Viscous profiles for traveling waves of scalar balance laws: The uniformly hyperbolic case

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1 Introduction

We are concerned with traveling wave solutions for scalar hyperbolic balance laws

$$u_t + f(u)_x = g(u), \quad x \in \mathbb{R}, \quad u \in \mathbb{R}. \tag{H}$$

The question whether these traveling waves can be obtained as the limit of traveling waves of the viscous balance law

$$u_t + f(u)_x = \varepsilon u_{xx} + g(u), \quad x \in \mathbb{R}, \quad u \in \mathbb{R} \tag{P}$$

when the viscosity parameter ε tends to zero is discussed in this article. More precisely, the search for such viscous profiles leads to a singular perturbation problem. Hyperbolic balance laws are extensions of hyperbolic conservation laws where a source term g is added. These reaction terms can model chemical reactions, combustion or other interactions [7], [1]. From the theoretical point of view, the source terms dramatically change the long-time behaviour of the equation compared to hyperbolic conservation laws. While for conservation laws the only traveling wave solutions are shock waves, balance laws exhibit different types of traveling waves. A classification of the traveling waves in the case of a convex flow function f has been done by Mascia [5]. We summarize his results in section 2.1.
Since hyperbolic balance laws are often considered as a simplified model for some parabolic (*viscous*) equation with a very small viscosity, it is important to know, whether traveling wave solutions of the hyperbolic equation correspond to traveling waves of the viscous equation. If this is true in a sense to be specified below, we say that the traveling wave admits a *viscous profile*.

In this paper we prove that under mild assumptions on f and g some types of waves of the hyperbolic equation admit a viscous profile.
The paper is organized as follows: In chapter 2 we introduce the notion of entropy traveling waves, make the meaning of viscous profiles more precise and state the main result. Since three different types of traveling waves occur, in chapters 3-5 the proofs are given for each case separately. The paper concludes with a short discussion.
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2 Entropy Traveling Waves

We assume the following about f and g:

(F) f is convex: $f \in C^2$, $f''(u) > 0$

(G) The zeroes of g are simple

We denote the zeroes of g with u_i where $i \in \mathcal{J} \subseteq \mathbb{Z}$ with indizes chosen such that

$$i \cdot \operatorname{sign}(g'(u_i)) > 0.$$

The set of all zeroes is called $\mathcal{Z}(g)$. Depending on the sign of g' the zeroes of g are divided into two sets :

$$\begin{aligned} \mathcal{R}(g) &:= \{u_i \in \mathcal{Z}(g) : g'(u_i) > 0\} \\ \mathcal{A}(g) &:= \{u_i \in \mathcal{Z}(g) : g'(u_i) < 0\} \end{aligned}$$

Like hyperbolic conservation laws, balance laws (H) do in general not possess global smooth solutions. Since passing to weak solutions destroys the uniqueness, an entropy condition has to be given which chooses the "correct" solution among all weak solutions. Here we define directly for traveling waves what is meant by such an entropy solution.

Definition 2.1 *An* **entropy traveling wave** *is a solution of (H) of the form $u(x,t) = u(\xi)$ with $\xi = x - st$ for some wave speed $s \in \mathbb{R}$ with the following properties:*

(i) u is piecewise C^1, i.e. $u \in C^1(\mathbb{R} \setminus J)$ and the set of accumulation points of J has only isolated points. At points where u is continuously differentiable it satisfies the ordinary differential equation

$$(f'(u(\xi)) - s)\ u'(\xi) = g(u(\xi)). \tag{1}$$

(ii) At points of discontinuity the one-sided limits $u(\xi+)$ and $u(\xi-)$ of u satisfy both the Rankine-Hugoniot condition

$$s\,(u(\xi+) - u(\xi-)) = f(u(\xi+)) - f(u(\xi-))$$

and the entropy condition

$$u(\xi+) \leq u(\xi-).$$

Due to the convexity assumption (F), for any $u \in \mathrm{IR}$ and any speed s there is at most one other value $h(u,s)$ which satisfies the Rankine-Hugoniot condition

$$\frac{f(u) - f(h(u,s))}{u - h(u,s)} = s.$$

If there is no such $h(u,s)$ we set

$$h(u,s) := \begin{cases} -\infty & \text{for} \quad f'(u) - s > 0 \\ +\infty & \text{for} \quad f'(u) - s < 0 \end{cases}$$

Definition 2.2 *A traveling wave u is said to be a* **heteroclinic wave** *if*

$$\begin{aligned} \lim_{\xi \to -\infty} u(\xi) &= u_i \\ \lim_{\xi \to +\infty} u(\xi) &= u_j \end{aligned}$$

for some u_i, $u_j \in \mathrm{IR}$.

Remark 2.3 *From (H) we can immediately conclude that $g(u_i) = g(u_j) = 0$. For this reason, we say that there is a connection between the equilibria u_i and u_j.*

2.1 Heteroclinic waves of the hyperbolic equation

Mascia [5] has classified the heteroclinic waves that occur for convex f. We collect here the results of [5, theorems 2.3-2.5] but sort them in a different way and make the statements on wave speeds more precise. To this end we distinguish three types of waves:

- Heteroclinic waves which exist for a whole interval of wave speeds s
- waves which can be found only if the speed s takes the discrete value $f'(u_i)$ for some i and
- undercompressive waves which do also show up only for particular shock speeds.

Proposition 2.4 *Heteroclinic connections from u_i to u_j that exist for a range of wave speeds are of the following types:*

(A1) Continuous monotone waves that connect adjacent equilibria

(i) $j = i+1$, $u_i \in \mathcal{A}(g)$ *and* $s \geq f'(u_{i+1})$

(ii) $j = i-1$, $u_i \in \mathcal{A}(g)$ *and* $s \geq f'(u_i)$

(iii) $j = i+1$, $u_i \in \mathcal{R}(g)$ *and* $s \leq f'(u_i)$

(iv) $j = i-1$, $u_i \in \mathcal{R}(g)$ *and* $s \leq f'(u_{i-1})$

(A2) Discontinuous heteroclinic waves

(i) $i > j$, $u_i \in \mathcal{A}(g)$, $u_j \in \mathcal{R}(g)$, $u_i \in (h(u_{j+1}, s), h(u_{j-1}, s))$

(ii) $i > j$, $u_i \in \mathcal{R}(g)$, $u_j \in \mathcal{R}(g)$, $(h(u_{j+1}, s), h(u_{j-1}, s)) \cap (u_{i-1}, u_{i+1}) \neq \emptyset$

(iii) $i > j$, $u_i \in \mathcal{R}(g)$, $u_j \in \mathcal{A}(g)$, $h(u_j, s) \in (u_{i-1}, u_{i+1})$.

Proposition 2.5 *Heteroclinic connections from u_i to u_j that exist only for a particular wave speed are of the following types:*

(B1) Continuous, monotone waves

(i) $j = i+2$, $u_i, u_j \in \mathcal{A}(g)$ *and* $s = f'(u_{i+1})$ *with an increasing profile*

(ii) $j = i-2$, $u_i, u_j \in \mathcal{R}(g)$ *and* $s = f'(u_{i-1})$

(B2) *(i)* $i \geq j$, $u_i \in \mathcal{A}(g)$, $u_j \in \mathcal{A}(g)$, $s = f'(u_{i+1})$ *and* $h(u_j, s) < u_{i+2}$,

(ii) $i > j$, $u_i \in \mathcal{A}(g)$, $u_j \in \mathcal{R}(g)$, $s = f'(u_{i+1})$ *and* $h(u_{j-1}, s) < u_{i+2}$,

(iii) $i > j$, $u_i \in \mathcal{R}(g)$, $u_j \in \mathcal{A}(g)$, $s = f'(u_{j-1})$ *and* $h(u_{j-2}, s) < u_{i-1}$,

(iv) $i \geq j$, $u_i \in \mathcal{A}(g)$, $u_j \in \mathcal{A}(g)$, $s = f'(u_{j-1})$ *and* $h(u_i, s) < u_{j-2}$.

(B3) *(i) Discontinuous waves that connect u_i to u_{i+2} with speed $s = f'(u_{i+1})$,*

(ii) Discontinuous waves that connect u_i to u_{i+1} with speed $s = f'(u_{i+1})$.

(C) Undercompressive shocks: $i > j$, u_i, $u_j \in \mathcal{A}(g)$, $s = \dfrac{f(u_i) - f(u_j)}{u_i - u_j}$.

2.2 Viscous Profiles

Unlike for viscosity solutions of hyperbolic conservation laws, we cannot get rid of the viscosity parameter ε by a simple scaling but have to discuss the full singularly perturbed system (P).
With the traveling wave ansatz $u(x,t) = u(x - st)$ we get from (P) the equation

$$\varepsilon u'' = (f'(u) - s)u' - g(u). \tag{2}$$

Here the prime denotes differentiation with respect to a new coordinate $\xi := x - st$. We are now able to define what we mean by a viscosity traveling wave solution.

Definition 2.6 *A traveling wave solution u_0 of (H) is called a* **viscosity traveling wave solution** *with wave speed s_0 if there is a sequence (u^{ε_n}) of solutions of (2) such that $\varepsilon_n \searrow 0$, $s_n \to s_0$ and $\|u^{\varepsilon_n} - u_0\|_{L^1(\mathbb{R})} \to 0$. The heteroclinic wave of the hyperbolic equation is said to admit a viscous profile.*

In this paper we will prove the admissibility for some of the heteroclinic waves. The main result is the following:

Theorem 2.7 *The heteroclinic waves of type (A1), (A2) and (C) admit a viscous profile.*

We concentrate on these types of traveling waves, since they fit into the classical theory of geometrical singular perturbation theory and can be treated in a similar way.
Although our main interest is in L^1-convergence, we will be more general and prove convergence in spaces with exponentially weighted norms. To this end, we define for $\beta \geq 0$ the norm

$$\|u\|_{L^1_\beta} := \int_{\mathbb{R}} (1 + e^{\beta|\xi|})|u(\xi)| \, d\xi$$

and the space

$$L^1_\beta := \{u \in L^1, \|u\|_{L^1_\beta} < \infty\}.$$

Obviously, the choice $\beta = 0$ is equivalent to the usual L^1-norm. We state now a short lemma which will simplify the later proofs.

Lemma 2.8 *For $\varepsilon \geq 0$, consider a family of functions $u_\varepsilon \in L^1(\mathbb{R})$. Assume that there exist limiting states*

$$u_\pm = \lim_{\xi \to \pm\infty} u_\varepsilon(\xi)$$

independent of ε and constants C, $c > 0$, $-\infty < \xi_- < \xi_+ < \infty$ such that the following conditions are satisfied:

(i) $|u_\varepsilon(\xi) - u_-| \leq Ce^{c\xi}$ for all $\xi \leq \xi_-$ and all $\varepsilon \geq 0$,

(ii) $|u_\varepsilon(\xi) - u_+| \leq Ce^{-c\xi}$ for all $\xi \geq \xi_+$ and all $\varepsilon \geq 0$,

(iii) For any $-\infty < a < b < +\infty$

$$\lim_{\varepsilon \searrow 0} \int_a^b |u_\varepsilon(\xi) - u_0(\xi)| \, d\xi = 0.$$

Then for any $\beta < c$ we have

$$\lim_{\varepsilon \searrow 0} \|u_\varepsilon - u_0\|_{L^1_\beta} = 0.$$

Proof: Given any integer n, we can find a_n such that

$$\int_{-\infty}^{a_n} Ce^{(\beta-c)\xi}\, d\xi \leq \frac{1}{10n}.$$

Using (i), we get by comparison

$$\int_{-\infty}^{a_n} |u_\varepsilon - u_-|(1+e^{\beta|\xi|})\, d\xi \leq \frac{1}{5n}.$$

Similarly, by (ii), we can find b_n with

$$\int_{b_n}^{+\infty} |u_\varepsilon - u_+|(1+e^{\beta|\xi|})\, d\xi \leq \frac{1}{5n}.$$

Using (iii), we can choose ε sufficiently small such that

$$\int_{a_n}^{b_n} |u_\varepsilon(\xi) - u_0(\xi)|\, d\xi \leq \frac{1}{5n(1+\max\{e^{\beta|a_n|}, e^{\beta|b_n|}\})}$$

and estimate the L^1_β-norm of $u_0 - u_\varepsilon$ as

$$\begin{aligned}
\|u_0 - u_\varepsilon\|_{L^1_\beta} &= \int_{-\infty}^{a_n} |u_- - u_\varepsilon(\xi)|(1+e^{\beta|\xi|})\, d\xi + \int_{-\infty}^{a_n} |u_- - u_0(\xi)|(1+e^{\beta|\xi|})\, d\xi \\
&\quad + \int_{a_n}^{b_n} |u_0(\xi) - u_\varepsilon(\xi)|(1+e^{\beta|\xi|})\, d\xi \\
&\quad + \int_{b_n}^{\infty} |u_+ - u_\varepsilon(\xi)|(1+e^{\beta|\xi|})\, d\xi + \int_{b_n}^{+\infty} |u_+ - u_0(\xi)|(1+e^{\beta|\xi|})\, d\xi \\
&\leq \frac{1}{n}
\end{aligned}$$

which completes the proof of the lemma.

□

This lemma, althoughsimple, shows the key ingredients in our later convergence proofs. Typically, (i) and (ii) will be consequences of the hyperbolicity of some fixed points, while (iii) is the point where one has to do some work.

2.3 Singular Perturbations

A convenient way to write the second-order equation (2) as a first-order system is the Liénard plane

$$\left.\begin{aligned} \varepsilon u' &= v + f(u) - su \\ v' &= -g(u). \end{aligned}\right\} \tag{3}$$

From this "slow-fast"-system two limiting systems can be derived which both capture a part of the behavior that is observed for $\varepsilon > 0$.

One is the "slow" system obtained by simply putting $\varepsilon = 0$:

$$\left.\begin{array}{rcl} 0 & = & v + f(u) - su \\ v' & = & -g(u). \end{array}\right\} \tag{4}$$

In this case the flow is confined to a curve

$$\mathcal{C}_s := \{(u, v) : v + f(u) - su = 0\}$$

that we call the **singular curve**. The other, "fast" system originates in a different scaling. With $\xi =: \varepsilon\eta$ and a dot denoting differentiation with respect to η we arrive at

$$\left.\begin{array}{rcl} \dot{u} & = & v + f(u) - su \\ \dot{v} & = & -\varepsilon g(u). \end{array}\right\} \tag{5}$$

In the limit $\varepsilon = 0$, equation (5) defines a vector field for which the singular curve $\mathcal{C}_s$ consists of equilibrium points only. This vector field is called the "fast" system. It points to the left below the curve $\mathcal{C}_s$ and to the right above.
Trajectories of the fast system connect only points for which $v + f(u) - su$ has the same values. This is exactly the Rankine-Hugoniot condition for waves propagating with speed s. Moreover the direction of the fast vector field is in accordance with the Oleinik entropy condition.
Geometric singular perturbation theory in the spirit of Fenichel [2] makes precise statements how the slow and the fast equations together describe the dynamics of (3) for small $\varepsilon > 0$. It is a strong tool in regions where the singular curve is normally hyperbolic, i.e. where the points on $\mathcal{C}_s$ are hyperbolic with respect to the fast field. The only non-hyperbolic point on $\mathcal{C}_s$ is the point where $f'(u) = s$. The heteroclinic waves of type (A1), (A2) and (C) stay away from these points and hence fit into the classical framework. The other cases involving non-hyperbolic points on the singular curve are more subtle and will be treated by blow-up techniques in a forthcoming paper [**?**].
The steady states of system (2) are exactly the points

$$\{(u, v) : \ (u, v) \in \mathcal{C}_s, \ u \in \mathcal{Z}(g)\}.$$

The linearization of (3) in a steady state $(u_i, -f(u_i) + su_i)$, possesses the eigenvalues

$$\lambda_i^{\pm} = \frac{f'(u_i) - s \pm \sqrt{(f'(u_i) - s)^2 - 4\varepsilon g'(u_i)}}{2\varepsilon} \tag{6}$$

which are real except for the case when $g'(u_i) > 0$ and

$$(f'(u_i) - s)^2 < 4\varepsilon g'(u_i). \tag{7}$$

We call the region in (s, ε)-parameter space where all eigenvalues associated to the steady states are real

$$R := \{(s, \varepsilon) : (f'(u_i) - s)^2 \geq 4\varepsilon g'(u_i) \ \forall i\}.$$

Note that any point on the axis $\varepsilon = 0$ can be approximated with a sequence of points from the interior of R.

3 Heteroclinic connections between adjacent equilibria

In this chapter we will prove that the heteroclinic waves of type (A1) possess a viscous profile.

Lemma 3.1 *The monotone heteroclinic waves of type (A1) admit a viscous profile.*

Proof: We concentrate on the case (A1)(i) since the other statements can be proved similarly. In that case, since $u_i \in \mathcal{A}(g)$ we know already that u_i is of saddle type, u_{i+1} is a sink and $g(u) < 0$ for $u \in (u_i, u_{i+1})$. The wave speed of the hyperbolic traveling wave will be denoted by s_0.
Two cases have to be distinguished, depending on the smoothness of the hyperbolic wave:
I. $s_0 > f'(u_{i+1})$**:** Since we want to apply lemma 2.8 we need to find a family (u_ε) of candidates for a viscous profile, i.e. a family of heteroclinic orbits of the system (3) for all sufficiently small ε. It turns out that such a family can be found by varying only ε while keeping s fixed at the value s_0 of the hyperbolic traveling wave.
To this end, we refine lemma 3.5 of [4] and show that for a (large) positive number k and all ε sufficiently small the region

$$P := \left\{ (u,v); u_i \le u \le u_{i+1}, k\varepsilon^2 g(u) \le v + f(u) - su - \varepsilon \frac{g(u)}{f'(u) - s} \le -k\varepsilon^2 g(u) \right\}$$

is positively invariant and contains a heteroclinic orbit from u_i to u_{i+1}. As a short notation, we will write

$$v_1(u) := \frac{g(u)}{f'(u) - s}. \tag{8}$$

The scalar product of the outer normal vector with the vector field along the upper boundary of P is

$$\begin{aligned} & \begin{pmatrix} f'(u) - s - \varepsilon v_1' + k\varepsilon^2 g'(u) \\ 1 \end{pmatrix}^T \cdot \begin{pmatrix} v + f(u) - su \\ -\varepsilon g(u) \end{pmatrix} \\ = & -\varepsilon^2 g(u) \left(\frac{(f'(u) - s) g'(u) - g(u) f''(u)}{(f'(u) - s)^3} + k(f'(u) - s) \right) + \mathcal{O}(\varepsilon^3) \\ < & \ 0 \end{aligned}$$

for k large enough and ε small, since both $g(u)$ and $f'(u) - s$ are negative on (u_i, u_{i+1}). An analogous calculation for the lower boundary of P completes the proof that P is positively invariant.
To establish the existence of a heteroclinic connection, it remains to show that a branch of the unstable manifold $W^u(u_i)$ of u_i enters P. The eigenvector associated

with the positive eigenvalue λ_i^+ of u_i is

$$\begin{pmatrix} 2 \\ \sqrt{(f'(u_i)-s)^2 - 4\varepsilon g'(u_i)} - (f'(u_i)-s) \end{pmatrix}$$

Expanding the square root with respect to ε one obtains for the slope of $W^u(u_i)$ in u_i the expression

$$-(f'(u_i)-s) - \frac{g'(u_i)}{f'(u_i)-s}\varepsilon - \frac{g'(u_i)^2}{4(f'(u_i)-s)^3}\varepsilon^2 + \mathcal{O}(\varepsilon^3).$$

It is easily checked that this is up to order ε the slope of the boundary of P at u_i. Now by choosing k larger, if necessary, it can be achieved that a branch of the unstable manifold $W^u(u_i)$ lies in P while P is still positively invariant. Since there are no equilibria in the interior of P there cannot be any periodic orbits in the interior of P, so by the Poincaré-Bendixson theorem there has to be a heteroclinic orbit u_ε from u_i to the only other equilibrium u_{i+1} on the boundary of P. Monotony of u_ε follows from the fact that it lies above the singular curve $\mathcal{C}_s$.
As indicated above, we want to apply lemma 2.8. First we parametrize all the heteroclinic orbits $u_\varepsilon(\xi)$ of the parabolic problem (P) and the heteroclinic orbit $u_0(\xi)$ of the hyperbolic equation (H) in a way such that

$$u_0(0) := u_\varepsilon(0) := \frac{u_i + u_{i+1}}{2}.$$

Then we fix some $\delta > 0$ with the property that for $u_i \le u \le u_i + \delta$ we have

$$-\frac{g(u)}{f'(u)-s} - k\varepsilon g(u) \le -c(u-u_i) \tag{9}$$

for the number k from the preceding lemma, some constant c_- and all $\varepsilon \le \varepsilon_1$. Similarly, we require for $u_{i+1} - \delta \le u \le u_{i+1}$

$$-\frac{g(u)}{f'(u)-s} - k\varepsilon g(u) \le -c(u_{i+1}-u).$$

By choosing ξ_- as the supremum

$$\xi_- := \sup_{0\le\varepsilon\le\varepsilon_1} \{\xi : u_\varepsilon(\xi) = u_i + \delta\}$$

and ξ_+ as

$$\xi_+ := \inf_{0\le\varepsilon\le\varepsilon_1} \{\xi : u_\varepsilon(\xi) = u_{i+1} - \delta\}$$

and a comparison argument we can make sure that the assumptions (i) and (ii) of lemma 2.8 are met for some constant $C > 0$. In other words, (i) and (ii) are simple consequences of the uniform hyperbolicity of the equilibria u_i and u_{i+1}. It remains to show that

$$\int_a^b |u_\varepsilon(\xi) - u_0(\xi)|\, d\xi \to 0 \quad \text{for all } -\infty < a < b < +\infty.$$

For $a \le \xi \le b$ we have

$$\begin{aligned} |u_\varepsilon(\xi) - u_0(\xi)| &\le \left| \int_0^\xi u_\varepsilon'(\eta) - u_0'(\eta)\, d\eta \right| \\ &\le \int_0^\xi |v_1(u_\varepsilon(\eta)) - v_1(u_0(\eta))| + k\varepsilon |g(u_\varepsilon(\eta))|\, d\eta \\ &\le \int_0^\xi \left(L|u_\varepsilon(\eta) - u_0(\eta)| + \varepsilon k \sup|g(u)|\right)\, d\eta \end{aligned}$$

where L is a Lipschitz constant for the function v_1 from (8) on the interval $[u_i, u_{i+1}]$ and the sup is taken over the same interval. In particular, this estimate is independent of a and b. Applying the Gronwall inequality we get

$$|u_\varepsilon(\xi) - u_0(\xi)| \le \varepsilon \frac{k \sup|g|}{L} \left(e^{L|\xi|} - 1\right)$$

for $\xi \in [a, b]$. Hence

$$\int_a^b |u_0(\xi) - u_\varepsilon(\xi)|\, d\xi \le \int_a^b \varepsilon \frac{k \sup|g|}{L} \left(e^{L|\xi|} - 1\right)\, d\xi \to 0$$

as $\varepsilon \searrow 0$. Consequently, assumption (iii) of lemma 2.8 been checked and as a consequence of this lemma, the hyperbolic wave of type (A1)(i) admits a viscous profile.

II. $s_0 = f'(u_{i+1})$: This limiting case has to be treated seperately because the traveling wave u_0 of the hyperbolic equation is only continuous but not C^1. Fixing a parametrization we have $u_0(\xi) \equiv u_{i+1}$ for $\xi \ge 0$ while for $\xi \le 0$ u_0 solves the differential equation

$$u_0'(\xi) = \begin{cases} \dfrac{g(u_0)}{f'(u_0) - s_0} & \text{for } u_0 \ne u_{i+1} \\[2ex] \dfrac{g'(u_{i+1})}{f''(u_{i+1})} & \text{for } u_0 = u_{i+1} \end{cases}$$

with $u_0(0) = u_{i+1}$. We approximate s_0 by a sequence s_n with $s_n \searrow s_0$ such that the corresponding traveling waves $u_0^{(n)}$ satisfy

$$\|u_0^{(n)} - u_0\|_{L^1} \le \frac{1}{2n}. \tag{10}$$

Since for each s_n the inequality of case I is satisfied, there exists ε_n with $\varepsilon_n \searrow 0$ such that the corresponding heteroclinic orbit u_{ε_n} from u_i to u_{i+1} with speed s_n satisfies

$$\|u_{\varepsilon_n} - u_0^{(n)}\|_{L^1} \leq \frac{1}{2n}.$$

Putting this estimate together with (10) shows that the heteroclinic wave u_0 admits a viscous profile.
We still have to show that (10) can be satisfied by an appropriate sequence $(u_0^{(n)})$. Note that in this step of the proof only traveling waves of the hyperbolic equations are involved. For δ sufficiently small we find $\underline{\xi} = \bar{\xi}(\delta) < 0$ such that $u_0(\bar{\xi}) = u_{i+1} - \delta$ and

$$\|u_0 - u_{i+1}\|_{L^1([\bar{\xi},+\infty))} = \int_{\bar{\xi}}^{\infty} |u_0(\xi) - u_{i+1}| \, d\xi \leq \frac{1}{10n}.$$

For $s - s_0$ small, let u_0^s be the solution of (1) with wave speed s and $u_0^s(\bar{\xi}) = u_{i+1} - \delta$. Since the linearization of (1) at $u = u_{i+1}$ tends to $-\infty$ as s approaches s_0, by choosing δ small enough, we can achieve that

$$\|u_0^s - u_0\|_{L^1([\bar{\xi},+\infty))} \leq \frac{1}{10n} \tag{11}$$

for all $|s - s_0|$ sufficiently small. Since for any s the heteroclinic orbits $u_0^s(\xi)$ converge to u_i exponentially as $\xi \to -\infty$ we find some $\underline{\xi}$ such that

$$\|u_0^s - u_0\|_{L^1((-\infty,\underline{\xi}])} \leq \frac{1}{10n} \tag{12}$$

for all wave speeds $s \geq s_0$ which are sufficiently close to s_0. On the intermediate part $[\underline{\xi}, \bar{\xi}]$ the vector fields for the wave speeds s and s_0 are $\mathcal{O}(|s - s_0|)$-close, hence by choosing $|s - s_0|$ small enough one can achieve

$$\|u_0^s - u_0\|_{L^1([\underline{\xi},\bar{\xi}])} \leq \frac{1}{10n}. \tag{13}$$

So by the choice $s_n := \sup\{\bar{s} \geq s_0;\ (11), (12), (13) \text{ hold for all } s_0 \leq s \leq \bar{s}\}$ we can satisfy (10).

$\square$

4 More discontinuous waves

This chapter is devoted to the waves of type (A2). We distinguish two cases depending on type of the equilibria involved. In the "Lax"-like situation the Morse indices differ by one, while the waves of type (A2)(ii) connect equilibria whose Morse indices differ by two. This is analogous to the case of overcompressive shock waves of hyperbolic conservation laws.

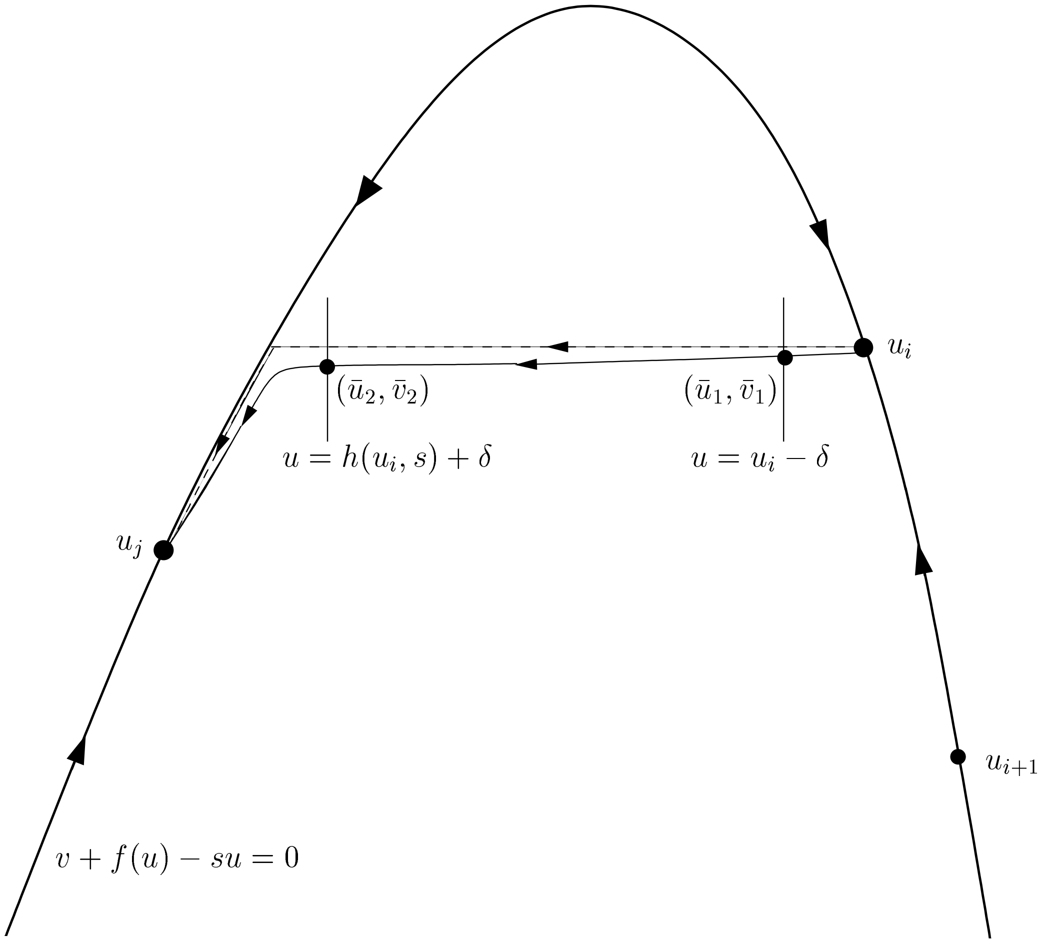

Figure 1: A "Lax" heteroclinic traveling wave (dashed) and its viscous counterpart

4.1 The "Lax" case

The heteroclinic waves of type (A2)(i) and (A2)(iii) are related via the symmetry $\xi \mapsto -\xi$. For this reason, we treat only waves of type (A2)(i), see figure 1.

We claim that for ε small and the correct value s a branch of the unstable manifold of u_i is a heteroclinic orbit from u_i to u_j and that these heteroclinic orbits provide a viscous profile. The existence of the heteroclinic orbit is shown as follows: The unstable manifold is C^1-close to the stable eigenspace, so the manifold leaves a small neighborhood of u_i with at the point $(\bar{u}_1, \bar{v}_1)$ with $\bar{u}_1 = u_i - \delta$ and $\bar{v}_1 = -f(u_i) + su_i + \mathcal{O}(\varepsilon)$, see figure 1. Outside a neighborhood of the singular curve $\mathcal{C}_s$ the vector field (3) is of order $\mathcal{O}(\varepsilon^{-1})$, so following the unstable manifold, a neighborhood of the other branch of $\mathcal{C}_s$ is reached at $(\bar{u}_2, \bar{v}_2)$ with $\bar{u}_2 = h(u_i, s) + \delta$ and $\bar{v}_2 = -f(u_i) + su_i + \mathcal{O}(\varepsilon)$. Near the singular curve the vector field can be transformed to a normal form due to Takens [8]. By calculations analogous to those in [3] it can be shown that it takes a "time" ξ of order $\mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon})$ until the trajectory reaches the invariant region of width $k\varepsilon^2$ that was already used in the proof of lemma 3.1. Moreover, it enters this region at a v-value $-f(u_i) + su_i + \mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon})$. The same asymptotics have been obtained by Mishchenko and Rozov [6]. After the manifold has entered the positively invariant invariant region near $\mathcal{C}_s$ it has to remain there forever and by the Poincaré-Bendixson theorem the branch of the unstable manifold

must converge to u_j.
We still have to show that the heteroclinic orbits u_ε yield a viscous profile for the heteroclinic wave of the hyperbolic equation. To this end, after chosing some $n \in \mathbb{N}$, we must find $\varepsilon_n > 0$ small such that $\|u_0 - u_\varepsilon\| \leq 1/n$, where u_ε is a parametrization of the heteroclinic orbit of (3) at $\varepsilon = \varepsilon_n$. We parametrize $u_0(\xi)$ and u_ε in the following way:

$$\begin{aligned} u_0(\xi) &= \begin{cases} u_i \text{ for } \xi \leq 0 \\ \leq h(u_i, s_0) \text{ for } \xi > 0 \end{cases} \\ u_0(0) &= u_i - \delta \end{aligned}$$

where we choose δ later. In any case we know from the exponential decay of u_ε and u_0 that conditions (i) and (ii) of lemma 2.8 are met.

4.2 The "overcompressive" case

Similarly as for overcompressive shocks of conservation laws, for a fixed wave speed s_0 we have a whole one-parameter-family of heteroclinic waves with a shock at $\xi = 0$, where the jump values $u(\xi+)$ plays the role of a parameter. To find heteroclinic waves of the parabolic equation (P) which provide a viscous profile for such a heteroclinic wave, we define $(u_\varepsilon, v_\varepsilon)$ as the solution of (3) with

$$u_\varepsilon(0) = \frac{u_i + u_j}{2}$$

and

$$v_\varepsilon(0) = -f(u(\xi+)) + su(\xi+) = -f(u(\xi-)) + su(\xi-)$$

where $u(\xi+)$, $u(\xi-)$ are the one-sided limits of the hyperbolic wave at the shock.

5 Undercompressive Shocks

In this chapter we consider the simple shock waves of type (C) which are of the form

$$u(x,t) = \begin{cases} u_j \text{ for } \ x - st < 0 \\ u_i \text{ for } x - st > 0 \end{cases}$$

with shock speed $s_0 = \frac{f(u_i)-f(u_j)}{u_i-u_j}$. Here the source term is only involved by the fact that shocks can connect only equilibria of the reaction dynamics. Since both equilibria are of saddle-type here, we call this shock undercompressive. In the traveling wave setting this correspond to an entropy solution

$$u(\xi) = \begin{cases} u_j \text{ for } \ \xi < 0 \\ u_i \text{ for } \xi > 0. \end{cases}$$

To show that the heteroclinic waves of type (C) admit a viscous profile, we consider the unstable manifold of u_i and the stable manifold of u_j. For $s < \frac{f(u_i)-f(u_j)}{u_i-u_j}$ and ε sufficiently small the unstable manifold of u_i passes below the stable manifold of u_j in the u-v-plane.
This implies that there exist a wave speed $s = s(\varepsilon)$ such that $W^u(u_i) \cap W^s(u_j) \neq \emptyset$. Since this intersection is one-dimensional, it must be a heteroclinic orbit u_ε. Also since for any fixed $s \neq s_0$ and ε small enough the unstable manifold of u_j and the stable manifold of u_i miss each other, the limiting relation

$$\lim_{\varepsilon \searrow 0} s(\varepsilon) = s_0$$

holds. To prove that the family of these heteroclinic orbits yields a viscous profile for the shock wave, we use again lemma 2.8 and parametrize the heteroclinic orbits in such a way that

$$u_\varepsilon(0) = \frac{u_i + u_j}{2}.$$

At both ends the convergence to the equilibria is exponentially fast with a rate of order $\mathcal{O}(c/\varepsilon)$, hence the assumptions (i) and (ii) of lemma 2.8 are met. To check assumption (iii) for some interval $a < 0 < b$, we fix some neighborhoods of u_j and u_i where the convergence is exponential. Outside this neighborhood $u' = \mathcal{O}(\varepsilon^{-1})$, hence there exist $\underline{(\xi)} < 0 < \bar{xi}$, both of order $\mathcal{O}(\varepsilon)$, such that for $\xi < \underline{\xi}$

$$u_\varepsilon(\xi) - u_j \leq Ce^{-c|\xi|/\varepsilon} \text{ for } \ \xi < \underline{\xi}$$

and

$$u_i - u_\varepsilon(\xi) \leq Ce^{-c\xi/\varepsilon} \text{ for } \ \xi > \bar{\xi}.$$

To prove assumption (iii) of lemma 2.8 we need to calculate

$$\begin{aligned}
& \int_a^0 |u_j - u_\varepsilon(\xi)| \, d\xi + \int_0^b |u_i - u_\varepsilon(\xi)| \, d\xi \\
= \; & \int_a^{\underline{\xi}} |u_j - u_\varepsilon(\xi)| \, d\xi + \int_{\underline{\xi}}^0 |u_j - u_\varepsilon(\xi)| \, d\xi \\
& + \int_0^{\bar{\xi}} |u_i - u_\varepsilon(\xi)| \, d\xi + \int_{\bar{\xi}}^b |u_i - u_\varepsilon(\xi)| \, d\xi \\
= \; & \mathcal{O}(\varepsilon)
\end{aligned}$$

using our estimates on the exponential decay and the size of $\underline{\xi}$ and $\bar{\xi}$. Note that we could similarly prove convergence even in the weighted spaces L^1_β with arbitrary large exponential weight β.

6 Discussion

We have in this paper shown that using methods of classical singular perturbation theory can be used to show that several heteroclinic waves of scalar balance laws admit a viscous profile. However, not all heteroclinic waves admit a viscous profile: There are discontinuous waves with more than one discontinuity, which can be shown not to possess a viscous profile by a simple application of the Jordan curve theorem. These and all other remaining types of heteroclinic waves will be treated in a forthcoming paper [**?**].

There are many obvious generalizations. For instance, the question of existence and viscous admissibility of heteroclinic traveling waves can be asked for systems of balance laws, too. While the existence part seems to be quite straightforward, the existence of viscous profiles will lead to singularly perturbed equations with many fast and many slow variables.

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