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Describing a class of global attractors via symbol sequences

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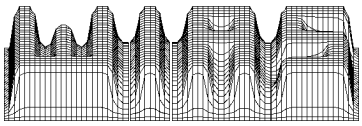

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Abstract

We study a singularly perturbed scalar reaction-diffusion equation on a bounded interval with a spatially inhomogeneous bistable nonlinearity. For certain nonlinearities, which are piecewise constant in space on k subintervals, it is possible to characterize all stationary solutions for small ε by means of sequences of k symbols, indicating the behavior of the solution in each subinterval. Determining also Morse-indices and zero numbers of the equilibria in terms of the symbol sequences, we are able to give a criterion for heteroclinic connections and a description of the associated global attractor for all k.

1 Introduction

Scalar reaction diffusion equations on a bounded interval provide an interesting class of infinite dynamical systems, which still allows to obtain a detailed qualitative understanding of the dynamics. Due to standard theorems they give under appropriate conditions rise to compact analytic semigroups, possessing a global attractor that contains all solutions which are uniformly bounded for all times. Moreover there have been a lot of investigations about stationary solutions, their stability and how their nodal properties determine the heteroclinic connections in the attractor (see [Pol02] and references there).

A particular interesting situation is the case where the effect of diffusion is very small compared to the size of the reaction terms. This is modeled by a small parameter ε in front of the diffusion term. Some examples like the Chafee-Infante equation [CI74] show that one must expect both the number of equilibria and the dimension of the global attractor to become unbounded as $\varepsilon \searrow 0$. On the other hand, there are situations where the dimension remains uniformly bounded, e.g. for viscous conservation laws with dissipative source terms, see [Hä98].

In this paper, we want to investigate the interaction of small diffusion with spatial inhomogeneities of the nonlinearity. To this end we study a special class of examples of the form

$$u_t = \varepsilon^2 u_{xx} + (1-u^2)(u-a(x)), \quad x \in (0,1) \tag{1}$$

under Neumann boundary conditions $u_x(0) = u_x(1) = 0$, with a bistable nonlinearity where the position of the unstable root $-1 < a(x) < 1$ may change in space.

It has been shown in [AMP87], that for small ε, all stable stationary solutions stay close to the stable roots -1 and 1 except for some transition layers, occurring at positions where $a(x) = 0$ and $a'(x) \neq 0$. In general, however, the number of stationary

solutions, together with the dimension of some unstable manifolds becomes infinite for small ε [ABF93]. Hence it is still impossible, to obtain a complete understanding of the set of all stationary solutions and the attractor in the limit $\varepsilon \searrow 0$.

Salazar and Solà-Morales suggested in [SS01] to study the situation of piecewise constant functions $a(x)$, being alternatingly smaller and bigger than 0. This means that $a(x) = c_i$ for $x \in [x_i, x_{i+1})$, where $0 = x_1 < x_2 < \ldots < x_k < x_{k+1} = 1$ is a given partition of the unit interval. For definiteness, we will always assume that $c_i \cdot (-1)^i > 0$. Equation (1) with piecewise constant $a(x)$ has first been considered by Rocha [Ro88] who showed that for ε sufficiently small there are exactly F_k stable stationary solutions, where F_k is the k-th Fibonacci number. In [SS01] it has been shown that for c_i sufficiently distant from 0, the number of equilibria and the unstable dimensions stay bounded in the limit $\varepsilon \to 0$ (see Theorem 1 in [SS01]). Later, together with Fiedler and Rocha, they proved in [FRSS01] some statements about the number and stability of stationary solutions of (1).

Building on these results we propose a more algebraic framework which allows to translate the information on the stationary solutions efficiently into information on the dynamics. We will give in Section 2 a characterization of all stationary profiles, using a description of transition layers and spikes by symbol sequences (Theorem 1). Then, in Section 3, we give an easy condition for heteroclinic connections of the equilibria in terms of their symbolic description (Theorem 2). To this end, we recall a general result about heteroclinic connections for such equations from [Wo00]. We show that corresponding order relations, as they are used in [Wo00], can be defined also for the symbol sequences. Moreover, the symbol sequences and their order relations can be used to obtain explicitly the permutation of the equilibria. This permutation is defined by the ordering of the stationary profiles at both ends of the interval [FuRo91]. It contains all information about nodal properties and can also be used to determine Morse-indices and heteroclinic connections [FR96]. Finally, we give in Section 4 a system of ordinary differential equations in k dimensions, which reproduces the dynamics of the equation (1).

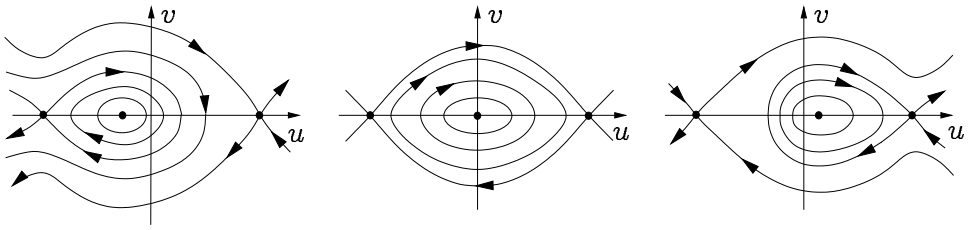
2 Describing stationary solutions by symbol sequences

Stationary solutions to (1) are given as solutions of the second order ODE boundary value problem

$$\begin{aligned} \varepsilon^2 u'' + (1-u^2)(u - a(x)) &= 0 \\ u'(0) = u'(1) &= 0 \end{aligned}$$

which can be written as a first order system for $\mathbf{u} = (u, v)$:

$$\left\{ \begin{array}{rcl} \varepsilon u' &=& v \\ \varepsilon v' &=& (u^2-1)(u - a(x)) \end{array} \right. \tag{2}$$



$c_i < 0$ $\qquad$ $c_i = 0$ $\qquad$ $c_i > 0$

Figure 1: The vectorfield for $a(x) \equiv c_i$

with boundary conditions

$$v(0) = v(1) = 0. \tag{3}$$

For each subinterval $[x_i, x_{i+1}]$, we can look at the phase portrait of this first order system, which is different for odd i, i.e. $a(x) \equiv c_i < 0$ and even i, i.e. $a(x) \equiv c_i > 0$: There are always three equilibria $(-1, 0)$, $(c_i, 0)$ and $(1, 0)$. For $c_i < 0$ there is a homoclinic orbit attached to the hyperbolic equilibrium $(-1, 0)$, while for $c_i > 0$ the homoclinic orbit is asymptotic to the other hyperbolic equilibrium $(1, 0)$ (see Figure 2). For $c_i = 0$ there exists a pair of heteroclinic orbits that connect $(-1, 0)$ and $(1, 0)$. The size of the homoclinic orbits is measured by its diameter

$$\gamma(c_i) = \frac{2}{3}\left(3 - |c_i| - \sqrt{c_i^2 + 3|c_i|}\right),$$

the distance between the asymptotic state and the point where the homoclinic intersects the u-axis.

The linearization at the equilibrium to which the homoclinic orbit is asymptotic possesses the real eigenvalues $\pm\sqrt{2(1 - |c_i|)}$. The eigenvalues of the linearization at the other hyperbolic equilibrium are $\pm\sqrt{2(1 + |c_i|)}$.

Salazar and Solà-Morales have obtained sufficient conditions to assure a definite limiting behavior for $\varepsilon \to 0$. We restate their result in our coordinates:

Proposition 2.1 ([SS01], Theorem 3) *Assume that*

$$c_i \cdot (-1)^i > 0, \tag{4}$$

$$\gamma(c_i) + \gamma(c_{i+1}) < 2 \quad \textit{for} \quad i = 1, 2, \ldots, k-1 \tag{5}$$

and

$$\left.\begin{array}{rcl} (x_2 - x_1)\sqrt{1 - |c_1|} & < & (x_3 - x_2)\sqrt{1 + |c_2|}, \\ (x_{i+1} - x_i)\sqrt{1 - |c_i|} & < & (x_i - x_{i-1})\sqrt{1 + |c_{i-1}|} \quad \text{for } i = 3, \ldots, k. \end{array}\right\} \tag{6}$$

Then for ε small enough the number N_k of stationary solutions of (1) does not depend on ε. N_k satisfies the recursion relation $N_{k+1} = N_k + 2N_{k-1}$ with $N_2 = 3$, $N_3 = 5$. Moreover, the Morse index of any stationary solution does not exceed $2k$.

Note that these conditions are satisfied for example if all subintervals are of equal length, and the c_i just satisfy (4) and (5).

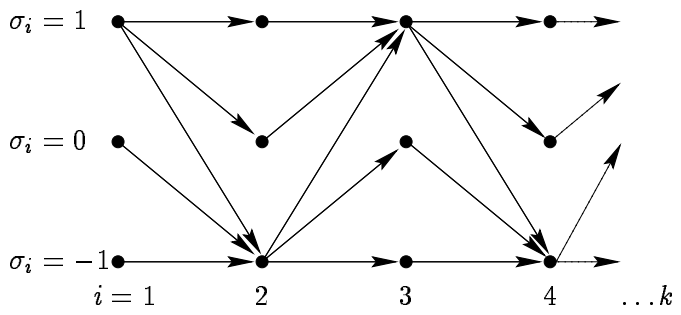


Figure 2: Each symbol sequence in $\mathcal{S}_k$ corresponds to a directed path in this graph

2.1 Abstract symbol sequences

In this section we describe a set of finite symbol sequences. Later, we show that these symbol sequences can be identified with the stationary solutions of (1), such that each symbol describes the behavior of the stationary profile in one subinterval.

Definition 2.2 *The set $\mathcal{S}_k$ consists of sequences $s := (\sigma_1, \ldots, \sigma_k)$ of symbols $\sigma_i \in \{-1, 0, 1\}$ satisfying the following rules:*

- *For odd i the symbol $\sigma_i = 1$ may be followed by any symbol σ_{i+1} , whereas for $\sigma_i = -1$ or 0, σ_{i+1} has to be -1.*
- *For even i the symbol $\sigma_i = -1$ may be followed by any symbol σ_{i+1} , whereas for $\sigma_i = +1$ or 0, σ_{i+1} has to be $+1$.*

For any $s = (\sigma_1, \ldots, \sigma_k) \in \mathcal{S}_k$ we denote with $i(s)$ the number of zeroes contained in this sequence.

In Figure 2 these transition rules are visualized by a directed graph. Now we decompose $\mathcal{S}_k$ into subsets

$$\mathcal{S}_k^j = \{s \in \mathcal{S}_k |\, i(s) = j\}.$$

The cardinalities of the sets $\mathcal{S}_k^j$ can be computed recursively:

Lemma 2.3 *We have the recursions:*

$$\begin{aligned} |\mathcal{S}_k| &= |\mathcal{S}_{k-1}| + 2|\mathcal{S}_{k-2}| && (7)\\ |\mathcal{S}_k^0| &= |\mathcal{S}_{k-1}^0| + |\mathcal{S}_{k-2}^0| && (8)\\ |\mathcal{S}_k^i| &= |\mathcal{S}_{k-1}^i| + |\mathcal{S}_{k-2}^i| + |\mathcal{S}_{k-2}^{i-1}|, \end{aligned}$$

subject to the (artificial) initial values $|\mathcal{S}_{-1}| = |\mathcal{S}_0| = |\mathcal{S}_{-1}^0| = |\mathcal{S}_0^0| = 1$ and $|\mathcal{S}_{-1}^i| = |\mathcal{S}_0^i| = 0$ for $i > 0$.

These recursions coincide with those, given in [FRSS01] for the numbers of stationary solutions with Morse index i in the case of k subintervals.

Proof: We proceed by induction. First, it is straightforward to check that the artificial initial values give the correct values for $k = 1$ and $k = 2$. Assume now that k is even and take an arbitrary sequence $s = (\sigma_1, \ldots, \sigma_k) \in \mathcal{S}_k$. Then for $\sigma_k = -1$ the predecessor σ_{k-1} is arbitrary, i.e. the sequence $(\sigma_1, \ldots, \sigma_{k-1})$ is arbitrary in $\mathcal{S}_{k-1}$, giving the first term in (7). For $\sigma_k = 0$ or $+1$, the transition rule requires $\sigma_{k-1} = -1$, and only the sequence $(\sigma_1, \ldots, \sigma_{k-2})$ is arbitrary in $\mathcal{S}_{k-2}$. Since we can extend in two ways with $\sigma_k = 0$ or $+1$, we get the factor 2 in the second term of (7). Not allowing the symbol 0 at all, leads to (8), where the factor 2 is missing. If the number of symbols 0 is fixed to some value i, then $\sigma_k = 0$ and $\sigma_k = -1$ give different contributions $|\mathcal{S}_{k-2}^{i-1}|$ and $|\mathcal{S}_{k-2}^{i}|$ since the remainder $(\sigma_1, \ldots, \sigma_{k-2})$ contains the symbol 0 either $i-1$ or i times, respectively. For odd k the same arguments work, interchanging -1 and $+1$. □

2.2 Stationary profiles

We show now that the symbol sequences in $\mathcal{S}_k$ can be used to describe the stationary profiles in the case of k subintervals for sufficiently small ε. With E_k^ε we denote the set of stationary solutions to (1) in the case of k subintervals.

Theorem 1 *Assume that the piecewise constant function $a(x)$ satisfies (5) and (6). Then there exists a $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the following statements are true:*

(i) There is a one-to-one-correspondence between the stationary solutions in E_k^ε and the symbol sequences in $\mathcal{S}_k$.

(ii) A stationary profile $w \in E_k^\varepsilon$ is characterized by its corresponding symbol sequence $s = (\sigma_1, \ldots, \sigma_k)$ in the following way: The i-th symbol σ_i describes the behavior of $w(x)$ in the i-th subinterval (x_i, x_{i+1}); $\sigma_i = -1$ corresponds to $w(x) \approx -1$,whereas $\sigma_i = 1$ corresponds to $w(x) \approx 1$. The symbol $\sigma_i = 0$ is associated with a spike-type or boundary-layer behavior in the corresponding interval.

(iii) The Morse-index (dimension of the unstable manifold) for a stationary profile $w \in E_k^\varepsilon$ is given by the number $i(s)$ of symbols 0 in the corresponding symbol sequence s.

Recall that for $x \in (x_i, x_{i+1})$, i.e. for $a(x) \equiv c_i$ equation (2) defines a Hamiltonian system which possesses a homoclinic orbit Γ_i asymptotic to the equilibrium $((-1)^i, 0)$.

For $i = 1, 2, \ldots, k$, let W_i^s and W_i^u be the stable and unstable manifold of the hyperbolic equilibrium $((-1)^{i+1}, 0)$, which is not contained in the closure of Γ_i.

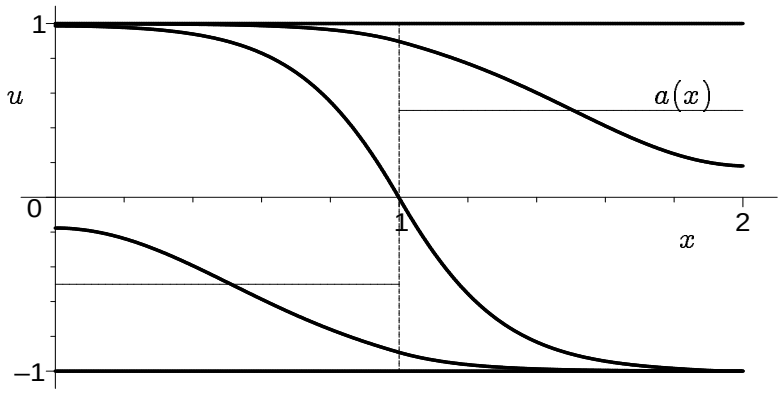


Figure 3: Numerically computed stationary profiles for two subintervals, $\varepsilon = 0.1$, $a(x) = \pm\frac{1}{2}$

Remark 2.4 *W_i^u and W_{i+1}^s intersect transversally, as can be checked from their representation as level sets of the corresponding Hamiltonian systems.*

It is easy to show that any solution of the boundary value problem (2), (3) must satisfy $u(x) \in [-1, 1]$ for all $x \in [0, 1]$. To see this, assume that $\mathbf{u}(x) = (u(x), v(x))$ is a trajectory satisfying the left boundary condition $v(0) = 0$ and that $u(\bar{x}) > 1$ for some $\bar{x}$. Then $v(\bar{x}) > 0$ since the region $\{u \geq 1, v \leq 0\}$ is negatively invariant and cannot be reached by a trajectory that starts from the u–axis. Similarly, the region $\{u > 1, v > 0\}$ is positively invariant which implies that the trajectory will never reach the u–axis again and therefore cannot satisfy the right boundary condition. The proof that $u(x) \geq -1$ is completely analogous.

In a next step we therefore describe all solutions of the system

$$\begin{aligned} \varepsilon u' &= v \\ \varepsilon v' &= (u^2 - 1)(u - a(x)) \end{aligned}$$

for which $u(x)$ remains in the interval $[-1, 1]$ as long as x is in some subinterval $[x_{i-1}, x_{i+1}]$. We do not aim at an optimal description of this set but rather prepare the setting for the proof of Theorem 1.

Let δ be small such that δ–neighborhoods of adjacent homoclinic orbits do not intersect, in other words,

$$\gamma(c_i) + \gamma(c_{i+1}) + 2\delta < 2$$

is satisfied for $i = 1, \ldots, k$. Moreover, we require that δ–neighborhoods of Γ_i and W_i^s do not intersect either. This can obviously be achieved by choosing δ sufficiently small. With such δ, let $\mathcal{H}_i$ be a δ–neighborhood of the equilibrium which is contained in the closure of the homoclinic Γ_i and $\mathcal{N}_i$ a δ–neighborhood of the intersection between W_i^s and the strip $\{-1 \leq u \leq 1\}$. The notation indicates that $\mathcal{H}_i$ is associated with the equilibrium where the **h**omoclinic orbit is, while $\mathcal{N}_i$ contains the equilibrium with **n**o homoclinic orbit.

Lemma 2.5 *Consider solutions of (2) where the c_i satisfy (5). Then for ε small enough the following holds:*

If a trajectory $\mathbf{u}(x)$ of (2) satisfies $u(x) \in [-1,1]$ for all $x \in [x_{i-1}, x_{i+1}]$ for some $2 \leq i \leq k$, then

$$\mathbf{u}(x_i) \in \mathcal{H}_i \cup \mathcal{N}_i.$$

Proof: We will determine separately the locus of "initial conditions" $\mathbf{u}(x_i)$ for which $u(x)$ remains in $[-1,1]$ for $x \in [x_i, x_{i+1}]$ and the locus of "terminal conditions" for which $u(x) \in [-1,1]$ for $x \in [x_{i-1}, x_i]$. The intersection of the two sets will be contained in $\mathcal{H}_i \cup \mathcal{N}_i$.

From the phase portrait for $x \in [x_i, x_{i+1}]$ we can immediately read off that for ε small we have $-1 \leq u(x_{i+1}) \leq 1$ only if

(i) the initial condition $\mathbf{u}(x_i)$ is close to the stable manifolds of the two hyperbolic equilibria, or

(ii) the initial condition $\mathbf{u}(x_i)$ lies in the interior of the homoclinic orbit Γ_i.

In particular, $\mathbf{u}(x_i)$ must for ε small lie in the union of W_i^s and some neighborhood of the interior of Γ_i.

Similarly, the condition $u(x) \in [-1,1]$ for $x \in [x_{i-1}, x_i]$ can only be satisfied for small ε if $\mathbf{u}(x_i)$ is close to one of the unstable manifolds or if $\mathbf{u}(x_i)$ lies in the interior of the homoclinic orbit Γ_{i-1}. In other words, $\mathbf{u}(x_i)$ has to lie in a neighborhood of W_{i-1}^u or in some neighborhood of the interior of Γ_{i-1}.

For ε sufficiently small, the intersection of the four sets consists of the union of $\mathcal{H}_i$, $\mathcal{H}_{i-1}$ and a small neighborhood of the intersection $W_{i-1}^u \cap W_i^s$. In particular, $\mathbf{u}(x_i)$ is therefore contained in $\mathcal{H}_i \cup \mathcal{N}_i$. □

We outline now our strategy for the proof of Theorem 1: Stationary solutions will be found via a shooting method. Denote with $\Phi_{i,j}^\varepsilon$ the flow of (2) from $x = x_i$ to $x = x_j$. We follow the image of the line $\{v = 0\}$, which corresponds to the left boundary condition, under the flow $\Phi_{1,j+1}^\varepsilon$ for $j = 1, \ldots, k$. Due to the previous lemma we know that we need only to keep track of those parts of the shooting curve which at $x = x_i$ lie within $\mathcal{H}_i \cup \mathcal{N}_i$. Preimages of these parts will be intervals $I_{\sigma_1,\ldots,\sigma_j}^\varepsilon$. As is suggested by this notation these intervals will connect the differential equation with the symbol sequences introduced earlier.

To get a precise correspondence between neighborhoods and symbols it is necessary to decompose $\mathcal{H}_i$ in two parts. The line $\{v = 0\}$ divides $\mathcal{H}_i$ in two sets: $\mathcal{H}_i^s$ is the part that has a non-empty intersection with W_{i-1}^s and $\mathcal{H}_i^u$ contains some part of W_{i-1}^u.

The basic tool will be the following lemma that describes the evolution of curves under the flow $\Phi_{i,i+1}^\varepsilon$ corresponding to one interval where $a(x) \equiv c_i$ is constant.

Lemma 2.6 (Transition Lemma) *(i) Suppose that the curve $C \subset \mathcal{N}_i$ is a graph over the u-axis which is transverse to the stable manifold W_i^s. Then for ε sufficiently small,*

- $\Phi_{i,i+1}^{\varepsilon}(C) \cap \mathcal{N}_{i+1}$ *is a graph over the u-axis transverse to* W_{i+1}^s,
- $\Phi_{i,i+1}^{\varepsilon}(C) \cap \mathcal{H}_{i+1}$ *is a curve which is C^1-close to W_i^u. Therefore, it intersects the homoclinic orbit Γ_{i+1} transversally and is again a graph over the u-axis .*

Moreover, $\Phi_{i,i+1}^{\varepsilon}$ preserves the order on C, that is, if (u_1, v_1) and (u_2, v_2) are two points on C with $u_1 < u_2$ then the u-coordinate of $\Phi_{i,i+1}^{\varepsilon}(u_1, v_1)$ is smaller than the u-coordinate of $\Phi_{i,i+1}^{\varepsilon}(u_2, v_2)$.

(ii) Suppose that the curve C is a graph over the u-axis in $\mathcal{H}_i^s$ which is transverse to the homoclinic orbit Γ_i. Then for ε sufficiently small,

- $\Phi_{i,i+1}^{\varepsilon}(C) \cap \mathcal{H}_{i+1} = \emptyset$,
- $\Phi_{i,i+1}^{\varepsilon}(C) \cap \mathcal{N}_{i+1}$ *has a connected component which is a graph over the u-axis transverse to* W_{i+1}^s.

Again, $\Phi_{i,i+1}^{\varepsilon}$ preserves the order on C.

(iii) Suppose that the curve C is a graph over the u-axis in $\mathcal{H}_i^u$ which is C^1–close to W_{i-1}^u. Then for ε sufficiently small

- $\Phi_{i,i+1}^{\varepsilon}(C) \cap \mathcal{H}_{i+1} = \emptyset$,
- $\Phi_{i,i+1}^{\varepsilon}(C) \cap \mathcal{N}_{i+1}$ *is a graph over the u-axis transverse to* W_{i+1}^s.

The order on C is reversed by $\Phi_{i,i+1}^{\varepsilon}$.

Proof: (i) is an immediate consequence of the λ-lemma which states that for ε small $\Phi_{i,i+1}^{\varepsilon}(C)$ will be expanded along the unstable manifold W_i^u. Because of the transverse intersection of W_i^u and W_{i+1}^s the curve $\Phi_{i,i+1}^{\varepsilon}(C)$ will also be transverse to W_{i+1}^s in $\mathcal{N}_{i+1}$. Similarly, since it is C^1–close to W_i^u in $\mathcal{H}_{i+1}$ it is automatically transverse to Γ_{i+1}. That $\Phi_{i,i+1}^{\varepsilon}$ preserves the order on C can be seen from the phase portrait, see Figure 4.

(ii) The first claim is a simple consequence of the fact that for $a(x) \equiv c_i$ there are no trajectories of (2) which lead from $\mathcal{H}_i$ to $\mathcal{H}_{i+1}$. As before, the λ-lemma implies that a small part of C near the intersection with Γ_i will be stretched to a curve that is close to the unstable manifold of the equilibrium contained in the closure of Γ_i. This in turn implies that it will be transverse to W_{i+1}^s in $\mathcal{N}_{i+1}$. Note that $\Phi_{i,i+1}^{\varepsilon}$ may in general map other parts of the curve C also back to $\mathcal{H}_i$. This "spiraling" inside of Γ_i, however, does not occur if the curve C is close enough to W_{i-1}^u. In the situation we are interested in, this closeness is achieved by condition (6) (see[SS01]).

(iii) Again, since no trajectories of (2) pass from $\mathcal{H}_i$ to $\mathcal{H}_{i+1}$ the first claim is obvious.

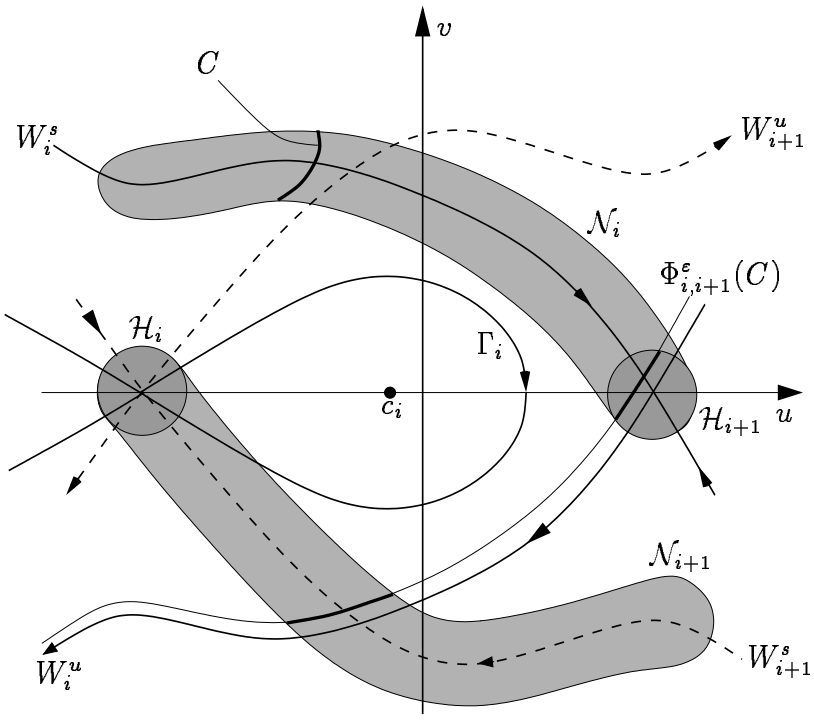


Figure 4: Effect of the flow $\Phi_{i,i+1}^\varepsilon$ on C in case (i)

This time we will use the λ-lemma in a different way than before. The stable manifold W_{i+1}^s is transverse to the homoclinic orbit Γ_i. (This can again be checked using level sets of the corresponding Hamiltonians.) The backward evolution $\Phi_{i+1,i}(W_{i+1}^s)$ will therefore stretch along Γ_i and along the other branch of the unstable manifold of the equilibrium. In particular, it will have a unique point of transverse intersection with the curve C. This in turn guarantees that $\Phi_{i,i+1}^\varepsilon(C) \cap \mathcal{N}_{i+1}$ consists of a curve that is transverse to W_{i+1}^s. Due to the excursion along Γ_i the order on the curve is reversed. □

From this lemma it becomes clear that $\mathbf{u}(x_i) \in \mathcal{N}_i$ corresponds to the symbol $\sigma_i = (-1)^{i+1}$ because in this case the trajectory will remain near the equilibrium without the homoclinic orbit for the major part of the interval $[x_i, x_{i+1}]$. Similarly, $\mathbf{u}(x_i) \in \mathcal{H}_i^s$ corresponds to the symbol $\sigma_i = (-1)^i$ and $\mathbf{u}(x_i) \in \mathcal{H}_i^u$ corresponds to the symbol $\sigma_i = 0$ alias an excursion along the homoclinic orbit Γ_i.

In the next step we use the transition lemma to find trajectories of (2) with a given itinerary $s \in \mathcal{S}_k$.

In addition to the sets $\mathcal{H}_i$ and $\mathcal{N}_i$ we choose a neighborhood U_ℓ of the intersection between the homoclinic orbit Γ_1 and the u-axis. We need this set to take care of the Neumann boundary conditions which allows for solutions with a sharp boundary layer at $x = 0$ corresponding to trajectories close to a homoclinic orbit in (2).

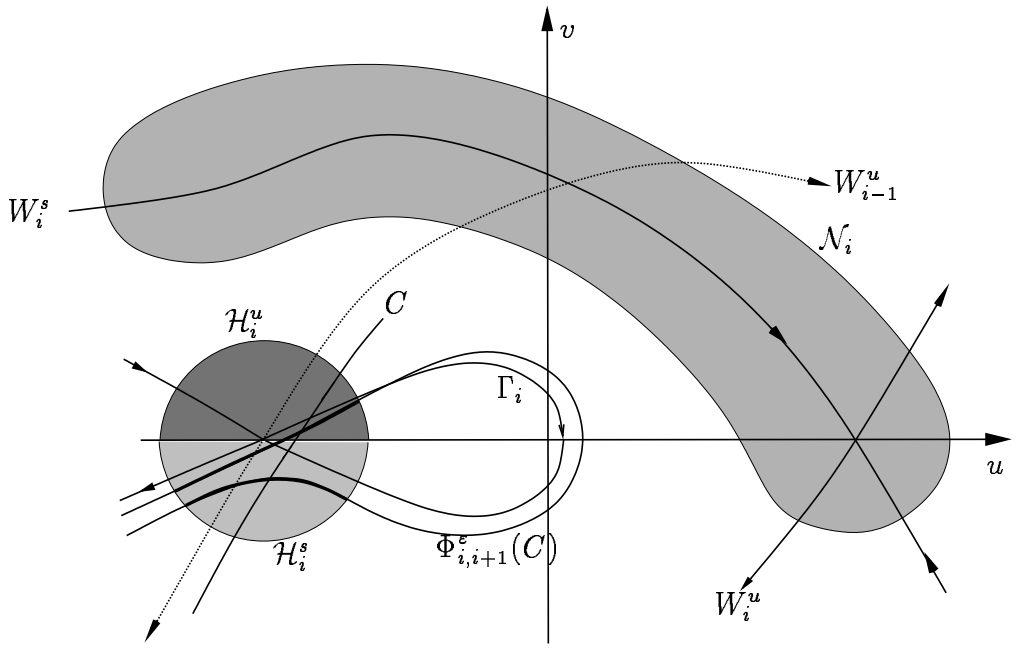


Figure 5: Effect of the flow $\Phi^\varepsilon_{i,i+1}$ on C in case (ii) and (iii)

Lemma 2.7 *Assume that (4) is satisfied by the constants c_i. Then for any ε sufficiently small and $s = (\sigma_1, \sigma_2, \ldots, \sigma_k) \in \mathcal{S}_k$, there exists some subinterval I_s^ε of the $u-$axis such that solutions of (2) with $\mathbf{u}(0) \in I_s^\varepsilon$ have the following properties:*

(i) $\mathbf{u}(x_i) \in \mathcal{N}_i$ *iff* $\sigma_i = (-1)^{i+1}$,

(ii) $\mathbf{u}(x_i) \in \mathcal{H}_i^s$ *iff* $\sigma_i = (-1)^i$,

(iii) $\mathbf{u}(0) \in U_\ell$ *if* $\sigma_1 = 0$ *and* $\mathbf{u}(x_i) \in \mathcal{H}_i^u$ *if* $\sigma_k = 0$ *for some* $2 \le i \le k$.

Proof: We construct a sequence of intervals $I^\varepsilon_{\sigma_1,\sigma_2,\ldots,\sigma_j}$ with the property that

$$I^\varepsilon_{\sigma_1,\ldots,\sigma_j,\sigma_{j+1}} \subset I^\varepsilon_{\sigma_1,\sigma_2,\ldots,\sigma_j}, \tag{9}$$

and such that $\mathbf{u}(0) \in I^\varepsilon_{\sigma_1,\sigma_2,\ldots,\sigma_j}$ implies that the conditions (i)-(iii) are satisfied for $1 \le i \le j$.

To begin with, we choose intervals

$$\begin{aligned} I^\varepsilon_{-1} &:= \mathcal{H}_1 \cap \{v = 0\}, \\ I^\varepsilon_0 &:= U_\ell \cap \{v = 0\}, \\ I^\varepsilon_1 &:= \mathcal{N}_1 \cap \{v = 0\} \end{aligned} \tag{10}$$

and study the evolution of these intervals under the flow $\Phi^\varepsilon_{1,2}$ to $x = x_2$. By the λ-lemma, if ε is small enough $\Phi_{1,2}(I^\varepsilon_{-1})$ is a curve which is C^1-close to the unstable

manifold $W^u(0)$ alias the homoclinic orbit Γ_1 and which may spiral inside the homoclinic orbit Γ_1. Hence, $\Phi^\varepsilon_{1,2}(I^\varepsilon_{-1}) \cap \mathcal{N}_2$ may consist of several components. As we want to avoid too much spiraling, we take as $I^\varepsilon_{-1,-1}$ the preimage of the connected component, containing the origin in $\Phi^\varepsilon_{1,2}(I^\varepsilon_{-1}) \cap \mathcal{N}_2$.

Similarly, $\Phi^\varepsilon_{1,2}(I^\varepsilon_0) \cap \mathcal{N}_2$ may consist of several pieces and among the corresponding preimages we take as $I^\varepsilon_{0,-1}$ the component which contains the intersection of Γ_1 with the u-axis.

Lemma 2.6(i) tells us that $\Phi^\varepsilon_{1,2}(I^\varepsilon_1)$ is a curve C^1-close to the unstable manifold W^u_1. This curve therefore intersects the sets $\mathcal{N}_2$ and $\mathcal{H}_2$ so that we can choose as $I^\varepsilon_{1,-1}$, $I^\varepsilon_{1,0}$ and $I^\varepsilon_{1,1}$ the preimage of the intersection with $\mathcal{N}_2$, $\mathcal{H}^u_2$ and $\mathcal{H}^s_2$, respectively.

From now on we proceed inductively and assume that all intervals $I^\varepsilon_{\sigma_1,\sigma_2,\ldots,\sigma_j}$ have already been constructed for some j. Moreover, we assume that $\Phi^\varepsilon_{1,j}(I^\varepsilon_{\sigma_1,\sigma_2,\ldots,\sigma_j})$ is

- a curve in $\mathcal{N}_j$ transverse to W^s_j if $\sigma_j = (-1)^{j+1}$
- a curve in $\mathcal{H}_j$ close to W^u_{j-1} if $\sigma_j = (-1)^j$ or $\sigma_j = 0$.

It is straightforward check that these assumptions are satisfied for $j = 2$.

We distinguish now three cases:

1. $\sigma_j = (-1)^{j+1}$
   In this case $\Phi^\varepsilon_{1,j}(I^\varepsilon_{\sigma_1,\sigma_2,\ldots,\sigma_j})$ is a curve in $\mathcal{N}_j$. By Lemma 2.6 the image of this curve under $\Phi^\varepsilon_{j,j+1}$ intersects both $\mathcal{N}_{j+1}$ and $\mathcal{H}_{i+1}$. Choose $I^\varepsilon_{\sigma_1,\ldots,\sigma_j,(-1)^j}$ as the preimage of the intersection with $\mathcal{N}_{j+1}$, then $\Phi^\varepsilon_{1,j+1}(I^\varepsilon_{\sigma_1,\ldots,\sigma_j,(-1)^j})$ is a curve transverse to W^s_{j+1} as desired. The part which intersects $\mathcal{H}_{j+1}$ is C^1-close to W^u_j. Choose as $I^\varepsilon_{\sigma_1,\ldots,\sigma_j,(-1)^j}$ the preimage of the part which lies in $\mathcal{H}^s_{j+1}$, and as $I^\varepsilon_{\sigma_1,\ldots,\sigma_j,0}$ the preimage of the part which lies in $\mathcal{H}^u_{j+1}$.

2. $\sigma_j = (-1)^j$
   By assumption, $\Phi^\varepsilon_{1,j}(I^\varepsilon_{\sigma_1,\sigma_2,\ldots,\sigma_j})$ is a curve in $\mathcal{H}^s_j$ close to W^s_{j-1} and therefore transverse to th homoclinic orbit Γ_j. From Lemma 2.6(ii) we know that the image $\Phi^\varepsilon_{1,j+1}(I^\varepsilon_{\sigma_1,\sigma_2,\ldots,\sigma_j})$ does not intersect $\mathcal{H}_{j+1}$ but $\mathcal{N}_{j+1}$. The preimage of this intersection will be denoted with $I^\varepsilon_{\sigma_1,\sigma_2,\ldots,\sigma_j,(-1)^j}$.

3. $\sigma_j = 0$
   We have seen in Lemma 2.6(iii) that for ε small there is a unique point on $I^\varepsilon_{\sigma_1,\sigma_2,\ldots,\sigma_j}$ that will be mapped to a point on W^s_{j+1} under $\Phi^\varepsilon_{j,j+1}$. Choose as $I^\varepsilon_{\sigma_1,\sigma_2,\ldots,\sigma_j,0}$ the preimage of the component of $\Phi^\varepsilon_{j,j+1}(I^\varepsilon_{\sigma_1,\sigma_2,\ldots,\sigma_j})$ which contains this point on W^s_{j+1}.

Inductively one can therefore find intervals which satisfy the conditions given above.
□

We have not yet accounted for the right boundary condition although we have already used all our information from the symbol sequences. We complete the proof of Theorem 1 by showing that for each interval I_s^ε there will be automatically one point of intersection of $\Phi_{1,k+1}^\varepsilon(I_s^\varepsilon)$ with the u-axis. To this end, we introduce a neighborhood U_r of the intersection between Γ_k and the u-axis. This neighborhood takes care of solutions with a sharp boundary layer at $x = 1$.

Proof of Theorem 1: Consider an arbitrary symbol sequence $s = (\sigma_1, \sigma_2, \ldots, \sigma_k) \in \mathcal{S}_k$. We will show that for ε sufficiently small $\Phi_{1,k+1}^\varepsilon(I_s^\varepsilon)$ has a point of intersection with the u-axis. This intersection will correspond to a stationary solution in E_k^ε. Uniqueness will follow from the result of Salazar and Sola-Morales who determined the exact number of stationary solutions under the assumptions (3)-(5).

For definiteness, we discuss the case that k is even only. The case that k is odd can be treated in a completely symmetric way. This assumption implies that in the last interval $[x_k, x_{k+1}]$ the homoclinic orbit Γ_k is attached to $u = 1$. We have to distinguish four cases:

Case 1: $\sigma_k = -1$

In this case, we know that $\Phi_{1,k}^\varepsilon(I_s)$ is a curve in $\mathcal{N}_k$ which is transverse to W_k^s. For ε small this curve will be stretched along W_k^u according to the λ-lemma. In particular it will have a point of intersection with $\{v = 0\}$.

Case 2: $\sigma_k = 1$

Here $\Phi_{1,k}^\varepsilon(I_s)$ is a curve in $\mathcal{H}_k^s$ close to W_{k-1}^u and transverse to the homoclinic orbit Γ_k. For ε small this curve will be stretched along the unstable manifold of the equilibrium $u = 1$ and will therefore have at least one intersection with $\{v = 0\}$.

Case 3: $\sigma_k = 0$

By Lemma 2.7 $\Phi_{1,k}^\varepsilon(I_s)$ is a curve in $\mathcal{H}_k^u$ which is transverse to W_k^s. By the λ-lemma this curve will be stretched along W_k^u under the flow $\Phi_{k,k+1}^\varepsilon$ for ε small enough. Therefore $\Phi_{1,k+1}^\varepsilon(I_s^\varepsilon) = (\Phi_{1,k}^\varepsilon \circ \Phi_{k,k+1}^\varepsilon)(I_s^\varepsilon)$ will have an intersection with the u-axis which corresponds to a hyperbolic stationary solution.

Uniqueness of stationary solutions with a given symbol sequence follows by referring to Theorem 3 of [SS01]. It is obvious that solutions with different symbol sequences are distinct. In Lemma 2.3 we have shown how to determine the number of different symbol sequences $|\mathcal{S}_k|$ recursively. It was shown in [SS01] that under the assumptions (3)-(5) and for ε small this is exactly the number of stationary solutions. This implies that, under these hypotheses, we have found all stationary solutions. This proves part (i) of Theorem 1.

Part (ii) follows from the considerations in the proof of Lemma 2.6. There we have determined the way how trajectories pass from the neighborhoods $\mathcal{N}_i$ and $\mathcal{H}_i$ to $\mathcal{N}_{i+1} \cup \mathcal{H}_{i+1}$.

For part (iii) note that the Morse index is exactly the number of clockwise half turns of a tangent vector to the interval I_s under the map $\Phi_{1,k+1}^\varepsilon$ (see [FuRo91]). By our considerations in Lemma 2.6 about the order-preservation along the curve under $\Phi_{i,i+1}^\varepsilon$ it is clear that the Morse index increases by 1 when the trajectory follows the

homoclinic loop Γ_i and remains the same in all other cases. The Morse index of the stationary solution can therefore be determined by counting the number of excursions along homoclinic loops, i.e. by counting the number of zeroes in the associated symbol sequence. □

3 Heteroclinic connections in the attractor

In this section, we will prove an explicit criterion, deciding whether two equilibria $w, \tilde{w} \in E_k^\varepsilon$ have a heteroclinic connection.

Recall that our choice of the nonlinearity in (1) satisfies a dissipativity condition, providing a global compact semiflow on the Banach space $X \subseteq W^{1,2}((0,1), \mathbb{R})$. This semiflow possesses a global attractor $\mathcal{A}$, i.e. a compact invariant set which attracts bounded subsets of X and which is maximal with this properties (see [Hal88], [BV92]). Moreover, due to the gradient structure of the system, the global attractor contains only equilibria and their unstable manifolds, which consist of heteroclinic connections. Stable and unstable manifolds always intersect transversely [An86]. Since we know from Proposition 2.1 that under the conditions (5), (6), sufficiently small ε, the equilibria do not change any more, we can describe the heteroclinic connections in $\mathcal{A}^\varepsilon$ independent of ε:

Theorem 2 *Assume that the piecewise constant function $a(x)$ satisfies (5) and (6) and $w, \tilde{w}$ are stationary solutions in E_k^ε. Then there exists a $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the following two statements are equivalent:*

(i) There is a heteroclinic connection from w to $\tilde{w}$

(ii) If the two corresponding symbol sequences $s = (\sigma_1, \dots \sigma_k)$ and $\tilde{s} = (\tilde{\sigma}_1, \dots \tilde{\sigma}_l)$ differ at any position i, that is $\sigma_i \neq \tilde{\sigma}_i$, then $\sigma_i = 0$.

Note that according to this theorem the equilibrium w connects exactly to those equilibria $\tilde{w}$ whose symbol sequence $\tilde{s}$ can be obtained by replacing in the sequence s symbols 0 by other symbols. Interpreting this according to Theorem 1, one can see that any heteroclinic solution can be described as follows: In one or several subintervals, where for $t \to -\infty$ a spike is located, this spike disappears and for $t \to \infty$ the profile becomes either close to constant 1 or 0 in the interior of the corresponding subintervals. In all other subintervals the shape of the solution remains nearly unchanged. This means that for small ε on the attractor the motion in different subintervals becomes nearly decoupled and in each subinterval there is a simple bistable dynamical behavior.

3.1 Determining heteroclinic connections by order structures

In order to prove Theorem 2, we want to apply now a general result on heteroclinic connections to our specific situation:

It has been shown in [Wo00] that for scalar parabolic equations

$$u_t = u_{xx} + f(u, u_x, x), \qquad u_x(0,t) = u_x(1,t) = 0, \qquad x \in [0,1] \tag{11}$$

with a dissipative nonlinearity and hyperbolic equilibria, the heteroclinic connections in the attractor can be described in a way which is similar to scalar ordinary differential equations. For those it is well known that two hyperbolic equilibria have a heteroclinic connection, if and only if there is no third equilibrium in between.

Indeed, due to [Wo00] a corresponding theorem can be formulated also for scalar parabolic equations, where, however, the underlying order structure has to be more complicated in order to cover also multidimensional structures in the attractor. Since nodal properties of the solutions play a central role in scalar parabolic equations, they are used to define the appropriate order relations. We recall here the basic definitions from [Wo00].

Definition 3.1 *(i) For any x-profile $w \in C^1[0,1]$, we denote with $z(w)$ the number of strict sign changes (zero number) of $w(x)$ in the interval $[0,1]$.*

(ii) A pair $w, \tilde{w}$ of stationary solutions to (11) with $z(w - \tilde{w}) = n$ is called n-ordered, and we write

$$w \prec_n \tilde{w},$$

if we have $w(0) < \tilde{w}(0)$.

(iii) A n-ordered pair $w \prec_n \tilde{w}$ of stationary solutions to (11) is called adjacent, if there is no third stationary solution $\hat{w}$ with $w \prec_n \hat{w} \prec_n \tilde{w}$.

Proposition 3.2 ([Wo00], Theorem 2.4) *Two hyperbolic equilibria solutions w and $\tilde{w}$ of (11) have a heteroclinic connection if and only if they are adjacent.*

In order to apply Proposition 3.2 to our specific situation, we have to recover the order of the stationary profiles at $x = 0$ and the zero numbers for pairs of stationary solutions. in terms of the corresponding symbol sequence in $\mathcal{S}_k$. This information is sufficient to obtain the order relations $\prec_n$ and hence the notion of adjacency, which is according to Proposition 3.2 the criterion for heteroclinic connections.

Definition 3.3 *On the set $\mathcal{S}_k$, we define recursively the total order $\prec$ by the following two rules:*

(i) $(-1, \ldots) \prec (0, \ldots) \prec (1, \ldots)$

(ii) $(\sigma_1, \sigma_2, \dots \sigma_k) \prec (\sigma_1, \tilde{\sigma}_2, \dots \tilde{\sigma}_k) \Leftrightarrow \begin{cases} (\sigma_2, \dots \sigma_k) \prec (\tilde{\sigma}_2, \dots \tilde{\sigma}_k) & \text{and} \quad \sigma_1 \neq 0 \\ (\tilde{\sigma}_2, \dots \tilde{\sigma}_k) \prec (\sigma_2, \dots \sigma_k) & \text{and} \quad \sigma_1 = 0 \end{cases}$

This means, to compare two sequences $s = (\sigma_1, \sigma_2, \dots, \sigma_k)$ and $\tilde{s} = (\tilde{\sigma}_1, \tilde{\sigma}_2, \dots, \tilde{\sigma}_k)$, one has to look at the first position i, where the two sequences differ, i.e.

$$\sigma_i \neq \tilde{\sigma}_i, \qquad \sigma_j = \tilde{\sigma}_j \quad \text{for} \quad j < i.$$

The order of the sequences is then determined by the ordering of σ_i and $\tilde{\sigma}_i$. But in contrast to usual lexicographic order, the number of symbols 0 appearing in the first $i-1$ identical symbols is taken into account: If this number is even, then the sequences s and $\tilde{s}$ are ordered in the same way as σ_i and $\tilde{\sigma}_i$ (according to $-1 < 0 < 1$). If this number is odd, the order of s and $\tilde{s}$ is reversed with respect to that of σ_i and $\tilde{\sigma}_i$.

Note that this definition can be applied to sequences of the symbols $\{-1, 0, 1\}$ independent of the transition rules from Definition 2.2. Especially, we can apply it to the reversed symbol sequences

$$R(s) := (\sigma_k, \dots, \sigma_1), \qquad s = (\sigma_1, \dots, \sigma_k) \in \mathcal{S}_k.$$

Obviously, for reversed sequences $R(s), s \in \mathcal{S}_k$ with even k, the transition rules for odd and even i are interchanged.

Using the ordering of the reversed sequences, we can now define a discrete counterpart of the zero number:

Definition 3.4 *For a pair of sequences $s = (\sigma_1, \dots, \sigma_k)$ and $\tilde{s} = (\tilde{\sigma}_1, \dots, \tilde{\sigma}_k)$ in $\mathcal{S}_k$ we denote by $t = (\sigma_1, \dots, \sigma_{k-1})$ and $\tilde{t} = (\tilde{\sigma}_1, \dots, \tilde{\sigma}_{k-1})$ the truncated sequences. The discrete zero number $z_d(s - \tilde{s})$ is then defined recursively by:*

(i) $z_d(s - \tilde{s}) = z_d(t - \tilde{t})$, *if $R(s)$ and $R(\tilde{s})$ are ordered in the same way as $R(t)$ and $R(\tilde{t})$*

(ii) $z_d(s - \tilde{s}) = z_d(t - \tilde{t}) + 1$, *if $R(s)$ and $R(\tilde{s})$ are ordered opposite to $R(t)$ and $R(\tilde{t})$*

(iii) $z_d(s - \tilde{s}) = i(t)$, *if $t = \tilde{t}$*

For $k = 2$, the zero number for any pair of sequences is zero.

Note that although in the case $k = 1$ there will be no relation to the stationary solutions of a corresponding PDE, we include this case in our definitions in order to achieve a convenient description of the structural properties of the symbol sequences. The following Lemma collects some basic properties of our recursive definition of the zero number for symbol sequences:

Lemma 3.5 *For two sequences $s, \tilde{s} \in \mathcal{S}_k$ we have*

(i) $z_d(s-\tilde{s}) = z_d(R(s)-R(\tilde{s}))$

(ii) $s \prec_n \tilde{s} \Longleftrightarrow \begin{cases} R(s) \prec_n R(\tilde{s}), & \text{if } n \text{ even} \\ R(\tilde{s}) \prec_n R(s), & \text{if } n \text{ odd} \end{cases}$

(iii) *If* $z_d(t-\tilde{t}) < z_d(s-\tilde{s})$, *then* $\sigma_{k-1} = \tilde{\sigma}_{k-1}$

Proof: To prove statement (i), we first consider sequences s and $\tilde{s}$, such that $\sigma_1 \neq \tilde{\sigma}_1$ and $\sigma_k \neq \tilde{\sigma}_k$, but $\sigma_i = \tilde{\sigma}_i$ for all $1 < i < k$. Computing for those sequences the zero number $z_d(s-\tilde{s})$ step by step, according to the recursion from Definition 3.4, we get a contribution $+1$ for all positions $1 < i < k$ with $\sigma_i = \tilde{\sigma}_i = 0$. These contributions sum up to $i(\sigma_2, \ldots, \sigma_{k-1})$. Moreover, we get one more contribution $+1$, exactly if

$$(\sigma_1 - \tilde{\sigma}_1)(\sigma_k - \tilde{\sigma}_k) < 0 \quad \text{and} \quad i(\sigma_2, \ldots, \sigma_{k-1}) \quad \text{even}$$

or

$$(\sigma_1 - \tilde{\sigma}_1)(\sigma_k - \tilde{\sigma}_k) > 0 \quad \text{and} \quad i(\sigma_2, \ldots, \sigma_{k-1}) \quad \text{odd}.$$

Exactly the same is obviously true for the zero number of the reversed sequences $z_d(R(s) - R(\tilde{s}))$. For an arbitrary pair of sequences s and $\tilde{s}$ we observe, that the zero number is given by summation over the zero numbers of all segments $(\sigma_i, \sigma_{i+1}, \ldots, \sigma_j)$, $i < j$ which are of the form, described above. This formula is obviously independent on the orientation of the sequences.

Statement (ii) follows inductively from Definition 3.4: For $k = 1$, all zero numbers are zero and $s \prec \tilde{s} \Leftrightarrow R(s) \prec R(\tilde{s})$. Assuming the statement to be true for $t \neq \tilde{t}$, both in $\mathcal{S}_{k-1}$, we get it immediately for s and $\tilde{s}$, using the recursive definitions 3.4 (i), (ii). For $t = \tilde{t}$, it follows immediately from 3.4 (iii) and 3.3 (ii).

Statement (iii), finally, is a consequence of the transition rules. Indeed, to obtain $R(s) \prec R(\tilde{s})$ for $R(\tilde{t}) \prec R(t)$, it is necessary to have

$$\sigma_k < \tilde{\sigma}_k \quad \text{or} \quad \sigma_k = \tilde{\sigma}_k = 0.$$

According to the transition rules, both is impossible if

$$\tilde{\sigma}_{k-1} < \sigma_{k-1}.$$

On the other hand, $R(\tilde{t}) \prec R(t)$ implies that $\tilde{\sigma}_{k-1} \leq \sigma_{k-1}$. So, we must have $\tilde{\sigma}_{k-1} = \sigma_{k-1}$. □

In the following lemma, we prove that the above defined zero number and order relation for symbol sequences indeed agree with their corresponding counterparts for the stationary solutions.

Lemma 3.6 *Assume that the piecewise constant function $a(x)$ satisfies (5) and (6) and ε is sufficiently small (cf. Theorem 1). Then for any two stationary solutions $w, \tilde{w} \in E_k^\varepsilon$ and the corresponding symbol sequences $s, \tilde{s} \in \mathcal{S}_k$ we have:*

(i) $w(0) < \tilde{w}(0) \Longleftrightarrow s \prec \tilde{s}$

(ii) $z(w - \tilde{w}) = z_d(s - \tilde{s})$

Proof : To prove (i), we show that for sufficiently small ε, the intervals of initial conditions I_s^ε and $I_{\tilde{s}}^\varepsilon$ are ordered according to the order of s and $\tilde{s}$. Since for $s \neq \tilde{s}$, I_s and $I_{\tilde{s}}$ are disjoint, there is an obvious notion for the order of these intervals of real numbers. For $k = 1$, this follows immediately from (10). We proceed now by induction over k. If the sequences s and $\tilde{s}$ differ at any position $i < k$, then the ordering of s and $\tilde{s}$ is already determined by the order of the initial parts $(\sigma_1, \ldots, \sigma_i)$ and $(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_i)$. By induction, the order for $I^\varepsilon_{\sigma_1,\ldots,\sigma_i}$ and $I^\varepsilon_{\tilde{\sigma}_1,\ldots,\tilde{\sigma}_i}$ is the same. According to (9), this ordering of the intervals carries over to I_s^ε and $I_{\tilde{s}}^\varepsilon$.

It remains to treat the case, where $\sigma_i = \tilde{\sigma}_i$ for all $i < k$. According to Lemma 2.6, the image $\Phi_{1,k}(I^\varepsilon_{\sigma_1,\ldots,\sigma_{k-1}})$ is a graph over the u-axis. It contains the images

$$\Phi_{1,k}(I^\varepsilon_{\sigma_1,\ldots,\sigma_{k-1},-1}), \quad \Phi_{1,k}(I^\varepsilon_{\sigma_1,\ldots,\sigma_{k-1},0}), \quad \Phi_{1,k}(I^\varepsilon_{\sigma_1,\ldots,\sigma_{k-1},1}),$$

ordered along the u-axis according to the last symbol. Tracing this ordering back to the order of the intervals

$$I^\varepsilon_{\sigma_1,\ldots,\sigma_{k-1},-1}, \quad I^\varepsilon_{\sigma_1,\ldots,\sigma_{k-1},0}, \quad I^\varepsilon_{\sigma_1,\ldots,\sigma_{k-1},1}$$

inside the interval $I^\varepsilon_{\sigma_1,\ldots,\sigma_{k-1}}$, we have to regard that according to Lemma 2.6 (iii), the graph has been reversed $i(\sigma_1, \ldots, \sigma_{k-1})$ times while being mapped iteratively to $\Phi_{1,k}(I^\varepsilon_{\sigma_1,\ldots,\sigma_{k-1}})$. This is exactly reflected by our definition of the order relation $\prec$ on the set of symbol sequences.

To prove (ii), we proceed again by induction, assuming that the statement is true for $k-1$: For any pair of sequences $t, \tilde{t} \in \mathcal{S}_{k-1}$ with corresponding stationary profiles $u(x), \tilde{u}(x)$, we assume that

$$z(u(x) - \tilde{u}(x)) = z_d(t - \tilde{t}). \tag{12}$$

For $k = 2$ it is easy to check from the phase portrait, that $z(u(x) - \tilde{u}(x)) = 0$ for any two stationary profiles $u(x), \tilde{u}(x) \in E_2^\varepsilon$ (see Figure 3). According to Definition 3.4 for $k = 2$ also $z_d(t - \tilde{t}) = 0$ for all $t, \tilde{t} \in \mathcal{S}_2$.

For given sequences $s, \tilde{s} \in \mathcal{S}_k$, we get the truncated sequences $t, \tilde{t} \in \mathcal{S}_{k-1}$. If $t \neq \tilde{t}$, we can apply the induction hypothesis in the following way: For $t, \tilde{t}$, there exist corresponding stationary profiles $w_T(x), \tilde{w}_T(x)$, $x \in [0, x_k]$, satisfying Neumann boundary conditions at 0 and x_k, with zero number given by (12).

We will establish now a relation between $z(w_T(x) - \tilde{w}_T(x))$ and $z(w(x) - \tilde{w}(x))$ in two steps: First, we restrict $w(x)$ and $\tilde{w}(x)$ to $[0, x_k]$ compare the zeroes there to those of $w_T(x) - \tilde{w}_T(x)$. Then, we account for the additional zeroes in the subinterval $[x_k, 1]$.

To this end, we consider two trajectories

$$\mathbf{u}_\alpha(x) = (u_\alpha(x), v_\alpha(x)), \ \mathbf{u}_\beta(x) = (u_\beta(x), v_\beta(x)), \qquad x \in [0, x_k]$$

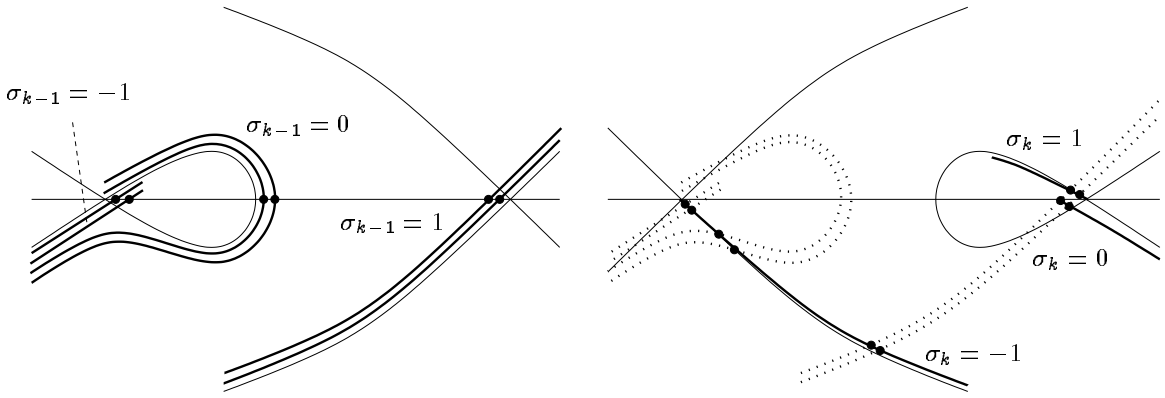


Figure 6: Schematic picture at $x = x_k$ for k even. Left: bold lines indicate possible locations of $\Phi_{0,k}(I_t)$ and $\Phi_{0,k}(I_{\tilde{t}})$; points indicate corresponding locations of $w_T(x_k)$ and $\tilde{w}_T(x_k)$. Right: segments of the preimage of $\{v = 0\}$ (bold), intersecting $\Phi_{0,k}(I_t)$ and $\Phi_{0,k}(I_{\tilde{t}})$ (dotted lines) at $w(x_k)$ and $\tilde{w}(x_k)$ (points)

with initial condtions

$$u_\alpha(0) = \alpha \in I_t^\varepsilon, \qquad v_\alpha(0) = 0,$$

and

$$u_\beta(0) = \beta \in I_{\tilde{t}}^\varepsilon, \qquad v_\beta(0) = 0.$$

The zero number $z(u_\alpha(x) - u_\beta(x))$, $x \in [0, x_k]$, is locally constant in α and β, unless $u_\alpha(x_k) = u_\beta(x_k)$. At a point where $u_\alpha(x_k) = u_\beta(x_k)$, we have to regard whether $\mathbf{u}_\alpha(x_k)$ and $\mathbf{u}_\beta(x_k)$ move around each other clockwise, which leads to increasing $z(u_\alpha - u_\beta)$, or anti-clockwise, which leads to decreasing $z(u_\alpha - u_\beta)$ (see [FR96]). At each such point the sign of $u_\alpha(x_k) - u_\beta(x_k)$ changes.

Note that not only $w_T(0) \in I_t^\varepsilon$ but also $w(0) \in I_s^\varepsilon \subset I_t^\varepsilon$. Changing α monotonically from $w_T(0)$ to $w(0)$, the profile $u_\alpha(x)$, changes from $w_T(x)$ to $w(x)$. Then, changing β monotonically from $\tilde{w}_T(0)$ to $\tilde{w}(0)$, we move the profile $u_\beta(x)$, $x \in [1, x_k]$ from $\tilde{w}_T(x)$ to $\tilde{w}(x)$.

The values $\mathbf{u}_\alpha(x_k)$ and $\mathbf{u}_\beta(x_k)$ are located on the curves $\Phi_{0,k}^\varepsilon(I_t)$ and $\Phi_{0,k}^\varepsilon(I_{\tilde{t}})$, which are close to the unstable manifold of one of the two fixed points, or to the homoclinic loop Γ_k. In Figure 6, we have drawn two instances of such curves for all choices of the $k-1$th symbol. For α and β varying as described above, we can observe in Figure 6 how the endpoints $\mathbf{u}_\alpha(x_k)$ and $\mathbf{u}_\beta(x_k)$ move from $w_T(x_k)$ and $\tilde{w}_T(x_k)$ (left hand side in the figure) to $w(x_k)$ and $\tilde{w}(x_k)$ (right hand side in the figure). The bold lines in the left part show segments of the image of the u-axis under the backward flow $\Phi_{k+1,k}$ in the kth subinterval. Each of the three segments corresponds to one possible choice of the kth symbol. The order of points on the segments corresponding to ± 1 is the same as on the u-axis, whereas on the middle segment the order is reversed. A zero of $w(x) - \tilde{w}(x)$ in the interval $[x_k, 1]$ is accompanied with a different sign of $w(x_k) - \tilde{w}(x_k)$ and $w(1) - \tilde{w}(1)$. Obviously, there is at most one additional zero in

this subinterval.

From this configuration, it is now easy to check that for $w_T(x_k) < \tilde{w}_T(x_k)$, we get

$$z(w - \tilde{w}) = z(w_T - \tilde{w}_T)$$

exactly, if $w(1) < \tilde{w}(1)$. If $\tilde{w}(1) < w(1)$, we have

$$z(w - \tilde{w}) = z(w_T - \tilde{w}_T) + 1.$$

Since the ordering of the endpoints corresponds to the ordering of the reversed symbol sequences, this shows the coincidence of the zero number with its discrete counterpart in the cases (i) and (ii) of Definition 3.4

It remains to treat the case, where $t = \tilde{t}$, i.e $\sigma_i = \tilde{\sigma}_i$, for $1 \leq i < k$ and $\sigma_k \neq \tilde{\sigma}_k$. In this case, both $\Phi_{1,k}(I_s^\varepsilon)$ and $\Phi_{1,k}(I_{\tilde{s}}^\varepsilon)$ are contained in $\Phi_{1,k}(I_t^\varepsilon)$. Due to the transition rules we have $\sigma_{k-1} = (-1)^k$, and $\Phi_{1,k}(I_t^\varepsilon)$ is close to W_{k-1}^u, which is a graph over the u-axis. But for any two trajectories $\mathbf{u}_\alpha(x)$, $\mathbf{u}_\beta(x)$ of (2) with initial conditions α, $\beta \in I_t^\varepsilon$, we have

$$z(u_\alpha(x) - u_\beta(x)) = i(\sigma_1, \ldots, \sigma_{k-1}). \tag{13}$$

This follows from the fact according to Lemma 2.6 (iii) that the flow $\Phi_{1,k}$ reverses the interval I_t^ε exactly $i(t)$ times. In the last subinterval $x \in [x_k, x_{k+1}]$, there are no additional zeros. Since (13) coincides with Definition 3.3 (iii), the lemma is true also in this case. □

Lemma 3.7 *For any pair of symbol sequences $s, \tilde{s} \in \mathcal{S}_k$, $s \neq \tilde{s}$, the following two statements are equivalent:*

(i) s and $\tilde{s}$ are adjacent and $i(s) > i(\tilde{s})$

(ii) For all i with $\sigma_i \neq \tilde{\sigma}_i$, we have $\sigma_i = 0$.

Proof : (i) implies (ii): To show this implication, we proceed as follows: We assume that with a pair of two sequences from $\mathcal{S}_k$, it is impossible to satisfy condition (ii) for both choices of s, $\tilde{s}$ from that pair. Then, we show by induction over k, that the sequences cannot be adjacent. Having obtained in this way, that for a pair of sequences, adjacency implies one of the two variants of condition (ii), we can use the additional information $i(s) > i(\tilde{s})$ from (i), to make the proper choice of s and $\tilde{s}$, and hence obtain that (i) implies (ii).

To prove the above assertion, we have to distinguish two cases. Moreover, we assume for definiteness, that $s \prec_n \tilde{s}$.

Case 1: With the pair of truncated sequences t, $\tilde{t} \in \mathcal{S}_{k-1}$ it is impossible to satisfy condition (ii) and $t \neq \tilde{t}$.

Then by induction they are not adjacent and hence there exists a sequence $\hat{t} = (\hat{\sigma}_1, \ldots, \hat{\sigma}_{k-1})$, satisfying

$$t \prec_m \hat{t} \prec_m \tilde{t}, \tag{14}$$

with

$$m := z_d(t - \tilde{t}) = z_d(s - \tilde{s}) \quad \text{or} \quad z_d(s - \tilde{s}) - 1.$$

This sequence $\hat{t} \in \mathcal{S}_{k-1}$ has now to be extended to $\hat{s} \in \mathcal{S}_k$ by an additional symbol $\hat{\sigma}_k$, such that

$$z_d(s - \hat{s}) = z_d(\tilde{s} - \hat{s}) = z_d(s - \tilde{s}) = n. \tag{15}$$

If this is possible, then we conclude that $t \prec \hat{t} \prec \tilde{t}$ implies $s \prec \hat{s} \prec \tilde{s}$, which together with (15) implies, that s and $\tilde{s}$ are not adjacent, too.

We show now, how to choose $\hat{\sigma}_k$: Using (14), Lemma 3.5 (ii) implies for m even that

$$R(t) \prec R(\hat{t}) \prec R(\tilde{t}). \tag{16}$$

and hence

$$\sigma_{k-1} \leq \hat{\sigma}_{k-1} \leq \tilde{\sigma}_{k-1} \tag{17}$$

For odd m the reversed inequalities are valid.

If $n = m + 1$, Lemma 3.5 (iii) implies that $\sigma_{k-1} = \tilde{\sigma}_{k-1}$. Due to (17), we get also

$$\hat{\sigma}_{k-1} = \sigma_{k-1} = \tilde{\sigma}_{k-1}.$$

Moreover, $n = m + 1$ implies that either $\sigma_k \neq \tilde{\sigma}_k$ or $\sigma_k = \tilde{\sigma}_k = 0$. In both cases, the transition rules allow also for $\hat{\sigma}_k = 0$. It is easy to check that this choice of $\hat{\sigma}_k$ always satisfies (15).

If $n = m$, we choose $\hat{\sigma}_k \in \{\sigma_k, \tilde{\sigma}_k\} \setminus \{0\}$. This set is nonempty, since $\sigma_k = \tilde{\sigma}_k = 0$ contradicts to $n = m$. Moreover, using (17), it is easy to check that there is always such a choice, which satisfies the transition rules. Equation (15) is satisfied for this choice, since we get from (16) immediately

$$R(s) \prec R(\hat{s}) \prec R(\tilde{s}),$$

which, together with (16), implies that no zero number changes occur by adding the k-th symbol.

Case 2: t and $\tilde{t}$ are equal, or satisfy condition (ii).

In this case, we look first at the reversed sequences (see Lemma 3.5). If they satisfy the setting for Case 1, we are done. The only possibility, where this fails is, if

$$\begin{aligned} &\sigma_1 = \tilde{\sigma}_k = 0 \\ &\tilde{\sigma}_1 \neq 0 \neq \sigma_k \\ &\quad \sigma_i = \tilde{\sigma}_i \qquad \text{for} \quad i = 2 \ldots k-1. \end{aligned}$$

Defining now

$$\hat{s} := (\tilde{\sigma}_1, \sigma_2, \ldots, \sigma_{k-1}, \sigma_k),$$

one can check easily that

$$z_d(s - \hat{s}) = z_d(\tilde{s} - \hat{s}) = z_d(s - \tilde{s}) = i(\sigma_2, \ldots, \sigma_{k-1})$$

and either $s \prec \hat{s} \prec \tilde{s}$ or $\tilde{s} \prec \hat{s} \prec s$. Hence there is no adjacency also in that case and the assertion is proved.

(ii) implies (i): First, we show that for any pair of sequences $s, \tilde{s} \in \mathcal{S}_k$, condition (ii) implies adjacency of s and $\tilde{s}$. We will prove this by showing inductively the following assertion: If $s, \tilde{s} \in \mathcal{S}_k$, satisfy condition (ii), then for all $\hat{s} \in \mathcal{S}_k$ with

$$s \prec \hat{s} \prec \tilde{s}, \tag{18}$$

the quantity

$$D_s := z_d(s - \hat{s}) - z_d(\tilde{s} - \hat{s}) \tag{19}$$

is greater than zero. For definiteness, we may assume $s \prec_n \tilde{s}$.

First, note that for $k = 1$ there is no $\hat{s}$, satisfying (18), and hence the assertion is trivially satisfied. If the truncated sequences $t, \tilde{t}$ are equal, then condition (ii) implies also that there is no $\hat{s}$, satisfying (18). Hence we may assume in the sequel that $k > 1$ and $t \neq \tilde{t}$. We distinguish now three cases:

Case 1: $t \neq \hat{t} \neq \tilde{t}$. Here, (18) implies

$$t \prec \hat{t} \prec \tilde{t}$$

and we can assume by induction that

$$D_t := z_d(t - \hat{t}) - z_d(\tilde{t} - \hat{t}) > 0. \tag{20}$$

From Lemma 3.5 (ii), we can conclude that exactly for D_t even, one of the inequalities

$$R(t) \prec R(\hat{t}) \prec R(\tilde{t}) \quad \text{or} \quad R(\tilde{t}) \prec R(\hat{t}) \prec R(t) \tag{21}$$

is true. Consequently, if (21) is satisfied we get from (20) that $D_t \geq 2$. But since $|D_t - D_s| \leq 1$ (see Definition 3.4), this proves our assertion in this case.

If (21) is not satisfied, then $D_s \geq D_t$ due to the following reason: To obtain $D_s < D_t$, we need that $z_d(\tilde{s} - \hat{s}) > z_d(\tilde{t} - \hat{t})$, whereas $z_d(s - \hat{s}) = z_d(t - \hat{t})$. Moreover, if $D_s = 0$, we obtain inequalities analogous to (21) for $s, \hat{s}, \tilde{s}$, which implies that

$$\sigma_k \leq \hat{\sigma}_k \leq \tilde{\sigma}_k \quad \text{or} \quad \tilde{\sigma}_k \leq \hat{\sigma}_k \leq \sigma_k.$$

Taking into account condition (ii), it follows that either $\hat{\sigma}_k = \tilde{\sigma}_k$ or $\hat{\sigma}_k = \sigma_k = 0$. But $\hat{\sigma}_k = \tilde{\sigma}_k$ is impossible since then $z_d(\tilde{s} - \hat{s}) = z_d(\tilde{t} - \hat{t})$. Also $\hat{\sigma}_k = \sigma_k = 0$ is impossible, because it implies $z_d(s - \hat{s}) > z_d(t - \hat{t})$. This finishes the case where $t \neq \hat{t} \neq \tilde{t}$.

Case 2: $\hat{t} = t, \quad \hat{t} \neq \tilde{t}$. In this case, we can argue in a similar way as above. First, we notice that

$$z_d(s - \hat{s}) = i(t). \tag{22}$$

Then, we show that

$$z_d(\tilde{t} - t) = z_d(\tilde{t} - \hat{t}) < i(t). \tag{23}$$

Indeed, computing $z_d(\tilde{t}-t)$ recursively according to Definition 3.4, there is a contribution $+1$, whenever $\sigma_i = \tilde{\sigma}_i = 0$. Further contributions $+1$ may occur only at positions, where $\sigma_i \neq \tilde{\sigma}_i$, which implies $\sigma_i = 0$, according to condition (ii). Only at the first position, where $\sigma_i \neq \tilde{\sigma}_i$, there is never a contribution to $z_d(\tilde{t}-t)$. From this, we can conclude (23), since the number of positions $i = 1 \ldots k-1$ where $\sigma_i = 0$ is just $i(t)$. Since

$$z_d(\tilde{s}-\hat{s}) \leq z_d(\tilde{t}-\hat{t}) + 1,$$

it follows immediately from (22) and (23) that $D_s \geq 0$. To obtain $D_s = 0$, we would need that $z_d(\tilde{s}-\hat{s}) = z_d(\tilde{t}-\hat{t}) + 1$. According to Definition 3.4, this implies that $\tilde{\sigma}_k \neq \hat{\sigma}_k$. In contradiction to that, we obtain again inequalities analogous to (21) for $s, \hat{s}, \tilde{s}$. As above, we conclude that either $\hat{\sigma}_k = \tilde{\sigma}_k$ or $\hat{\sigma}_k = \sigma_k = 0$. But $\hat{\sigma}_k = \sigma_k$ is excluded here, since it would imply $\hat{s} = \tilde{s}$.

Case 3: $\hat{t} = \tilde{t}, \quad \hat{t} \neq t$. Here, we have by definition that

$$z_d(\hat{s}-\tilde{s}) = i(\tilde{t}.)$$

Due to condition (ii), at all positions $i = 1 \ldots k-1$, where $\tilde{\sigma}_i = \hat{\sigma}_i = 0$, we have $\sigma_i = 0$ as well, and hence

$$z_d(t-\tilde{t}) = z_d(t-\hat{t}) \geq i(\tilde{t}).$$

Hence $D_s \geq 0$, and $D_s = 0$ is only possible, if

$$z_d(s-\hat{s}) = z_d(t-\hat{t}) = i(\tilde{t}). \tag{24}$$

Additionally, we conclude as above, that either $\hat{\sigma}_k = \tilde{\sigma}_k$ or $\hat{\sigma}_k = \sigma_k = 0$. The first of these possibilities contradicts to $\hat{s} \neq \tilde{s}$, and the latter one gives

$$z_d(s-\hat{s}) = z_d(t-\hat{t}) + 1$$

in contradiction to (24). This finishes the proof for this case. Hence the assertion that $D_s > 0$ for all $\hat{s}$ between s and $\tilde{s}$ is proved. Since for $s \neq \tilde{s}$, (ii) implies obviously that $i(s) > i(\tilde{s})$, we get that (ii) implies (i), and the proof of Lemma 3.7 is finished. □

Proof of Theorem 2: First, we recall that due to the Morse-Smale property of the system (see [An86]), a heteroclinic connection from w to $\tilde{w}$ implies for the Morse-indices

$$i(w) > i(\tilde{w}). \tag{25}$$

Due to Proposition 3.2, the heteroclinic connection implies also adjacency of w and $\tilde{w}$. According to Lemma 3.6, this is equivalent to adjacency of the corresponding symbol sequences s and $\tilde{s}$. Due to Theorem 1 (iii), inequality (25) implies also

$$i(s) > i(\tilde{s}).$$

This, together with the adjacency of the sequences, is due to Lemma 3.7 finally equivalent to the condition that for all $j \in \{1, \ldots, k\}$

$$\sigma_j \neq \tilde{\sigma}_j \Longrightarrow \sigma_j = 0,$$

which is exactly our condition (ii) in Theorem 2. □

3.2 The Permutation of the equilibria

As an important tool for the investigation of scalar parabolic equations of the form (11) Fusco and Rocha introduced in [FuRo91] the permutation of the equilibria. For a given equation with hyperbolic equilibria, this permutation ρ is defined by first numbering all equilibria profiles according to their order at the left boundary $x = 0$ of the interval

$$w_1(0) < w_2(0) < \ldots < w_n(0),$$

and then looking how this order has changed at the right boundary $x = 1$:

$$w_{\rho(1)}(1) < w_{\rho(2)}(1) < \ldots < w_{\rho(n)}(1)$$

The permutation ρ contains all information about the nodal properties of the equilibria profiles. It has been shown that ρ can be used do determine the Morse indices and the heteroclinic connections of the equilibria [FR96]. Moreover, the permutation determines the attractor up to C^0 orbit equivalence [FR99]. How the permutation is related in general to the order relations $\prec_n$, which we used here, has been discussed in [Wo00].

Since in [FRSS01], this permutation has also been used to study the specific class of equations which is the subject of the present paper, we remark that, using the results from the previous section, we get immediately the following result:

Corollary 3.8 *Assume that the function $a(x)$ is piecewise constant on k subintervals and satisfies (5) and (6). Then for small enough $\varepsilon > 0$ the permutation ρ_k of the equilibria can be obtained from the symbol sequences in $\mathcal{S}_k$ in the following way: Numbering all sequences in $\mathcal{S}_k$ according to*

$$s_1 \prec s_2 \prec \ldots \prec s_n,$$

the permutation ρ_k is given by the order of the reversed sequences:

$$R(s_{\rho_k(1)}) \prec R(s_{\rho_k(2)}) \prec \ldots \prec R(s_{\rho_k(n)})$$

4 An ODE model

Using the information we have obtained so far, we can now construct a model for the global attractor $\mathcal{A}^\varepsilon$.

Theorem 3 *For $0 < \varepsilon < \varepsilon_0$ the attractor $\mathcal{A}^\varepsilon$ is connection equivalent to the global attractor $\mathcal{M}$ of the following model o.d.e. with $y = (y_1, y_2, \ldots, y_k) \in \mathbb{R}^k$:*

$$\left.\begin{array}{rcl} \dot{y}_1 &=& y_1(1-y_1^2) \\ \dot{y}_2 &=& y_2(1-y_2^2) + (y_1-1)(y_2+1) \\ \dot{y}_3 &=& y_3(1-y_3^2) - (y_2+1)(y_3-1) \\ \vdots && \vdots \\ \dot{y}_k &=& y_k(1-y_k^2) + (-1)^k(y_{k-1} - (-1)^k)(y_k + (-1)^k). \end{array}\right\} \qquad (26)$$

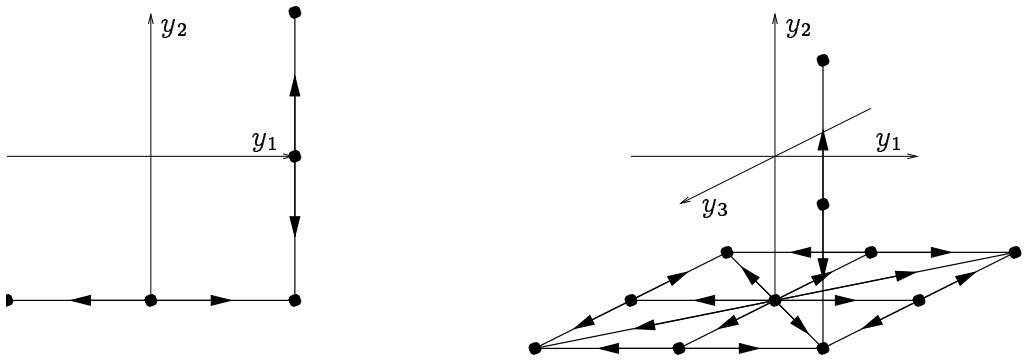


Figure 7: Heteroclinic connections on the attractor for $k = 2$ and $k = 3$

Although we expect that $\mathcal{M}$ is at least C^0-orbit equivalent to $\mathcal{A}^\varepsilon$ we do only prove the weaker statement of connection equivalence here.

Lemma 4.1 *The stationary solutions of (26) are precisely the vectors*

$$(y_1, y_2, \ldots, y_k) \in \mathcal{S}_k.$$

Proof: Looking for stationary solutions we have to solve first the equation $y_1(1 - y_1^2) = 0$, hence $y_1 = -1$ or 0 or $+1$. Concerning the second equation, we have to distinguish two cases. If $y_1 = 1$, then the second equation reduces to $y_2(1 - y_2^2) = 0$ which implies that $y_2 \in \{-1, 0, +1\}$. If, however, $y_1 = 0$ or $y_1 = -1$, then the second equation reads

$$0 = (y_2 + 1)(-y_2^2 + y_2 + y_1 - 1) = (y_2 + 1)(-(y_2 - \frac{1}{2})^2 + \frac{1}{4} + y_1 - 1).$$

Clearly, this implies that $y_2 = -1$ since the term in brackets does not vanish for $y_1 \leq 0$. One can now proceed by induction assuming that we have already found out that $y_i \in \{-1, 0, +1\}$. If $y_i = (-1)^{i+1}$ then the $(i+1)$-st equation reads $y_{i+1}(1 - y_{i+1}^2) = 0$. Hence y_{i+1} can take any value in $\{-1, 0, +1\}$. If $y_i = (-1)^i$ or $y_i = 0$ then we have to solve

$$0 = (y_{i+1} - (-1)^i)(-y_{i+1}^2 + (-1)^i y_{i+1} + (-1)^{i+1} y_i - 1).$$

As the second term does not vanish for $y_i = (-1)^i$ or $y_i = 0$, we must have $y_{i+1} = (-1)^i$. Comparing with the definition of $\mathcal{S}_k$ we see that $(y_1, y_2, \ldots, y_k) \in \mathcal{S}_k$. □

Proof of Theorem 3: We proceed again by induction to show that two equilibria $(y_1, y_2, \ldots, y_k)$ and $(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_k)$ are connected by a heteroclinic orbit if and only if

$$y_i \neq \hat{y}_i \implies y_i = 0 \tag{27}$$

holds for $i = 1, 2, \ldots, k$. That the claim holds for $k = 1$ and $k = 2$ can verified directly from the corresponding phase portraits.

Assume now that the statement is true up to $k = n$ and consider equation (26) with $k = n + 1$. From the last equation we can immediately read off that the hyperplane $\{y_{n+1} = (-1)^n\}$ is invariant and the restriction of the flow to this hyperplane is exactly (26) with $k = n$. From the induction hypothesis we know all about the heteroclinic orbits within this hyperplane. These heteroclinic orbits connect equilibria with $y_{n+1} = \hat{y}_{n+1} = (-1)^n$ which satisfy (27).

Another invariant hyperplane is $\{y_n = (-1)^{n+1}\}$. Within this hyperplane the flow is given by the system

$$\begin{aligned}
\dot{y}_1 &= y_1(1 - y_1^2) \\
\dot{y}_2 &= y_2(1 - y_2^2) + (y_1 - 1)(y_2 + 1) \\
\dot{y}_3 &= y_3(1 - y_3^2) - (y_2 + 1)(y_3 - 1) \\
&\vdots \qquad \vdots \\
\dot{y}_{n-1} &= y_{n-1}(1 - y_{n-1}^2) + (-1)^{n-1}(y_{n-2} - (-1)^{n-1})(y_{n-1} + (-1)^{n-1}) \\
\dot{y}_{n+1} &= y_{n+1}(1 - y_{n+1}^2).
\end{aligned}$$

The last equation is decoupled, so the flow is a direct product of the flow (26) with $k = n - 1$ and the flow generated by the last equation. It is therefore obvious that (27) has to be satisfied for $1 \leq i \leq n - 1$ and also for $i = n + 1$ while $y_n = \hat{y}_n$.

To show that there are no other heteroclinic orbits outside the invariant planes $\{y_{n+1} = (-1)^n\}$ and $\{y_n = (-1)^{n+1}\}$ it suffices to calculate the eigenvalue of the linearization of the equilibria in the transverse direction. It turns out that all equilibria are stable in the transverse direction so there cannot be any heteroclinic orbits outside the invariant hyperplanes. □

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