlying random walk is recurrent, the limit-distribution is not in the class of classical extreme value distributions.

Keywords: extremes, stationary sequences, random walk, random scenery
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1 Introduction

Extreme value theory has been developed for independent identically distributed random-variables by Gnedenko in the forties of the past century (see Gnedenko (1943)). He obtained limit-theorems toward special types of random-variables called the extreme value distributions. Later those theorems were generalised to stationary sequences of dependent random-variables. In order to address those problems some concepts of fading dependence were introduced. Among the most advanced of those concepts are the conditions $D(u_n)$ and $D'(u_n)$ which were introduced by Leadbetter to describe weak-mixing type dependence for the tails of stationary sequences. If they are satisfied, the random-sequence behaves essentially like the independent

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sequence with the same individual distributions (see Leadbetter et al. (1983)). Another concept, which quantifies the amount of clustering in the dependent sequence is the extremal index. If the extremal index is larger than zero, the asymptotics of the extremes for the dependent sequence can be compared to the independent situation (see Leadbetter et al. (1983)).

In this article we investigate the behaviour of a special dependent stationary sequence. The example is a random walk in a random scenery, where the random walk is assumed to be in the domain of attraction of a stable Lévy-process and the scenery consists of random-variables which are in the domain of attraction of an extreme value distribution. We will see that if the underlying random walk is recurrent, the sequence does not satisfy the conditions $D(u_n)$ introduced by Leadbetter. However, we can prove a limit-theorem in this situation. In the transient case the resulting sequence satisfies the conditions $D(u_n)$ but not the condition $D'(u_n)$. We also compute the extremal index for the random walk in random scenery. It turns out to be zero when the natural scaling of the underlying random-walk is larger than one.

Our investigation is motivated by the work of Kesten and Spitzer (1979). They proved limit theorems for the sums of stationary sequences arising from recurrent random walks in random sceneries. The resulting limits turned out to be new kinds of self-similar processes with strong dependence in the recurrent case.

We now come to the definition of the model that we will investigate in this article. Let $\{X_k, k \in \mathbb{N}\}$ be a sequence of centered, integer-valued, iid random-variables with the property that for $S_n = X_1 + ... + X_n$ and all $x \in \mathbb{R}$ one has

$$\mathbb{P}\left(n^{-1/\alpha} S_n \leq x\right) \longrightarrow F_\alpha(x) \quad \text{as } n \to \infty,$$

where F_α is the distribution-function of a stable law with characteristic function given by

$$\varphi(\theta) = \exp\left(-|\theta|^\alpha (C_1 + iC_2 \mathrm{sgn}\theta)\right); \alpha \in (0, 2].$$

We will denote by $\{Y(t), t \geq 0\}$ the right continuous α-stable Lévy-process with characteristic function given by $\varphi(t\theta)$. It is well known that the processes

$$S^{(n)}(t) := n^{-1/\alpha} S_{[nt]} \tag{1}$$

converge in distribution toward Y with respect to the Skorohod-topology (see Kesten and Spitzer (1979)).

Let $\{\xi(k), k \in \mathbb{Z}\}$ be a family of $\mathbb{R}$-valued iid random-variables which we assume to be independent of the sequence $\{X_k, k \in \mathbb{N}\}$. The sequence of random-variables $\{\xi(S_n), n \in \mathbb{N}\}$ is called a random walk in random scenery in the literature. The sequence $\{\xi(S_n), n \in \mathbb{N}\}$ is a stationary sequence with some non-trivial long-range dependence.

If the random-variables $\{\xi(n), n \in \mathbb{Z}\}$ belong to the domain of attraction of a stable law with exponentβ and if $\alpha \neq 1$ Kesten and Spitzer (1979) proved a limit theorem for the sum

$$W_n := \xi(S_1) + ... + \xi(S_n).$$

It turns out that the scaled sequence $W^{(n)}(t) := n^{-\delta} W_{[nt]}$ converges in distribution toward a self-similar process. The scaling exponent δ is $1 - 1/\alpha + 1/\alpha\beta$ if $\alpha \in (1, 2]$ and is β if $\alpha \in (0, 1)$. The case $\alpha = 1$ is more difficult, since the underlying random walk then is only zero-recurrent. Beside the fundamental work of Kesten and Spitzer, a lot of refinements and generalisations in various directions were obtained by other authors (see Lang and Nguyen (1983), Shieh (1995), Maejima (1996), Arai (2001), Saigo and Takahashi (2005)).

In this article we investigate the asymptotic behaviour of the maxima

$$K_n := \max\{\xi(S_1), \xi(S_2), ..., \xi(S_n)\}.$$

For this we will assume that the distribution function F of the random variables $\{\xi(n), n \in \mathbb{Z}\}$ is in the domain of attraction of an extreme value distribution $G(x)$. This means that there exist two sequences $\{a_n > 0, n \in \mathbb{N}\}$ and $\{b_n \in \mathbb{R}, n \in \mathbb{N}\}$ such that for

$$M_n := \max\{\xi(1), ..., \xi(n)\}$$

we have

$$\mathbb{P}\left((M_n - b_n)/a_n \leq x\right) = (F(a_n x + b_n))^n \longrightarrow G(x) \quad \text{as } n \to \infty.$$

Three possible limit-distributions G emerge from the application of the convergence of type-theorem: the Fréchet distribution, the Weibull distribution and the Gumbel distribution (see Resnick (1987)). The corresponding domains of attraction are well known and can be found in Resnick (1987).

To the distribution-function G, we associate an extreme value process having finite dimensional distributions defined as follows

$$G_{t_1,...,t_k}(x_1, ..., x_k) := G^{t_1}\left(\bigwedge_{i=1}^{k} x_i\right) G^{t_2-t_1}\left(\bigwedge_{i=2}^{k} x_i\right) \cdot ... \cdot G^{t_k - t_{k-1}}(x_k).$$

The resulting stochastic process $\{Z(t), t > 0\}$ is a Markov-process with non-decreasing paths. A version of this process exists in $D(0, \infty)$. We define the sequence

$$Z^{(n)}(t) := (M_{[nt]} - b_n)/a_n.$$

It is well known that $Z^{(n)}$ converges in distribution toward Z with respect to the Skorohod-topology (see Resnick (1987) p.211).

The difference between the sequence $M_n = \max\{\xi(1), ..., \xi(n)\}$ and the sequence

$K_n = \max\{\xi(S_1), ..., \xi(S_n)\}$ is due to the fact that the random walk $\{S_n, n \in \mathbb{N}\}$ visits certain sites several times. It is obvious that the distribution of K_n depends on the number of sites that the random walk $\{S_k, k \in \mathbb{N}\}$ has visited until time n. In fact it is the range

$$R_n := \text{card}\{S_1, ..., S_n\}$$

of the underlying random walk $\{S_k, k \in \mathbb{N}\}$ which determines the behaviour of the sequence $\{K_n, n \in \mathbb{N}\}$. The asymptotic behaviour of the range for rescaled integer-valued random walks $S^{(n)}$ defined in (1) can be found in Le Gall and Rosen (1991). They present the following results, if $S^{(n)}$ converges toward an α-stable Lévy-process Y:
R1. If $\alpha < 1$ then we have that

$$\frac{1}{n} R_{[nt]} \longrightarrow qt \quad \mathbb{P} - \text{almost surely as } n \to \infty,$$

where $q := \mathbb{P}(S_k \neq 0; k \in \mathbb{N})$.
R2. If $\alpha = 1$ then we have that

$$\frac{h(n)}{n} R_{[nt]} \longrightarrow t \quad \text{in } L^p(\Omega, \mathbb{P}) \text{ as } n \to \infty,$$

where

$$h(n) := 1 + \sum_{k=1}^{n} \mathbb{P}(S_k = 0) \quad \text{is the truncated Green function.}$$

R3. If $1 < \alpha \leq 2$ then we have for all $L \in \mathbb{N}$ and all $t_1 < t_2 < ... < t_L$ that

$$n^{-1/\alpha}(R_{[nt_1]}, ..., R_{[nt_L]}) \longrightarrow (m(Y(0, t_1)), ..., m(Y(0, t_L))) \text{ in distribution as } n \to \infty,$$

where m denotes the Lebesgue measure on $\mathbb{R}$.

Remark: The first statement was proved by Kesten, Spitzer and Whitman for $0 < \alpha \leq 2$. However, q is equal to zero in the transient case, i.e. $\alpha > 1$ (see Spitzer (1976)). For $2/3 \leq \alpha < 1$ one knows that $h(n) \to q^{-1}$ as $n \to \infty$, and (R1) yields the almost sure convergence of $\frac{h(n)}{n} R_{[nt]}$ toward t (see Le Gall and Rosen (1991)). The case $\alpha = 1$ is a particular case since in that situation it is in general not known whether the random walk is transient or recurrent. The second and the third statement were proved by Le Gall and Rosen (1991). We mention that Le Gall and Rosen (1991) only state the marginal convergence in R3. However, their proof also covers the joined convergence described in R3. The reason is the following: To prove the convergence in distribution of $R_{[nt]}$ toward $m(Y(0, t))$ they use Skorohod representation theorem to introduce a process $\{\tilde{S}^{(n)}(t); t \geq 0\}$ which has the same distribution as $\{S^{(n)}(t); t \geq 0\}$ and converges almost surely toward $\{Y(t); t \geq 0\}$

with respect to the Skorohod topology. For all $n \in \mathbb{N}$ the associated range processes $\{\tilde{R}_t^{(n)}; t \geq 0\}$ then have the same distributions as $\{R_{[nt]}; t \geq 0\}$. Le Gall and Rosen then prove that $n^{-1/\alpha}\tilde{R}_t^{(n)}$ converge in L^1 toward $m(Y(0,t))$ for all $t \geq 0$. This also implies the L^1-convergence of the vectors $n^{-1/\alpha}(\tilde{R}_{t_1}^{(n)}, ..., \tilde{R}_{t_L}^{(n)})$ toward the vector $(m(Y(0,t_1)), ..., m(Y(0,t_L)))$, which yields R3.

We are now in the position to state the first main result.

Theorem 1 *For $\alpha \leq 1$ the sequence*

$$K^{(n)}(t) := \left(\max\left\{\xi(S_1), ..., \xi(S_{[nt]})\right\} - b_{m(n)}\right) / \, a_{m(n)}$$

converges in distribution toward the extreme value process Z associated to the extreme value distribution G, where

$$m(n) := \begin{cases} [qn] & for\ \alpha < 1 \\ [n/h(n)] & for\ \alpha = 1 \end{cases}.$$

This theorem is a classical result in the sense that the limit distribution is again an extreme value distribution. Only the scaling has to be modified according to the behaviour of the range of the underlying random walk. We will see in the final section that the sequence $\{\xi(S_n), n \in \mathbb{N}\}$ satisfies the conditions $D(u_n)$ but not the condition $D'(u_n)$ for appropriate sequences $\{u_n, n \in \mathbb{N}\}$ when $\alpha < 1$. Furthermore, we will see in the final section, that the extremal index of the sequence $\{\xi(S_n), n \in \mathbb{N}\}$ can be computed and is equal to q. However, q turns out to be zero for $\alpha > 1$. This explains, why we have to modify the scaling in the second statement. Subsequently, we will see that for $\alpha > 1$ the sequence $\{\xi(S_n), n \in \mathbb{N}\}$ does not satisfy the condition $D(u_n)$ for the sequence $u_n := a_{[n^{1/\alpha}]}x + b_{[n^{1/\alpha}]}$. However, we can prove the following limit-theorem:

Theorem 2 *For $\alpha > 1$ the sequence*

$$K_t^{(n)} := \left(\max\left\{\xi(S_1), ..., \xi(S_{[nt]})\right\} - b_{[n^{1/\alpha}]}\right) \Big/ \, a_{[n^{1/\alpha}]}$$

converges in distribution toward the stochastic process

$$K(t) := Z(m(Y(0,t))).$$

It is important to notice that the limit-distribution in Theorem 2 is not of extreme value type. We will discuss this in the final section of this article.

The stationary sequence $\{\xi(S_n), n \in \mathbb{N}\}$ is dependent due to the recurrence of an underlying random walk. There have been some investigations on stationary sequences with dependence resulting from an underlying Markov structure. The most popular one of those concepts is the chain dependent sequence, which has been studied extensively in extreme value theory (see Resnick (1972), Denzel and O'Brien (1975), Turkman and Oliveira (1992)). However, we will see in the final section that the sequence $\{\xi(S_n), n \in \mathbb{N}\}$ is not chain dependent. A generalisation of chain dependence has been introduced in Turkman and Walker (1983). Nevertheless, the underlying process in their model has only a finite state-space and an invariant measure, which is not the case for the integer-valued random walk studied in this article.

2 Proof of the limit-theorems

In many extreme value situations there exists an underlying point-process. Often it is more suitable to prove limit theorem on the level of those point-processes and then use the continuous mapping theorem in order to understand the behaviour of the extremes. We will follow this approach in our subsequent investigation. We first define the stopping-times

$$\tau_k := \inf\{m \in \mathbb{N}; \mathrm{card}\{S_1, ..., S_m\} \geq k\}$$

and note that

$$K_n = \max\{\xi(S_{\tau_k}); \tau_k \leq n\}.$$

Moreover, we define

$$Q^{(n)}(t_1, t_2] := \mathrm{card}\{m \in \mathbb{N}; nt_1 < m \leq nt_2, S_m \notin \{S_1, ..., S_{m-1}\}\}.$$

The process $S^{(n)}$ defined in (1) visits a new site during the time-interval $(t_1, t_2]$ if and only if there exists an integer k such that $\tau_k/n \in (t_1, t_2]$. This implies that the total number of new sites visited by $S^{(n)}$ during the time-interval $(t_1, t_2]$ is $\sum_k \mathbb{I}_{(t_1,t_2]}(\tau_k/n)$. We therefore have the following identity

$$Q^{(n)}(t_1, t_2] = \sum_k \mathbb{I}_{(t_1,t_2]}(\tau_k/n) = R_{[nt_2]} - R_{[nt_1]}.$$

The next lemma states the independence of the $\xi(S_k)$-sequence and τ_k-sequence.

Lemma 1 *For all $L \in \mathbb{N}$ and all measurable sets $B_k \subset \mathbb{N}$, $A_k \subset \mathbb{R}$ with $1 \leq k \leq L$ we have*

$$\begin{aligned} &\mathbb{P}(\tau_k \in B_k, \xi(S_{\tau_k}) \in A_k, 1 \leq k \leq L) \\ &\qquad = \mathbb{P}(\xi(k) \in A_k, 1 \leq k \leq L)\mathbb{P}(\tau_k \in B_k, 1 \leq k \leq L). \end{aligned}$$

Proof: We use the independence of the random walk and the scenery to prove

$$\begin{aligned}
\mathbb{P}(\tau_k \in B_k, \xi(S_{\tau_k}) \in A_k) &= \sum_{m \in B_k} \mathbb{P}(\tau_k = m, \xi(S_m) \in A_k) \\
&= \sum_{m \in B_k} \sum_{z \in \mathbb{Z}} \mathbb{P}(\tau_k = m, S_m = z, \xi(z) \in A_k) \\
&= \sum_{m \in B_k} \sum_{z \in \mathbb{Z}} \mathbb{P}(\tau_k = m, S_m = z) \mathbb{P}(\xi(k) \in A_k) \\
&= \mathbb{P}(\tau_k \in B_k) \mathbb{P}(\xi(k) \in A_k).
\end{aligned}$$

The general case follows from a similar proof. □

For an $L \in \mathbb{N}$ we denote by $\mathbb{P}_\xi$ the joint distribution of $\{\xi(k), 1 \le k \le L\}$ on $\mathbb{R}^L$.

Lemma 2 *For every $L \in \mathbb{N}$ and every bounded measurable function*

$$f : \mathbb{R}^L \times \mathbb{N}^L \to \mathbb{R}; ((x_k), (m_k)) \mapsto f((x_k), (m_k))$$

we have

$$\mathbb{E}\left[f((\xi(S_{\tau_k})), (\tau_k))\right] = \mathbb{E}\left[\int_{\mathbb{R}^L} f((x_k), (\tau_k)) \mathbb{P}_\xi(d(x_k))\right].$$

Proof: In order to avoid notational overload we just prove a simplified statement for the one dimensional marginal distributions. Let

$$f(x, m) = \sum_{i=1}^{M} \sum_{j=1}^{K} \alpha_{ij} \mathbb{I}_{A_i}(x) \mathbb{I}_{B_j}(m)$$

be a step-function over measurable sets $A_i, 1 \le i \le M$ in $\mathbb{R}$ and $B_j, 1 \le j \le K$ in $\mathbb{N}$. Then it follows by the previous lemma that

$$\begin{aligned}
\mathbb{E}\left[f(\xi(S_{\tau_k}), \tau_k)\right] &= \sum_{i=1}^{M} \sum_{j=1}^{K} \alpha_{ij} \mathbb{P}(\xi(S_{\tau_k}) \in A_i, \tau_k \in B_j) \\
&= \sum_{i=1}^{M} \sum_{j=1}^{K} \alpha_{ij} \mathbb{P}(\xi(k) \in A_i) \mathbb{P}(\tau_k \in B_j) \\
&= \mathbb{E}\left[\int_{\mathbb{R}} f(x, \tau_k) \mathbb{P}_{\xi(k)}(dx)\right].
\end{aligned}$$

The result now follows from a monotone class argument. □

Now we use the sequence $\{\xi(S_{\tau_k}), k \in \mathbb{N}\}$ to construct a sequence of random measures on a suitable state space. We denote by P_n the distribution of the random variable $(\xi(1)-b_n)/a_n$. If $\xi(1)$ is in the domain of attraction of the extreme value distribution G, it is well known that

$$nP_n((x,\infty)) = n\mathbb{P}((\xi(1)-b_n)/a_n > x) \longrightarrow -\log(G(x)) \quad \text{as } n\to\infty.$$

This can be rephrased as the vague convergence of nP_n toward a suitable measure ν on a suitable topological space E. If G is a Fréchet distribution this holds for the right-compactified interval $E := (0,\infty]$ and $\nu(x,\infty] := x^{-\gamma}$; if G is a Weibull distribution this holds for $E := (-\infty, 0]$ and the measure $\nu(x,0] := (-x)^{-\gamma}$; if G is a Gumbel distribution this holds for the right-compactified interval $E := (-\infty,\infty]$ and $\nu(x,\infty] := e^{-x}$ (see Resnick (1987) p.210).
For a Borel-measure κ on $\mathbb{R}^+\times E$ and a measurable function $f : \mathbb{R}^+\times E \to [0,\infty)$ we define

$$\kappa(f) := \int_{\mathbb{R}^+}\int_E f(s,x)\kappa(ds,dx).$$

To the intensity-measure $\mu := m\times\nu$ on $\mathbb{R}^+\times E$, there exists a Poisson point-process N which is characterised by its Laplace functional through

$$\mathbb{E}\left[\exp\left(-N(f)\right)\right] = \exp\left(-\int_{\mathbb{R}^+}\int_E (1-e^{f(s,x)})\nu(dx)ds\right).$$

We denote by $\mathcal{M}_p$ the set of point-measures on $\mathbb{R}^+\times E$ and remark that $\mathcal{M}_p$ is a closed subset of the set $\mathcal{M}_+$ of Borel-measures with respect to the vague topology (see Resnick (1987) p.140). We now are in position to state the theorem for the transient situation:

Theorem 3 *For $\alpha \le 1$ the point-processes*

$$N^{(n)} := \sum_k \delta_{(\tau_k/n,(\xi(S_{\tau_k})-b_{m(n)})/a_{m(n)})}$$

converge weakly toward the Poisson point-process N with intensity measure $m\times\nu$, where

$$m(n) := \begin{cases} [qn] & for\ \alpha < 1 \\ [n/h(n)] & for\ \alpha = 1 \end{cases}.$$

In the following let N_Y be the Poisson point-process on $\mathbb{R}^+\times E$ with random intensity measure

$$\mu(dt,dx) = m_Y(dt)\times\nu(dx),$$

where $m_Y(t) := m(Y(0,t))$ is a random distribution-function on $\mathbb{R}^+$. Such point-processes are called Cox-processes in the literature (see Daley and Vere-Jones (1988) p.261). For an arbitrary continuous function $f : \mathbb{R}^+\times E \to \mathbb{R}$ with compact support, the Laplace-functional of N_Y is given by the following expression

$$\begin{aligned}\mathcal{L}(f) &:= \mathbb{E}\left[\exp\left(-N_Y(f)\right)\right]\\ &= \mathbb{E}\left[\exp\left(-\int_{\mathbb{R}^+}\int_E \left(1-\exp\left(-f(s,x)\right)\right)\nu(dx)m_Y(ds)\right)\right].\end{aligned}$$

Theorem 4 *If we define* $\tilde{a}_n := a_{[n^{1/\alpha}]}$ *and* $\tilde{b}_n := b_{[n^{1/\alpha}]}$*, the point-processes*

$$N^{(n)} := \sum_k \delta_{(\tau_k/n,(\xi(S_{\tau_k})-\tilde{b}_n)/\tilde{a}_n)}$$

on $\mathbb{R}^+ \times E$ *converge weakly toward the point-process* N_Y*.*

Proof of Theorem 3 and Theorem 4: We give a detailed proof for Theorem 4. A proof for Theorem 3 can be obtained from this by changing the scale parameters and by using (R1) resp. (R2) instead of (R3) in the subsequent proof.

The Laplace-functional of $N^{(n)}$ is given by the following expression

$$\begin{aligned}\mathcal{L}_n(f) &= \mathbb{E}\left[\exp\left(-N^{(n)}(f)\right)\right]\\ &= \mathbb{E}\left[\exp\left(-\sum_k f(\tau_k/n,(\xi(S_{\tau_k})-\tilde{b}_n)/\tilde{a}_n)\right)\right].\end{aligned}$$

We have to prove that the Laplace functionals of $N^{(n)}$ converge toward the Laplace functional of N_Y (see Resnick (1987) p.153)

Let $f : \mathbb{R}^+ \times E \to [0,\infty)$ be a continuous function with compact support contained in $(t_0, T] \times E$. For $L \in \mathbb{N}$ let $\{0 < t_0 < ... < t_L = T\}$ be a partition with the property $t_{i+1} - t_i < 1/m$. One can define the following truncated function

$$f_m(t,x) := \sum_{i=0}^{L-1} \mathbb{1}_{(t_i,t_{i+1}]}(t) g_i(x),$$

where $g_i(x) := \inf_{t_i \le s \le t_{i+1}} f(s,x)$. We know that $f_m \uparrow f$ uniformly in $\mathbb{R}^+ \times E$ as $m \to \infty$. The Laplace-functional in f_m has the following form

$$\begin{aligned}\mathcal{L}_n(f_m) &= \mathbb{E}\left[\exp\left(-\sum_k f_m(\tau_k/n,(\xi(S_{\tau_k})-\tilde{b}_n)/\tilde{a}_n)\right)\right]\\ &= \mathbb{E}\left[\exp\left(-\sum_k\sum_i g_i((\xi(S_{\tau_k})-\tilde{b}_n)/\tilde{a}_n)\mathbb{1}_{(t_i,t_{i+1}]}(\tau_k/n)\right)\right]\end{aligned}$$

We apply Lemma 2 and obtain

$$\mathcal{L}_n(f_m) = \mathbb{E}\left[\int_{\mathbb{R}^L} \exp\left(-\sum_k\sum_i g_i(x_k-\tilde{b}_n)/\tilde{a}_n)\mathbb{1}_{(t_i,t_{i+1}]}(\tau_k/n)\right)\mathbb{P}_\xi(d(x_k))\right].$$

Now, one can use the iid-property of the sequence $\{\xi(k), k \in \mathbb{N}\}$ to obtain

$$\begin{aligned}\mathcal{L}_n(f_m) &= \mathbb{E}\left[\prod_k \int_E \exp\left(-\sum_i g_i((x-\tilde{b}_n)/\tilde{a}_n)\mathbb{1}_{(t_i,t_{i+1}]}(\tau_k/n)\right)\mathbb{P}_{\xi(1)}(dx)\right]\\ &= \mathbb{E}\left[\prod_k\prod_i\left(1-\int_E\left(1-\exp\left(-g_i(z)\mathbb{1}_{(t_i,t_{i+1}]}(\tau_k/n)\right)\right)P_{[n^{1/\alpha}]}(dz)\right)\right].\end{aligned}$$

For the following we define

$$\lambda_{i,k;n} := \int_E \exp\left(-g_i(z)\mathbb{1}_{(t_i,t_{i+1}]}(\tau_k/n)\right) P_{[n^{1/\alpha}]}(dz)$$

and

$$\varphi_{i,k;n} := \exp\left(-n^{-1/\alpha}\int_E \left(1-\exp\left(-g_i(z)\mathbb{1}_{(t_i,t_{i+1}]}(\tau_k/n)\right)\right)\nu(dz)\right)$$

As nP_n converges toward ν in the vague topology we have that uniformly for all $1 \le i \le L$ and all $k \in \mathbb{N}$

$$-(1-\lambda_{i,k;n}) \sim \log(\varphi_{i,k;n}) \quad \text{as } n \to \infty.$$

This implies

$$\log(\lambda_{i,k;n}) \sim \log(\varphi_{i,k;n}) \quad \text{as } n \to \infty.$$

From this we deduce that

$$\sup_{i,k}\left|\frac{\log(\lambda_{i,k;n}) - \log(\varphi_{i,k;n})}{\log(\varphi_{i,k;n})}\right| \longrightarrow 0 \quad \text{as } n \to \infty.$$

We note that

$$\begin{aligned}
&\left|\sum_{i,k}\log(\lambda_{i,k;n}) - \sum_{i,k}\log(\varphi_{i,k;n})\right| \\
\le\;& R_{[nT]}\sup_{i,k}|\log(\varphi_{i,k;n})|\sup_{i,k}\left|\frac{\log(\lambda_{i,k;n}) - \log(\varphi_{i,k;n})}{\log(\varphi_{i,k;n})}\right|,
\end{aligned}$$

The fact that the functions $x \mapsto g_i(x)$ are compactly supported implies

$$\sup_{i,k}\left|\int_E \left(1-\exp\left(-g_i(z)\mathbb{1}_{(t_i,t_{i+1}]}(\tau_k/n)\right)\right)\nu(dz)\right| < C \quad \text{for all } n \in \mathbb{N}$$

which yields that

$$\left|R_{[nT]}\sup_{i,k}|\log(\varphi_{i,k;n})|\right| \le Cn^{-1/\alpha}R_{[nT]} \quad \text{for all } n \in \mathbb{N}.$$

Moreover, we know from (R3) that $n^{-1/\alpha}R_{[nT]}$ converges in distribution toward $m(Y(0,T))$. Hence, it follows that

$$\mathbb{P}\left(\left|R_{[nT]}\log(\varphi_{i,k;n})\right| > N\right) \longrightarrow \mathbb{P}(m(Y(0,T)) > N) \quad \text{as } n \to \infty.$$

Since the right side can be arbitrarily small by choice of large $N \in \mathbb{N}$, we obtain that

$$\left| \sum_{i,k} \log(\lambda_{i,k;n}) - \sum_{i,k} \log(\varphi_{i,k;n}) \right| \longrightarrow 0 \quad \text{in probability as} \quad n \to \infty,$$

which is equivalent to

$$\left| \prod_{i,k} \lambda_{i,k;n} - \prod_{i,k} \varphi_{i,k;n} \right| \longrightarrow 0 \quad \text{in probability as} \quad n \to \infty.$$

Taking expectations and using dominated convergence shows that the sequence

$$\mathcal{L}_n(f_m) = \mathbb{E}\left[\prod_k \prod_i \left(1 - \int_E \left(1 - \exp\left(-g_i(z) \mathbb{1}_{(t_i, t_{i+1}]}(\tau_k/n) \right) \right) P_{[n^{1/\alpha}]}(dz) \right) \right]$$

has the same limit as

$$\begin{aligned} & \mathbb{E}\left[\prod_k \prod_i \exp\left(-n^{-1/\alpha} \int_E \left(1 - \exp\left(-g_i(z) \mathbb{1}_{(t_i, t_{i+1}]}(\tau_k/n) \right) \right) \nu(dz) \right) \right] \\ = \; & \mathbb{E}\left[\exp\left(-\sum_k \sum_i n^{-1/\alpha} \int_E \left(1 - \exp\left(-g_i(z) \mathbb{1}_{(t_i, t_{i+1}]}(\tau_k/n) \right) \right) \nu(dz) \right) \right] \\ = \; & \mathbb{E}\left[\exp\left(-\sum_k \sum_i n^{-1/\alpha} \int_E \left(1 - \exp\left(-g_i(z) \right) \right) \mathbb{1}_{(t_i, t_{i+1}]}(\tau_k/n) \nu(dz) \right) \right] \\ = \; & \mathbb{E}\left[\exp\left(-\sum_i n^{-1/\alpha} \int_E \left(1 - \exp\left(-g_i(z) \right) \right) \nu(dz) \left(R_{[nt_{i+1}]} - R_{[nt_i]} \right) \right) \right]. \end{aligned}$$

It follows then from (R3) that the previous sequence converges toward

$$\begin{aligned} & \mathbb{E}\left[\exp\left(-\sum_i \int_E \left(1 - \exp\left(-g_i(z) \right) \right) \nu(dz) (m(Y(0, t_{i+1})) - m(Y(0, t_i))) \right) \right] \\ = \; & \mathbb{E}\left[\exp\left(-\int_{\mathbb{R}^+} \int_E \left(1 - \exp\left(-f_m(z,t) \right) \right) \nu(dz) m_Y(dt) \right) \right], \end{aligned}$$

which is just $\mathcal{L}(f_m)$. Since $f_m \to f$ in sup-norm as $m \to \infty$, it follows that

$$\mathcal{L}_n(f_m) \longrightarrow \mathcal{L}_n(f) \;\; \text{as } m \to \infty$$

uniformly in $n \in \mathbb{N}$. Moreover, we just have proved that for all $m \in \mathbb{N}$

$$\mathcal{L}_n(f_m) \longrightarrow \mathcal{L}(f_m) \;\; \text{as } n \to \infty.$$

Therefore, we have

$$\mathcal{L}_n(f) \longrightarrow \mathcal{L}(f) \text{ as } n \to \infty.$$

This proves Theorem 4. $\square$

Proof of Theorem 1 and Theorem 2: In order to prove Theorem 1 we define the map

$$\mathfrak{F} : \mathcal{M}_p(\mathbb{R}^+ \times E) \to D(0,\infty); N = \sum_k \delta_{t_k, j_k} \mapsto \left(t \mapsto \bigvee_{0<t_k \leq t} j_k \right).$$

It can be proved that $\mathfrak{F}(N^{(n)}) = K^{(n)}$ (see Resnick (1987) p.209). Moreover, $\mathfrak{F}$ is continuous $\mathbb{P}_N$ almost surely, where $\mathbb{P}_N$ denotes the distribution of the point-process N on $\mathcal{M}_p(\mathbb{R}^+ \times E)$ (see Resnick (1987) p.214). The continuous mapping theorem and the previous theorem, then imply that $K^{(n)}$ converges toward $Z = \mathfrak{F}(N)$.

For Theorem 2 we use the random-transformations

$$\mathfrak{f} : \mathbb{R}^+ \times E \to \mathbb{R}^+ \times E; (t,x) \mapsto (m(Y(0,t)), x).$$

If we use the transformation-formula for Poisson point-processes we can see that $N_Y = \mathfrak{f}(N)$ (see Resnick (1987) p.134). With the representation $N = \sum_k \delta_{(t_k, j_k)}$ this implies

$$N_Y = \sum_k \delta_{(m(Y(0,t_k)), j_k)}.$$

It then follows that

$$\mathfrak{F}(N_Y)(t) = \bigvee_{0<m(Y(0,t_k)) \leq t} j_k = Z(m(Y(0,t))).$$

$\square$

3 The long-range dependence

As we already mentioned in the introduction, the sequence does not satisfy the weak-mixing conditions introduced by Leadbetter, if the underlying random-walk is recurrent. We investigate this behaviour in this section.

3.1 The condition $D(u_n)$

The condition $D(u_n)$ is a condition, which assures a mixing type behaviour for the tails of the joint distributions of a stationary sequence of random-variables. For a sequence of random-variables $\{\Xi_i, i \in \mathbb{N}\}$ we denote its joint-distribution function

by $F_{i_1,...,i_n}; 1 \leq i_1 < ... < i_n$. Let $\{u_n, n \in \mathbb{N}\}$ be a given sequence of increasing positive real numbers.
A stationary sequence of random variables $\{\Xi_i, i \in \mathbb{N}\}$ *satisfies condition* $D(u_n)$ *if for any integers* $1 \leq i_1 < ... < i_p < j_1 < ... < j_q \leq n$ *with the property* $j_1 - i_p \geq l$ *we have*

$$|F_{i_1,...,i_p,j_1,...,j_q}(u_n) - F_{i_1,...,i_p}(u_n)F_{j_1,...,j_q}(u_n)| \leq \alpha_{n,l}$$

where $\alpha_{n,l} \to 0$ *as* $n \to \infty$ *for some sequence* $l = o(n)$.

The condition $D(u_n)$ is used to prove that the limit-distribution of the maxima of a stationary sequence has the same type as the independent sequence with the same marginal distributions (see Leadbetter et al. (1983) p.57).

Proposition 1 *For* $\alpha > 1$ *the stationary sequence* $\{\xi(S_n), n \in \mathbb{N}\}$ *does not satisfy the condition* $D(u_n)$ *with* $u_n = a_{[n^{1/\alpha}]}x + b_{[n^{1/\alpha}]}$.

Proof: We first note that for $u_n = a_{[n^{1/\alpha}]}x + b_{[n^{1/\alpha}]}$ we have

$$n^{1/\alpha}(1 - F(u_n)) \longrightarrow -\log G(x) \text{ as } n \to \infty.$$

From this it follows that as $n \to \infty$

$$(F(u_n))^{n^{1/\alpha}} = \exp(n^{1/\alpha}\log(1 - (1 - F(u_n)))) \sim \exp(-n^{1/\alpha}(1 - F(u_n))) \longrightarrow G(x).$$

We know from (R3) that $R^{(n)} := n^{-1/\alpha}R_n$ converges in distribution toward $R := m(Y(0,1))$. Therefore, there exist random-variables $\tilde{R}^{(n)}$ and $\tilde{R}$ with the same distribution as $R^{(n)}$ resp. R such that $\tilde{R}^{(n)}$ converges toward $\tilde{R}$ almost surely (see Billingsley (1986) p.343). It then follows that $\mathbb{P}$-almost surely

$$\left((F(u_n))^{n^{1/\alpha}}\right)^{\tilde{R}^{(n)}} \longrightarrow (G(x))^{\tilde{R}} \quad \text{as } n \to \infty.$$

Hence we obtain by dominated convergence and Lemma 1 that

$$\begin{aligned}
F_{1,...,n}(u_n) &= \mathbb{P}(\xi(S_1) \leq u_n, ..., \xi(S_n) \leq u_n) \\
&= \mathbb{E}\left[\mathbb{P}(\xi(S_1) \leq u_n, ..., \xi(S_n) \leq u_n | R_n)\right] \\
&= \mathbb{E}\left[(F(u_n))^{R_n}\right] \\
&= \mathbb{E}\left[(F(u_n))^{n^{1/\alpha}\tilde{R}^{(n)}}\right]
\end{aligned}$$

converges toward

$$\mathbb{E}\left[(G(x))^{\tilde{R}}\right] = \mathbb{E}\left[(G(x))^{m(Y(0,1))}\right].$$

If $l = o(n)$ we can prove in the same way that

$$F_{1,...,n,n+l,...,2n}(u_n) \longrightarrow \mathbb{E}\left[(G(x))^{m(Y(0,2))}\right] \quad \text{as } n \to \infty$$

and

$$F_{n+l,...,2n}(u_n) = F_{1,...,n-l}(u_n) \longrightarrow \mathbb{E}\left[(G(x))^{m(Y(0,1))}\right] \quad \text{as } n \to \infty$$

It then follows that $|F_{1,...,n,n+l,...,2n}(u_n) - F_{1,...,n}(u_n)F_{n+l,...,2n}(u_n)|$ converges toward

$$\begin{aligned} & \mathbb{E}\left[(G(x))^{m(Y(0,2))}\right] - \left(\mathbb{E}\left[(G(x))^{m(Y(0,1))}\right]\right)^2 \\ > \; & \mathbb{E}\left[(G(x))^{m(Y(0,1))+m(Y(1,2))}\right] - \mathbb{E}\left[(G(x))^{m(Y(0,1))}\right]^2 = 0, \end{aligned}$$

where we use the fact that $m(Y(0,2))$ is strictly smaller than $m(Y(0,1))+m(Y(1,2))$ almost surely and the independence of $m(Y(0,1))$ and $m(Y(1,2))$. This proves that the condition $D(u_n)$ does not hold for the sequence $u_n = a_{[n^{1/\alpha}]}x + b_{[n^{1/\alpha}]}$. □

Proposition 2 *For $\alpha \le 1$ the sequence $\{\xi(S_n), n \in \mathbb{N}\}$ satisfies the condition $D(u_n)$ with*

$$u_n := \begin{cases} a_{[qn]}x + b_{[qn]} & for \;\; \alpha < 1 \\ a_{[n/h(n)]}x + b_{[n/h(n)]} & for \;\; \alpha = 1 \end{cases}.$$

Proof: Assume that for a fixed $\epsilon > 0$ there exist for every $n \in \mathbb{N}$ a family of integers $1 \le i_1 < ... < i_p < j_1 < ... < j_q \le 2n$ with the property $j_1 - i_p \ge l(n)$, where $l(n)/n \to 0$ as $n \to \infty$ and

$$|F_{i_1,...,i_p,j_1,...,j_q}(u_n) - F_{i_1,...,i_p}(u_n)F_{j_1,...,j_q}(u_n)| > \epsilon.$$

We define $R_{j_1,...,j_q} := \text{card}\{S_{j_1},...,S_{j_q}\}$, $R_{i_1,...,i_p} := \text{card}\{S_{i_1},...,S_{i_p}\}$ and

$$R^{i_1,...,i_p}_{j_1,...,j_q} := R_{j_1,...,j_q} + R_{i_1,...,i_p} - \text{card}\{S_{i_1},...,S_{i_p},S_{j_1},...,S_{j_q}\}.$$

It then follows that

$$\begin{aligned} & (F(u_n))^{\text{card}\{S_{i_1},...,S_{i_p},S_{j_1},...,S_{j_q}\}} - (F(u_n))^{R_{j_1,...,j_q}+R_{i_1,...,i_p}} \\ = \; & (F(u_n))^{R_{j_1,...,j_q}+R_{i_1,...,i_p}}\left((F(u_n))^{-R^{i_1,...,i_p}_{j_1,...,j_q}} - 1\right) \\ \le \; & \left((F(u_n))^{-R^{1,...,i_p}_{i_p+l,...,2n}} - 1\right). \end{aligned} \tag{2}$$

We note that i_p is a sequence of integers, where the n-th element is bounded by $2n$. It follows that i_p/n must have convergent subsequences. We can therefore assume without loss of generality that $i_p/n \to u$ as $n \to \infty$. It follows from (R1) that the sequence

$$\begin{aligned} \frac{1}{[nq]}R^{1,...,i_p}_{i_p+l,...,2n} &= \frac{1}{[nq]}\left(R_{1,...,i_p} + R_{i_p+l,...,2n} - R_{1,...,2n}\right) \\ &\sim \frac{1}{[nq]}\left(R_{1,...,[nu]} + R_{[nu]+l,...,2n} - R_{1,...,2n}\right) \end{aligned}$$

converges almost surely toward $u + (2n - u) - 2n = 0$ as $n \to \infty$. As we have

$$qn(1 - F(u_n)) \longrightarrow -\log G(x) \quad \text{as } n \to \infty,$$

it follows that

$$(F(u_n))^{nq} \sim \exp(-nq(1 - F(u_n))) \longrightarrow G(x) \quad \text{as } n \to \infty.$$

Together, this implies that $\mathbb{P}$-almost surely

$$\left((F(u_n))^{-R^{1,\dots,i_p}_{i_p+l,\dots,2n}} - 1\right) \longrightarrow 0 \quad \text{as } n \to \infty. \tag{3}$$

We note that

$$\begin{aligned}
& F_{j_1,\dots,j_q}(u_n) F_{i_1,\dots,i_p}(u_n) \\
= \; & \mathbb{E}\left[(F(u_n))^{R_{j_1,\dots,j_q}}\right] \mathbb{E}\left[(F(u_n))^{R_{i_1,\dots,i_p}}\right] \\
= \; & \mathbb{E}\left[\mathbb{E}\left[(F(u_n))^{R_{j_1,\dots,j_q}} | S_{i_1}, \dots, S_{i_p}\right] (F(u_n))^{R_{i_1,\dots,i_p}}\right] \\
= \; & \mathbb{E}\left[\mathbb{E}\left[(F(u_n))^{R_{j_1,\dots,j_q}} (F(u_n))^{R_{i_1,\dots,i_p}} | S_{i_1}, \dots, S_{i_p}\right]\right] \\
= \; & \mathbb{E}\left[(F(u_n))^{R_{j_1,\dots,j_q}} (F(u_n))^{R_{i_1,\dots,i_p}}\right].
\end{aligned}$$

This together with (2) and (3) implies that

$$\begin{aligned}
& \left| F_{i_1,\dots,i_p,j_1,\dots,j_q}(u_n) - F_{i_1,\dots,i_p}(u_n) F_{j_1,\dots,j_q}(u_n) \right| \\
= \; & \mathbb{E}\left[(F(u_n))^{\text{card}\{S_{i_1},\dots,S_{i_p},S_{j_1},\dots,S_{j_q}\}} - (F(u_n))^{R_{j_1,\dots,j_q} + R_{i_1,\dots,i_p}}\right] \\
\leq \; & \mathbb{E}\left[\left((F(u_n))^{-R^{1,\dots,i_p}_{i_p+l,\dots,2n}} - 1\right)\right]
\end{aligned}$$

converges toward zero. This contradicts to the initial assumption and proves the first statement of the proposition. The second statement follows in the same way by using (R2) in stead of (R1). □

3.2 The condition $D'(u_n)$ for $\alpha < 1$

The following condition is called $D'(u_n)$ in the literature:
A stationary sequence of random variables $\{\Xi_i, i \in \mathbb{N}\}$ satisfies condition $D'(u_n)$ if

$$\limsup_{n\to\infty} \; n \sum_{j=2}^{[n/k]} \mathbb{P}(\Xi_1 > u_n, \Xi_j > u_n) \longrightarrow 0 \text{ as } k \to \infty.$$

The condition $D'(u_n)$ together with the condition $D(u_n)$ imply that

$$\mathbb{P}(\max\{\Xi_1, ..., \Xi_n\} \leq u_n) - \mathbb{P}(\max\{\tilde{\Xi}_1, ..., \tilde{\Xi}_n\} \leq u_n) \longrightarrow 0 \quad \text{as } n \to \infty,$$

where $\{\tilde{\Xi}_n, n \in \mathbb{N}\}$ is an independent sequence with the same marginals as $\{\Xi_n, n \in \mathbb{N}\}$ (see Leadbetter et al. (1983) p.61).

Proposition 3 *For $\alpha < 1$ the sequence $\{\xi(S_n), n \in \mathbb{N}\}$ does not satisfy the condition $D'(u_n)$ with $u_n = a_{[qn]}x + b_{[qn]}$.*

Proof: Due to the definition of the sequences u_n we have that

$$qn(1 - F(u_n)) \longrightarrow -\log(G(x)) \quad \text{as } n \to \infty.$$

We have

$$\begin{aligned}
& n \sum_{j=1}^{[n/k]} \mathbb{P}(\xi(S_1) > u_n, \xi(S_j) > u_n) \\
= \; & n \sum_{j=1}^{[n/k]} \mathbb{P}(\xi(S_1) > u_n, \xi(S_j) > u_n | S_j = S_1) \mathbb{P}(S_j = S_1) \\
& + n \sum_{j=1}^{[n/k]} \mathbb{P}(\xi(S_1) > u_n, \xi(S_j) > u_n | S_j \neq S_1) \mathbb{P}(S_j \neq S_1) \\
= \; & n(1 - F(u_n)) \sum_{j=1}^{[n/k]} \mathbb{P}(S_j = S_1) + n(1 - F(u_n))^2 \sum_{j=1}^{[n/k]} \mathbb{P}(S_j \neq S_1).
\end{aligned}$$

Since we have

$$0 \leq \sum_{j=1}^{[n/k]} \mathbb{P}(S_j \neq S_1) \leq \left[\frac{n}{k}\right]$$

it follows that

$$\lim_{k \to \infty} \lim_{n \to \infty} n(1 - F(u_n))^2 \sum_{j=1}^{[n/k]} \mathbb{P}(S_j \neq S_1) = -\lim_{k \to \infty} \log(G(x))/k = 0.$$

For the second term one has that

$$\begin{aligned}
\lim_{n \to \infty} n(1 - F(u_n)) \sum_{j=1}^{[n/k]} \mathbb{P}(S_j = S_1) \quad & \geq \quad -\log(G(x)) \mathbb{P}(\exists j \in \mathbb{N} : S_j = S_1) \\
& = \quad \log(G(x))(1 - q).
\end{aligned}$$

This is larger than zero since for $\alpha < 1$ we have $0 < q < 1$. □

Remark: This also follows from the fact that the extremal index is q (see Leadbetter (1983) p.59).

3.3 The extremal index

The extremal index is a measure for the dependence in the tails of a stationary sequence. This concept appeared in the work of Newell (1964) and Loynes (1965) for m-dependent variables and was later named extremal index in Leadbetter (1983). *A* $\theta \in [0,1]$ *is called extremal index for the stationary sequence* $\{\Xi_i, i \in \mathbb{N}\}$ *if for every* $\tau > 0$
(i) there exists a sequence $v_n \uparrow \infty$ *such that* $n(1-F(v_n)) \to \tau$,
(ii) $\mathbb{P}(\max\{\Xi_1, ..., \Xi_n\} \leq v_n) \to e^{-\tau\theta}$.

The extremal index is an indicator for the existence of clusters of exceedances (see Hsing et al (1988)). Usually for a model with extremal index $\theta \in (0,1)$ the observed cluster-size is $1/\theta$. In our situation the expected number of visits of a site is $1/q$. This motivates the following proposition.

Proposition 4 *The extremal index of the sequence* $\{\xi(S_n), n \in \mathbb{N}\}$ *is equal to* $q = \mathbb{P}(S_k \neq 0, k \in \mathbb{N})$.

Proof: Let $\{v_n, n \in \mathbb{N}\}$ be a sequence such that

$$n(1-F(v_n)) \longrightarrow \tau \quad \text{as } n \to \infty.$$

We know from the theorem of Kesten, Spitzer and Whitman, that $\frac{1}{n}R_n$ converges almost surely toward $q = \mathbb{P}(S_k \neq 0, k \in \mathbb{N})$ as $n \to \infty$ (see Spitzer p.38). Then it follows from Lemma 1 that

$$\begin{aligned}
\mathbb{P}(\max\{\xi(S_1), ..., \xi(S_n)\} \leq v_n) &= \mathbb{P}(\xi(S_1) \leq v_n, ..., \xi(S_n) \leq v_n) \\
&= \sum_{k=1}^{n} \mathbb{P}(\xi(S_{\tau_1}) \leq v_n, ..., \xi(S_{\tau_k}) \leq v_n, R_n = k) \\
&= \sum_{k=1}^{n} \mathbb{P}(\xi(1) \leq v_n, ..., \xi(k) \leq v_n)\mathbb{P}(R_n = k) \\
&= \mathbb{E}\left[(F(v_n))^{R_n}\right]
\end{aligned}$$

We note that by the assumption on the sequence $\{v_n, n \in \mathbb{N}\}$ as $n \to \infty$ we have that

$$(F(v_n))^n = \exp(n \log(1-(1-F(v_n)))) \sim \exp(-n(1-F(v_n))) \longrightarrow \exp(-\tau).$$

This implies together with the result of Kesten, Spitzer and Whitman that $\mathbb{P}$-almost surely

$$(F(v_n))^{R_n} = ((F(v_n))^n)^{\frac{1}{n}R_n} \longrightarrow \exp(-q\tau) \quad \text{as } n \to \infty.$$

Hence by dominated convergence we have

$$\mathbb{P}(\max\{\xi(S_1), ..., \xi(S_n)\} \leq v_n) = \mathbb{E}\left[(F(v_n))^{R_n}\right] \longrightarrow \exp(-q\tau) \quad \text{as } n \to \infty.$$

This proves that the extremal index is q. □

Remark: We note that $q = 0$ is equivalent to the random walk being recurrent, i.e. $\alpha > 1$.

3.4 The type of the limit-distribution for $\alpha > 1$

It is of course interesting whether in the recurrent case the limit-distribution has one of the extremal types. This is however not the case:

Proposition 5 *The distribution H of $Z(m(Y(0,1)))$ is not an extreme value distribution.*

Proof: If G is the distribution function associated to the extreme value process Z, we have that

$$H(x) = \mathbb{P}(Z(m(Y(0,1))) \leq x) = \mathbb{E}\left[(G(x))^{m(Y(0,1))}\right].$$

We first concentrate on the case where $\{\xi(k), k \in \mathbb{Z}\}$ is in the domain of attraction of a Fréchet-distribution Φ_γ. We then have for all $x \geq 0$ that

$$H(x) = \mathbb{E}\left[(\Phi_\gamma(x))^{m(Y(0,1))}\right] = \int_0^\infty \exp\left(-tx^{-\gamma}\right) \mathbb{P}_{m(Y(0,1))}(dt).$$

From this expression, we see that the support of H is equal to $[0, \infty)$. If we assume that H is an extreme value distribution, it must be a Fréchet-distribution $\Phi_{\gamma'}$. Further, it is easy to see from the previous expression that $1 - H$ is regular varying with exponent γ. This implies that $\gamma' = \gamma$ and the representation $H(x) = \Phi_\gamma(ax+b)$ with suitable constants $a > 0$ and $b \in \mathbb{R}$. Since the support of H is $[0, \infty)$ it follows that $b = 0$. Moreover, one can see from the previous equation that

$$\exp(-ax) = H(x^\gamma) = \int_0^\infty \exp(-tx)\mathbb{P}_{m(Y(0,1))}(dt),$$

which is the Laplace transform of the distribution of $m(Y(0,1))$. This is a contradiction, since this would imply that $m(Y(0,1))$ were constant. Therefore, H can not be an extreme value distribution.

A similar reasoning works if $\{\xi(k), k \in \mathbb{Z}\}$ is in the domain of attraction of a Weibull-distribution Ψ_γ. Again one assumes that H is an extreme value distribution. Comparison of the support shows that H must be a Weibull-distribution $\Psi_{\gamma'}$. A regular variation argument shows that $\gamma' = \gamma$. And a change of variable unveils

an equality involving the Laplace-transform of $m(Y(0,1))$, which would imply that $m(Y(0,1))$ is constant. This is obviously wrong and therefore H can not be an extreme value distribution.
If $\{\xi(k), k \in \mathbb{Z}\}$ is in the domain of attraction of a Gumbel-distribution, and if H would be an extreme value distribution, then it could only be a Gumbel-distribution. This follows from the fact that H has full support on $\mathbb{R}$. A suitable change of variables again leads to an equation involving the Laplace transform of $m(Y(0,1))$. The expression for the Laplace transform would imply that $m(Y(0,1))$ were constant, which is not true. □

3.5 Chain-dependent Markov-processes

Chain-dependent Markov-processes are special stationary sequences, where the dependence comes from an underlying Markovian structure. Their extreme value behaviour has been studied extensively in the past three decades (see Resnick (1972), Denzel and O'Brien (1975), Turkman and Oliveira (1992), Fereira (1998)). It is worth mentioning that applications of chain-dependent processes to meteorology have been described by Katz (1977). In the following let $\mathcal{Z}$ be a countable set.

A double sequence of random variables $\{\Xi_n, \zeta_n, n \in \mathbb{N}\}$ *is called a chain-dependent Markov-process with state-space* $\mathbb{R} \times \mathcal{Z}$*, if for all* $i, j \in \mathcal{Z}$ *and* $x \in \mathbb{R}$

$$\mathbb{P}(\Xi_n \leq x, \zeta_n = j | \Xi_0, \zeta_0, ..., \Xi_{n-1}, \zeta_{n-1} = i) = \mathbb{P}(\Xi_n \leq x, \zeta_n = j | \zeta_{n-1} = i),$$

where the right hand does not depend on n*.*

Proposition 6 *The double sequence* $\{\xi(S_n), S_n, n \in \mathbb{N}\}$ *is not a chain-dependent Markov-process.*

Proof: Let i, j be integers such that $\mathbb{P}(S_n = j | S_{n-1} = i) \neq 0$. Then we have

$$\begin{aligned} 0 &= \mathbb{P}(\xi(S_n) \leq x, S_n = j | \xi(S_1) > x, S_1 = j, S_{n-1} = i) \\ &\neq \mathbb{P}(\xi(S_n) \leq x, S_n = j | S_{n-1} = i). \end{aligned}$$

□

A generalisation of chain dependent stationary sequences was described in Turkman and Walker (1983). The process $\{\xi(S_n), S_n, n \in \mathbb{N}\}$ satisfies some of the structural properties of the sequences $\{\Xi_n, \zeta_n, n \in \mathbb{N}\}$ considered there. It satisfies the ζ_n-dependence of Ξ_n, i.e.:

$$\mathbb{P}(\Xi_n \leq x | \zeta_1 = s_1, ..., \zeta_n = s_n) = \mathbb{P}(\Xi_n \leq x | \zeta_n = s_n),$$

and the conditional independence of $\Xi_1, ..., \Xi_n$ from $\{S_{n+k}, k \in \mathbb{N}\}$, i.e.:

$$\begin{aligned} & \mathbb{P}(\Xi_1 \leq x_1, ..., \Xi_n \leq x_n | \zeta_1 = s_1, ..., \zeta_{n+k} = s_{n+k}) \\ = & \mathbb{P}(\Xi_1 \leq x_1, ..., \Xi_n \leq x_n | \zeta_1 = s_1, ..., \zeta_n = s_n). \end{aligned}$$

However, they only considered sequences of random variables $\{\zeta_n, n \in \mathbb{N}\}$ with a finite state-space $\mathcal{Z}$. This is not the case in our situation.

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4 References

Arai T. (2001): A class of semi-selfsimilar processes related to random walks in random scenery, *Tokyo J. Math.*, **24**, 69-85.

Billingsley P. (1986): *Probability and measure*, Second edition, Wiley Series in Probability and Mathematical Statistics: Probability, New York.

Daley D., Vere-Jones D. (1988): *An introduction to the theory of point processes*, Springer Series in Statistics. Springer-Verlag, New York.

Denzel G., O'Brien G. (1975): Limit theorems for extreme values of chain-dependent processes, *Ann. Probab*, **3**, 773-779.

Fereira H. (1998): Doubly stochastic compound Poisson-processes in extreme value theory, *Portugaliae Mathematica*, **55**, 465-474.

Gnedenko B. (1943): Sur la distribution limité du terme d'une série aléatoire, *Ann. Math.*, **44**, 423-453.

Hsing T., Hüsler J., Leadbetter M. (1988): On the exceedance point process for a stationary sequence, *Probab. Theory Related Fields*, **78** (1988), 97-112.

Katz R. (1977): An application of chain-dependent processes to meteorology, *J. Appl. Prob.*, **14**, 598-603.

Kesten H., Spitzer F. (1979): A limit theorem related to a new class of self-similar processes, *Z. Wahrsch. Verw. Gebiete*, **50**, 5-25.

Lang R., Nguyen X.-X. (1983): Strongly correlated random fields as observed by a random walker, *Z. Wahrsch. Verw. Gebiete*, **64**, 327–340.

Leadbetter M. (1983): Extremes and local dependence in stationary sequences, *Z. Wahrsch. Verw. Gebiete*, **65**, 291-306.

Leadbetter M., Lindgren G., Rootzén H. (1983): *Extremes and Related Properties of Random Sequences and Processes*, Applied Probability, Springer-Verlag, New-York.

Le Gall J.-F., Rosen J. (1991): The range of stable random walks, *Ann. Probab.*,

19, 650-705.

Loynes R. (1965): Extreme values in uniformly mixing stationary stochastic processes, *Ann. Math. Stat.*, **36**, 993-999.

Maejima M. (1996): Limit theorems related to a class of operator-self-similar processes, *Nagoya Math. J.*, **142**, 161–181.

Newell G. (1964): Asymptotic extremes for m-dependent random variables, *Ann. Math. Stat.*, **35**, 1322-1325.

Resnick S. (1972): Stability of maxima of random variables defined on a Markov-chain, *Adv. Appl. Probab.*, **4**, 285-295.

Resnick S. (1987): *Extreme Values, Regular Variation and Point Processes*, Applied Probability, Springer-Verlag, New-York.

Saigo T., Takahashi H. (2005): Limit theorems related to a class of operator semi-selfsimilar processes, *J. Math. Sci. Univ. Tokyo*, **12**, 111–140.

Shieh N.-R. (1995): Some self-similar processes related to local times, *Statist. Probab. Lett.*, **24**, 213-218.

Spitzer F. (1976): *Principles of Random Walk*, Graduate Texts in Mathematics, Springer Verlag New-York.

Turkman K., Oliveira M. (1992): Limit laws for the maxima of chain-dependent sequences with positive extremal index, *J. Appl. Probab.*, **29**, 222-227.

Turkman K., Walker A. (1983): Limit laws for the maxima of a class of quasi-stationary sequences, *J. Appl. Probab.*, **20**, 814-821.