# Heteroclinic orbits between rotating waves in hyperbolic balance laws

Jörg Härterich
Freie Universität Berlin
Arnimallee 2-6
D-14195 Berlin
Germany
email: haerter@math.fu-berlin.de

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**Abstract**

We deal with the large-time behaviour of scalar hyperbolic conservation laws with source terms

$$u_t + f(u)_x = g(u),$$

which are often called hyperbolic balance laws. Fan and Hale have proved existence of a global attractor $\mathcal{A}_0$ for this equation with $x \in S^1$. $\mathcal{A}_0$ consists of spatially homogenous equilibria, a large number of rotating waves and of heteroclinic orbits between these objects. In this paper, we solve the connection problem and show which equilibria and rotating waves are connected by a heteroclinic orbit. Apart from existence results, our approach via generalized characteristics gives also geometric information about the heteroclinic solutions, e.g. about the shock curves and their strength.

## 1 Introduction

Global attractors have been proved to exist for many different p.d.e.'s. However, only in some cases more than bare existence, regularity and estimates for the Hausdorff dimension are known. One of the exceptions is the class of scalar hyperbolic balance laws

$$u_t + f(u)_x = g(u), \quad u \in \mathbb{R}, \ x \in S^1 \cong \mathbb{R}/\mathbb{Z} \tag{1}$$

with a convex flux $f$. We will consider this equation together with the initial condition

$$u(x, t_0) = u_0(x) \in BV(S^1) \tag{2}$$

As for conservation laws there are no global classical solutions of (1), even for smooth initial data. On the other hand, weak solutions are not unique. For this reason, an entropy condition is imposed that selects one of the possible weak solutions.

**Definition 1.1** *We call $u \in L^\infty([t_0, \infty) \times \mathbb{R}_+)$ a* solution *of (1),(2) if it satisfies the equation in a weak sense, if $u(\cdot, t) \in BV(\mathbb{R})$ for every $t > t_0$ and if for $t > t_0$ the one-sided limits $u(x+, t)$, $u(x-, t)$ satisfy the entropy condition*

$$u(x+, t) \le u(x-, t).$$

We will also assume that the solutions are continuous from the left with respect to $x$. On the functions $f$ and $g$ we impose the following conditions:

$$f \in C^2, \ f' \text{ is strictly increasing} \tag{F}$$

$$u \cdot g(u) < 0 \text{ for all } |u| \text{ large (dissipativeness)} \tag{D}$$

$$g \in C^1, \ g \text{ possesses only simple zeroes } \ u_1, u_2, \dots u_{2n+1} \tag{G}$$

Krushkov [7] has proved the following important existence result:

**Proposition 1.2 ([7])** *Assume (F),(D) and (G). Then (1),(2) has a unique solution.*

Hence, (1) defines a flow on $BV(S^1)$ which we denote with $\Phi_t(u_0)$. Fan and Hale [4] proved the existence of a global attractor for (1) in $L^p(S^1) \cap BV(S^1)$ with $1 \le p < \infty$:

**Proposition 1.3 ([4], Theorem 3.6)** *Assume again (F),(D) and (G). Then the set*

$$\mathcal{A}_0 := \{u_0 \in BV(\mathbb{R}) : \Phi_t(u_0) \textit{ is defined and bounded for all } \ t \in \mathbb{R}\}$$

*is the global attractor of (1) in $L^p(S^1)$, i.e. it is invariant and attracts bounded sets in $L^p(S^1)$.*

In other words, the global attractor consists exactly of all global bounded solutions. By a global solution we mean a solution $U(x,t)$ of (1) such that for $-\infty < t_0 \leq t_1 < \infty$, $U(x,t_1)$ is the solution at time $t = t_1$ corresponding to the initial condition $U(x,t_0)$ at $t = t_0$.
The aim of this article is to give a closer description of the set $\mathcal{A}_0$ and the dynamics on it. For a single trajectory several authors have shown a type of Poincaré-Bendixson theorem:

**Proposition 1.4 ([3, 10, 13])** *As $t \to \infty$, any solution tends either to a homogenous steady state $u \equiv u_i$ or to a rotating wave, i.e. a solution of the form*

$$u(x,t) = \Psi(x - st)$$

*where the velocity $s$ can only take one of the values $f'(u_{2i})$, $1 \leq i \leq n$. For any globally defined bounded solution the same holds as $t \to -\infty$.*

Hence the global attractor $\mathcal{A}_0$ consists of homogenous steady states, rotating waves and heteroclinic orbits between these objects. Here we call a global solution $h(x,t)$ a *heteroclinic orbit* that *connects* the state $\Psi_{-\infty}(x,t)$ to the (not necessarily different) state $\Psi_{+\infty}(x,t)$ if $h$ tends to $\Psi_{\pm\infty}$ as the time t tends to $\pm\infty$.
The large-time behaviour is thus quite different from the situation of hyperbolic conservation laws when the source term $g$ is not present. In that case any solution decays to a spatially homogenous state [8, 9].
There are exactly $2n+1$ homogenous steady states $u \equiv u_i$. Note that exactly $n+1$ of them with $i$ odd are asymptotically stable. This follows easily from the monotonicity property stated below in proposition 2.2. We will call them the *stable steady states* compared to the unstable ones $u \equiv u_{2i}$.
In contrast, there are infinitely many (in fact, uncountably many) rotating waves oscillating around each of the values $u_{2i}$. Sinestrari proved in [12] that for any possible wave speed $f'(u_{2i})$ and any closed set $\mathcal{Z} \subseteq S^1$ there exists a unique rotating wave solution $u(x,t) = u_{\mathcal{Z}}(x - f'(u_{2i})t)$ with the property that

$$\{\xi \in S^1; u_{\mathcal{Z}}(\xi) = u_{2i}\} = \mathcal{Z},$$

and all rotating wave solutions of (1) are of this form. We will describe these rotating waves in more detail below.
Note that we will sometimes consider the homogenous states $u \equiv u_{2i}$ as a rotating wave with $\mathcal{Z} = S^1$. An important question is now, which steady states or rotating waves are connected to which ones by a heteroclinic orbit and how this connecting orbit looks like. Fan and Hale could show the following partial results:

**Proposition 1.5 ([4], Theorem 3.7)**

*(i) If a heteroclinic orbit connects two rotating waves then they have the same velocity $s$.*

*(ii) If a heteroclinic orbit connects a rotating wave with speed $f'(u_{2i})$ to a steady state $u \equiv u_j$ then $|2i - j| = 1$.*

In fact, for any global solution $u(x,t)$ we have the inclusions

$$u_{2i-1} \leq u(x,t) \leq u_{2i+1}$$

for some $i$ and almost every $(x,t)$. This implies that we can seperately look for heteroclinic orbits from or to rotating waves with given wave speed $f'(u_{2i})$. The global attractor is then obtained by gluing the parts associated with the wave speeds $f'(u_{2i})$ and $f'(u_{2i+2})$ at the homogenous equilibrium $u \equiv u_{2i+1}$. For this reason, we will from now on assume that $g$ possesses exactly three zeroes $u_1, u_2$ and $u_3$. Moreover, we will also assume $f'(u_2) = 0$ since this situation can always be achieved if $x$ is replaced by $x - f'(u_2)t$. Hence, all rotating waves have zero wave speed and are therefore equilibrium solutions of the new equation. Nevertheless, we will still call them rotating waves to distinguish them from the spatially homogenous equilibrium solutions. An important role is played by the set

$$\mathcal{Z}(u(\cdot,t)) := \{x \in S^1; u(x,t) = u_2\} \tag{3}$$

This set is a kind of Lyapunov functional :

**Proposition 1.6 ([13])** *For any solution $u$ and $t_1 > t_2$*

$$\mathcal{Z}(u(\cdot,t_1)) \subseteq \mathcal{Z}(u(\cdot,t_2)).$$

This implies:

**Proposition 1.7 ([4])** *If a heteroclinic orbit connects the rotating waves $\Psi_{-\infty}$ and $\Psi_{+\infty}$, then*

$$\mathcal{Z}(\Psi_{+\infty}) \subseteq \mathcal{Z}(\Psi_{-\infty}). \tag{4}$$

In this paper we want to show that (4) is not only necessary but also sufficient. We split this statement into three different cases that will be treated seperately.

**Theorem A** *For any rotating wave $\Psi_{-\infty}$ there exist heteroclinic orbits which connect $\Psi_{-\infty}$ to the homogenous states $u \equiv u_1$ and $u \equiv u_3$.*

**Theorem B** *For any rotating wave $\Psi_{+\infty}$ there exist (many) heteroclinic orbits that connect the spatially homogenous solution $u \equiv u_2$ to $\Psi_{+\infty}$.*

**Theorem C** *Suppose that for two rotating waves $\Psi_{-\infty}$ and $\Psi_{+\infty}$ condition (4) holds. Then there is a heteroclinic solution that approaches $\Psi_{\pm\infty}$ as the time $t$ tends to $\pm\infty$.*

After introducing some tools in the next chapter, these three statements will be proved in chapter 3. A concrete example with a model equation and some illustrations is given in chapter 4 before we conclude with a short discussion.

# 2 Preliminaries

## 2.1 Strong monotonicity

Already Kruzhkov in the early seventies noticed that the semiflow associated with (1) preserves the order of solutions.

**Proposition 2.1 ([7])** *If $u$, $v$ are solutions of (1), (2) with initial data $u_0$, $v_0$ and if*

$$u_0(x) \leq v_0(x) \text{ a.e.}$$

*then $u(x,t) \leq v(x,t)$ a.e. for all $t > t_0$.*

In fact, a stronger version holds in our case.

**Lemma 2.2** *If*

$$u_0(x) \leq v_0(x) \qquad \forall x \in S^1$$

*then $u(x,t) \leq v(x,t)$ for all $t > t_0$ and all $x \in S^1$.*

**Proof:** By the entropy condition and the left-continuity of $u$, one can conclude that

$$\mathcal{R}(u(\cdot,t)) := \{u(x,t);\ x \in S^1\}$$

is an interval.
Assume now that $u(x_0,t_0) < v(x_0,t_0)$ for some $(x_0,t_0)$. Then there exists $\delta > 0$ such that $u(x_0,t_0) + \delta \notin \mathcal{R}(u(\cdot,t_0))$. Again by the left-continuity of $u$ the set where $u$ is strictly less than $v$ has positive measure in contradiction to Kruzhkovs monotonicity statement.

$\square$

## 2.2 Generalized characteristics

A main tool for our construction of connecting orbits are generalized and backward characteristics, which were introduced by Dafermos [2].

**Definition 2.3** *A Lipschitz continuous curve* $x = \zeta(t)$ *defined on some interval* $[t_1, t_2]$ *is called a* generalized characteristic *if for almost all* $t \in [t_1, t_2]$ *the differential inclusion*

$$\zeta'(t) \in [f'(u(\zeta(t)-, t)), f'(u(\zeta(t)+, t))]$$

*holds. It is called* genuine *if*

$$u(\zeta(t)-, t) = u(\zeta(t)+, t) \text{ for almost all } t \in [t_1, t_2].$$

Generalized characteristics can be used to derive properties of the solution once existence has been proved.

**Proposition 2.4** *For any* $(x, t)$ *where* $u$ *satisfies the entropy condition (in particular, for* $t > t_0$*), there is a unique forward characteristic.*

However, at least equally important are the backward characteristics. Dafermos showed the following proposition which states that the value of $u$ along a backward characteristic satisfies an ordinary differential equation with an appropriate terminal condition.

**Proposition 2.5** *For* $(x_0, t_0) \in S^1 \times \mathbb{R}$ *let* $(v_\pm(\cdot; x_0, t_0), \zeta_\pm(\cdot; x_0, t_0))$ *be the solutions of the characteristic equation*

$$\left.\begin{array}{rcl} v'(t; x_0, t_0) &=& g(v(t; x_0, t_0)) \\ \zeta'(t; x_0, t_0) &=& f'(v(t; x_0, t_0)) \end{array}\right\} \tag{5}$$

*subject to the terminal condition*

$$v_\pm(t_0; x_0, t_0) = u(x_0\pm, t_0)\ ,\ \zeta_\pm(t_0; x_0, t_0) = x_0.$$

*Then*

$$u(\zeta_\pm(t; x_0, t_0), t) = v_\pm(t; x_0, t_0)\ \ \forall t \leq t_0.$$

*and* $u(\cdot, t)$ *is continuous at* $\zeta_\pm(t; x_0, t_0)$*. In particular, if* $u(x_0-, t_0) = u(x_0+, t_0)$*, there is a unique backward characteristic emanating from the point* $(x_0, t_0)$*.*

The curves $x = \zeta_-(t; x_0, t_0)$ and $x = \zeta_+(t; x_0, t_0)$ are called the *minimal* and *maximal backward characteristic* emanating from $(x_0, t_0)$. So the preceding lemma tells that the minimal and maximal backward characteristics are genuine.
We will supress the dependence on $x_0$ and $t_0$ in some places where it should cause no confusion.

**Proposition 2.6** *Genuine characteristics can intersect each other only at their endpoints.*

In particular, given a global solution $u$, two genuine backward characteristics $\zeta_1(t; x_1, t_1)$ and $\zeta_2(t; x_2, t_2)$ can intersect only if $t_1 = t_2$ and if $x_1 = x_2$ since otherwise there would be no unique forward characteristic emanating from the point of intersection.

**Lemma 2.7** *Assume that the point $(x_0, t_0)$ lies on a shock curve. Then*

$$x_0 \notin \mathcal{Z}(u(t, \cdot)) \qquad \forall t > t_0.$$

**Proof:** Assume that there is some $t_1 > t_0$ such that $x_0 \in \mathcal{Z}(u(t_1, \cdot))$. By the definition of $\mathcal{Z}$ and left continuity of $u$ we have $u(x_0, t_1) = u_2$ and hence the minimal backward characteristic is a straight line $\zeta_-(t; x_0, t_1) \equiv x_0$. However this a genuine characteristic which implies that $u$ is continuous along this line. This contradicts our assumption that at $t = t_0$ there is a shock at $x = x_0$.

$\square$

Due to the convexity of $f$ for any $x, y \in S^1$ there is a uniquely determined value $\phi(x - y)$ such that the solution of the characteristic equation (5) with terminal condition

$$\begin{aligned} v(t_0) &= \phi(x - a_j) \\ \zeta(t_0) &= x \end{aligned}$$

satisfies

$$\lim_{t \to -\infty} \zeta(t) = y. \tag{6}$$

Geometrically, $\phi(x - y)$ is the value one has to prescribe at $x$ in order that the backward characteristic converges to $y$.
Note that $\phi$ depends only on the difference $x - y$ and is implicitly given via the relation

$$\int_{u_2}^{\phi(\xi)} \frac{f'(v)}{g(v)} dv = \xi. \tag{7}$$

Hence $\phi$ is continuous and monotone increasing with $\phi(0) = u_2$ and

$$\begin{array}{l} \lim_{\xi \to -\infty} \phi(\xi) = u_1, \\ \lim_{\xi \to +\infty} \phi(\xi) = u_3. \end{array}$$

**Lemma 2.8** *Let $(v, \zeta)$ be a solution of the characteristic equation (5) with*

$$v(t_0) = \phi(\zeta(t_0) - y) \qquad \textit{for some } t_0.$$

*Then*

$$v(t) = \phi(\zeta(t) - y) \qquad \textit{for all } t.$$

*In particular if $\zeta$ is a genuine characteristic for a solution $u$ of the hyperbolic balance law, then*

$$u(\zeta(t), t) = \phi(\zeta(t) - y).$$

**Proof:**

$$\begin{array}{rrcl} & \zeta(t) - \zeta(t_0) & = & \displaystyle\int_{t_0}^{t} f'(v(s))ds \\ & & = & \displaystyle\int_{v(t_0)}^{v(t)} \frac{f'(\nu)}{g(\nu)} d\nu \\ & & = & \displaystyle\int_{u_2}^{v(t)} \frac{f'(\nu)}{g(\nu)} d\nu - \int_{u_2}^{v(t_0)} \frac{f'(\nu)}{g(\nu)} d\nu \\ & & = & \displaystyle\int_{u_2}^{v(t)} \frac{f'(\nu)}{g(\nu)} d\nu - \int_{u_2}^{\phi(\zeta(t_0)-y)} \frac{f'(\nu)}{g(\nu)} d\nu \\ \Rightarrow & \zeta(t) - y & = & \displaystyle\int_{u_2}^{v(t)} \frac{f'(\nu)}{g(\nu)} d\nu \\ \Rightarrow & v(t) & = & \phi(\zeta(t) - y) \end{array}$$

where we have used (7) in the last two lines together with the positivity of the integrand.

$\square$

**Remark:** Geometrically speaking, $(v(t), \zeta(t))$ is the unstable manifold of the stationary solution $(u_2, y)$ of (5).

# 3 The Proofs

## 3.1 Heteroclinic orbits connecting rotating waves to spatially homogenous states

This section contains the proof of theorem A. We are going to construct a global solution $U$ of (1) that, as $t \to -\infty$, approaches a rotating wave $\Psi_{-\infty}$ with some prescribed $\mathcal{Z}_{-\infty} := \mathcal{Z}(\Psi_{-\infty}(x))$ while for $t \to +\infty$ it tends to the homogenous state $u \equiv u_3$. Constructing a heteroclinic orbit from $\Psi_{-\infty}$ to $u \equiv u_1$ is completely analogous.

Since $\mathcal{Z}_{-\infty}$ is a closed set, we can write $S^1 \setminus \mathcal{Z}_{-\infty}$ as a (finite or countable) union of open intervals:

$$S^1 \setminus \mathcal{Z}_{-\infty} = \cup_{j \in \mathcal{J}} (a_j, b_j).$$

Assume now that there is a shock at some position $x \in (a_j, b_j)$ with the additional property that the minimal backward characteristic converges to $a_j$ while the maximal characteristic converges to $b_j$. This is a reasonable assumption since along backward characteristics the value $v$ converges to $u_2$ for $t \to -\infty$, so the characteristic itself should converge to some point of $\mathcal{Z}_{-\infty}$. By (6), the left and right states at such a shock have to be $\phi(x - a_j)$ and $\phi(x - b_j)$. The Rankine-Hugoniot condition requires then that the shock propagates with velocity

$$s(x; a_j, b_j) = \frac{f(\phi(x - a_j)) - f(\phi(x - b_j))}{\phi(x - a_j) - \phi(x - b_j)}. \tag{8}$$

We first show the following

**Lemma 3.1** *The vector field given on $(a_j, b_j)$ by the ordinary differential equation*

$$\dot{x} = s(x; a_j, b_j) \tag{9}$$

*has a unique unstable fixed point $\bar{x}_j \in (a_j, b_j)$.*

**Proof:** By convexity of $f$ and monotonicity og $\phi$, the function $s$ is strictly increasing in $x$. Moreover, at $x = a_j$ and $x = b_j$ we have

$$\begin{aligned} & \phi(b_j - a_j) > u_2 = \phi(0) > \phi(a_j - b_j) \\ \Rightarrow \quad & s(a_j; a_j, b_j) < 0 \text{ and } s(b_j; a_j, b_j) > 0 \end{aligned}$$

Hence $s$ possesses exactly one zero on $(a_j, b_j)$.

□

**Lemma 3.2** *The rotating wave $\Psi_{-\infty}$ satisfies*

$$\Psi_{-\infty}(x) := \begin{cases} \phi(x-a_j), & a_j < x \le \bar{x}_j \\ \phi(x-b_j), & \bar{x}_j < x < b_j \\ u_2, & x \in \mathcal{Z}_{-\infty} \end{cases}$$

**Proof:** All shocks of $\Psi_{-\infty}$ have to be stationary. Moreover, since the minimal and maximal backward characteristics emanating from a shock at $x = \bar{x}_j$ have to converge to some element of $\mathcal{Z}_{-\infty}$ the left and right state at the shock are $\Psi_{-\infty}(\bar{x}_j, t) \equiv \phi(x - a_j)$ and $\Psi_{-\infty}(\bar{x}_j+, t) \equiv \phi(x - b_j)$. The lemma is then a consequence of lemma 2.8.

□

Now we are able to construct the heteroclinic orbit from $\Psi_{-\infty}$ to $u \equiv u_3$. For any $j \in \mathcal{J}$ we choose $\tilde{x}_j(0) \in (\bar{x}_j, b_j)$ arbitrary. The $\tilde{x}_j(0)$ will be the shock positions at $t = 0$. We continue these shocks backward in time. In order to satisfy the Rankine-Hugoniot condition, let $\tilde{x}(t), t \le 0$, be the backward solution of (9) with terminal condition $\tilde{x}_j(0)$ at $t = 0$. By lemma 3.1

$$\lim_{t \to -\infty} \tilde{x}_j(t) = \bar{x}_j.$$

For $t = 0$ we set

$$U(x,0) := \begin{cases} \phi(x-a_j), & a_j < x \le \tilde{x}(0) \\ \phi(x-b_j), & \tilde{x}(0) < x < b_j \\ u_2, & x \in \mathcal{Z}_{-\infty} \end{cases} \tag{10}$$

For $t > 0$ let $U$ to be the solution of (1) with initial condition $U(x, 0)$ from (10). For $t < 0$ it will be proved that $U$ is already determined by its values on the line $t = 0$ and the one-sided limits along the shock curve $\gamma_j := \{(x, t); x = \tilde{x}_j(t), t \le 0\}$. The key observation is formulated in the next lemma.

**Lemma 3.3** *For any $j \in \mathcal{J}$ the set $(a_j, b_j) \times (-\infty, 0] \setminus \gamma_j$ is the union of genuine backward characteristics.*

**Proof:** Since $\phi$ is continuous, by proposition 2.5 there is a unique genuine backward characteristic emanating from each of the points $(x, 0)$ with $a_j < x < b_j$ and $x \ne \tilde{x}_j(0)$. In addition, from each point on the shock curve $\gamma_j$ there is a minimal and a maximal backward characteristic which are both genuine. These backward characteristics solve the same ordinary differential equation with a terminal conditions that depend continuously on the end point. Hence, there are no gaps between these characteristic curves. More precisely, the region $\{(x,t) - a_j < x < \tilde{x}_j(0),\ t < 0\}$

is filled with genuine backward characteristics. In fact, these characteristics are all translates of each other since the terminal condition does only depend on $x$ but not on $t$. Similarly, the region $\{(x,t); \tilde{x}_j(0) < x < b_j,\ t < 0\}$ is filled with backward characteristics.

□

Therefore we can define $U$ via the solution of (5) along these backward characteristics. In fact, by proposition 2.8 we get for $t < 0$

$$U(x,t) := \begin{cases} \phi(x-a_j), & a_j < x \leq \tilde{x}(t) \\ \phi(x-b_j), & \tilde{x}(t) < x < b_j \\ u_2, & x \in \mathcal{Z}_{-\infty} \end{cases}$$

which implies immediately convergence of $U$ to $\Psi_{-\infty}$ in $L^p$, $1 \leq p < \infty$ as $t \to -\infty$ because the $\tilde{x}_j$ converge to $\bar{x}_j$.

Vice versa, if we denote with $V$ the solution of the hyperbolic balance law (1) with initial condition $V(x,t_-) = U(x,t_-)$ for some arbitrary but fixed $t_- < 0$ then solving (1) by the method of characteristics it is immediately clear that at any time $t_- < t \leq 0$ the equality $V(x,t) = U(x,t)$ holds. For $t > 0$ nothing has to be proved since for $t$ positive $U$ is a solution by construction.

The only thing that remains to be shown for theorem A is the fact that $U$ converges to $u_3$ as the time $t$ tends to $+\infty$. We will first consider another solution $W$ whose behaviour is easier to analyze than that of $U$. The advantage of $W$ over $U$ is the fact that there is only one moving shock while all other shocks remain stationary. It will turn out that the moving shock "devours" all other shocks in finite time.

**Lemma 3.4** *Let $W$ be the solution of (1) with initial condition $W(x,0)$ given by*

$$W(x,0) := \begin{cases} \Psi_{-\infty}(x), & x \notin (a_1,b_1) \\ U(x,0), & x \in (a_1,b_1) \end{cases} \tag{11}$$

*where the interval $(a_1,b_1)$ is chosen with maximal length:*

$$b_1 - a_1 \geq b_j - a_j \qquad \forall j \in \mathcal{J}.$$

*Then there exists some time $T > 0$ and $\delta > 0$ such that for all $x \in S^1$ we have*

$$u_2 + \delta < W(x,T) < u_3.$$

**Proof:** By lemma 2.8 we find that along any forward characteristic $\zeta(t)$ either $W(\zeta(t),t) = \phi(\zeta(t)-a_j)$ or $W(\zeta(t),t) = \phi(\zeta(t)-b_j)$ or $W(\zeta(t),t) = u_2$ as long as

the characteristic does not hit a shock curve. Note that the choice of $W$ prevents forward characteristics from crossing each other. So we can follow the forward characteristics up to a collision with a shock curve and find thereby the left and right states at the shock.
We consider the (unique) genuine forward characteristic $\zeta(t)$ emanating from $t = 0, x = \tilde{x}_1(0)$. Note that uniqueness is guaranteed by the fact that $W(x,0)$ satisfies the entropy condition.
This generalized characteristic $\zeta(t)$ is a shock curve. As long as $a_1 \in \mathcal{Z}(W(t,\cdot))$, i.e. as long as no shock enters the interval $(a_1, b_1)$ from the left, the left state along $\zeta$ is $W(\zeta(t)-, t) = \phi(\zeta(t) - a_1)$. This is again a consequence of lemma 2.8 since the minimal backward characteristic $\zeta_-(\zeta(t), t)$ hits the line $t = 0$ somewhere between $a_1$ and $\tilde{x}(0)$ where $W(x, 0)$ takes the value $\phi(x - a_1)$.
The right state however depends in a more involved way on the location $\zeta(t)$: For $\zeta(t) \in (a_j, \bar{x}_j]$ the right state is

$$W(\zeta(t)+, t) = \phi(\zeta(t) - a_j)$$

since in this case a characteristic hits the shock which for $t \to -\infty$ tends to $a_j$. Hence the resulting shock speed is

$$s = \frac{f(\phi(\zeta(t) - a_1)) - f(\phi(\zeta(t) - a_j))}{\phi(\zeta(t) - a_1) - \phi(\zeta(t) - a_j)}.$$

We want to show that this shock speed is bounded from below by some constant $s_0 > 0$. Since both $\phi(\zeta(t) - a_1)$ and $\phi(\zeta(t) - a_j)$ are bigger than $u_2$, by convexity of $f$ we have the lower bound

$$s > s_0 := f'(\phi(\zeta(t) - a_1)) > 0.$$

Similarly, for $\zeta(t) \in (\bar{x}_j, b_j)$ we have as the right state at the shock

$$W(t, \zeta(t)+) = \phi(\zeta(t) - b_j)$$

with a resulting shock speed

$$\begin{aligned} s &= \frac{f(\phi(\zeta(t) - a_1)) - f(\phi(\zeta(t) - b_j))}{\phi(\zeta(t) - a_1) - \phi(\zeta(t) - b_j)} \\ &\geq \frac{f(\phi(\zeta(0) - a_1)) - f(\phi(\bar{x}_j - b_j))}{\phi(\zeta(0) - a_1) - \phi(\bar{x}_j - b_j)} \\ &\geq \frac{f(\phi(\zeta(0) - a_1)) - f(\phi(\bar{x}_1 - b_1))}{\phi(\zeta(0) - a_1) - \phi(\bar{x}_1 - b_1)} > 0 \end{aligned}$$

again by convexity of $f$.
Finally, for $\zeta(t) \in \mathcal{Z}(W(t,\cdot))$ the right state is simply

$$W(t, \zeta(t)+) = u_2$$

and the shock speed is

$$s = \frac{f(\phi(\zeta(t) - a_1))}{\phi(\zeta(t) - a_1) - u_2} \geq \frac{f(\phi(b_1 - a_1))}{\phi(b_1 - a_1) - u_2} > 0.$$

In all possible cases we have therefore a $j$-independent, positive lower bound on the shock speed $s$. This implies that after some finite time $T_0$ the shock curve $\zeta(t)$ reaches the position $a_1 + 1$. For $t > T_0$ by lemma 2.7 the set $\mathcal{Z}(W(t,\cdot))$ is empty. Since the range $\mathcal{R}(W(\cdot,t)$ is an interval, we must either have $W(x,t) < u_2$ or $W(x,t) > u_2$ for all $x \in S^1$ and $t > T_0$. It is easy to see that the latter must happen since the left state $W(t,\zeta(t)-) = \phi(\zeta(t) - a_1) > u_2$.
Fix $T > T_0$, then there exists some $\delta > 0$ such that $W(x,T) > u_2 + \delta$. Otherwise there would exist some $x_0 \in S^1$ with $W(x_0+,T) = u_2$ because the range of $W(\cdot,t_0)$ is an interval. In that case $W$ would take the value $u_2$ along the maximal backward characteristic which contradicts $\mathcal{Z}(W(t,\cdot)) = \emptyset$ for $T_0 < t < T$. This completes the proof of the lemma.

□

**Lemma 3.5** *$U(x,t)$ converges to $u_3$ in $L^\infty$ as $t \to \infty$.*

**Proof:** Obviously, $u_3 > U(x,0) \geq W(x,0)$ since $\phi(x - a_j) > u_2 > \phi(x - b_j)$ for all $x \in (a_j, b_j)$ by (7), and since the shock positions $\tilde{x}_j(0)$ of $U(x,0)$ have been chosen to lie to the right of the stationary shock positions $\bar{x}_j$ of $\Psi_{-\infty}$. The strong monotonicity statement from proposition 2.2 implies $u_3 > U(x,t) \geq W(x,t)\ \forall t > 0$, in particular $u_3 > U(x,T) > u_2 + \delta$. Using again monotonicity to compare $U$ and the solution $\bar{W}$ with initial data $\bar{W}(x,T) = u_2 + \delta$ we get uniform convergence to $u_3$ since $\bar{W}(x,t)$ remains spatially homogenous and solves the ordinary differential equation $\frac{d}{dt}\bar{W}(x,t) = g(\bar{W}(x,t))$ .

□

This completes the proof of theorem A.

## 3.2 Heteroclinic orbits connecting a homogenous state to a rotating wave

In this section we will prove theorem B and show that there is an abundance of orbits connecting the homogenous state $u \equiv u_2$ to any given rotating wave $\Psi_{+\infty}$.

Similar as above we define $\mathcal{Z}_{+\infty} := \mathcal{Z}(\Psi_{+\infty})$ and decompose the complement of $\mathcal{Z}_{+\infty}$ as an at most countable union of open intervals $\mathcal{Z}_{+\infty} := \cup_{j\in\mathcal{J}}(a_j, b_j)$.
We present three different ways of constructing heteroclinic connections $U$, one which keeps the number of shocks fixed for all $t \in \mathbb{R}$, another where for $t \to -\infty$ the number of shocks grows to infinity while their strength decreases and a third method where each shock is generated by a centered compression wave.

**First method:**
In each interval $(a_j, b_j)$ let $\bar{x}_j := \bar{x}_j(a_j, b_j)$ as before be again the position of the stationary shock from lemma 3.1. To keep the shock at this location for all $t \in \mathbb{R}$ we will choose the left and right states in such a way that they satisfy the Rankine-Hugoniot condition for a stationary shock, i.e.

$$f(U(\bar{x}_j-, t)) = f(U(\bar{x}_j+, t)). \tag{12}$$

To get convergence to a homogenous state for $t \to -\infty$ we require

$$\lim_{t\to-\infty} |U(\bar{x}_j-, t) - U(\bar{x}_j+, t)| = 0.$$

The two conditions together imply that both $U(\bar{x}_j-, t)$ and $U(\bar{x}_j+, t)$ must approach $u_2$ as $t$ tends to $-\infty$. For this reason we choose $U(\bar{x}_j-, t)$ with the following to properties:

(i) $U(\bar{x}_j-, t)$ is continuous with respect to $t$

(ii) $U(\bar{x}_j-, t)$ is monotone increasing with $\lim_{t\to-\infty} U(\bar{x}_j-, t) = u_2$ and $\lim_{t\to+\infty} U(\bar{x}_j-, t) = \phi(\bar{x}_j - a_j)$.

Since f is convex, the state $U(\bar{x}_j+, t)$ on the right side of the shock is uniquely determined by the Rankine-Hugoniot condition (12) and depends continuously on $t$. Our claim is now that a global solution is defined by the minimal and maximal backward characteristics emanating from the shock curve $x \equiv \bar{x}_j$. This however is not hard to see, as again the characteristic curves solve an ordinary differential equation where the terminal condition depends continuously on the end point. Hence there are no gaps between the backward characteristics. The properties (i) and (ii) of $U(\bar{x}_j-, t)$ ensure that the whole region $\{(x, t) - a_j < x < b_j, t < 0\}$ is filled by backwards characteristics.
It remains to prove the convergence part of theorem B. We start with $t \to -\infty$. First we fix some $\varepsilon > 0$. Then there exists a time $t_1 < 0$ such that

$$u_2 - \frac{\varepsilon}{2} < U(\bar{x}_j+, t) \le u_2 \le U(\bar{x}_j-, t) < u_2 + \frac{\varepsilon}{2}$$

for all $t < t_1$. Moreover there is a time $\Delta t$ such that the backward solution of the characteristic equation

$$\dot{v} = g(v)$$

with any terminal condition $v(0) \in [\phi(\bar{x}_j - b_j), \phi(\bar{x}_j - a_j)]$ satisfies

$$u_2 - \frac{\varepsilon}{2} < v(t) < u_2 + \frac{\varepsilon}{2} \tag{13}$$

for all $t < -\Delta t$. By a proper choice of the $U(\bar{x}_j-, t)$ we may also assume that the previous inequalities hold uniformly for $j \in \mathcal{J}_0$ where $\mathcal{J}_0$ is a subset of $\mathcal{J}$ with

$$\sum_{j\in\mathcal{J}_0} (b_j - a_j) < \frac{\varepsilon}{2(u_3 - u_1)}.$$

We claim now that for $t < t_1 - \Delta t$ and $x \notin \cup_{j\in\mathcal{J}_0}(a_j, b_j)$ we have

$$u_2 - \frac{\varepsilon}{2} < U(x,t) < u_2 + \frac{\varepsilon}{2}.$$

To see this consider the forward characteristic from a point $(x_0, t_0)$ with $t_0 < t_1 - \Delta t$. Without restriction we assume $x_0 \in (a_j, \bar{x}_j]$ with $j \notin \mathcal{J}_0$ since for $x_0 \in \mathcal{Z}_{+\infty}$ we would have $u(x_0, t_0) = u_2$ and for $x_0 \in (\bar{x}_j, b_j)$ one can argue analogously. The forward characteristic will hit the shock curve $x \equiv \bar{x}_j$ at some point $(\bar{x}_j, t)$. Either we have $t < t_1$, then $u_2 \le u(x_0, t_0) < u(\bar{x}_j-, t) < u_2 + \frac{\varepsilon}{2}$ by monotonicity of solutions of the characteristic equation, or we have $t > t_1$. In this case $t - t_0 > \Delta t$ and we are done by (13).
Hence for $t < t_1 - \Delta t$

$$\begin{aligned}
||U(x,t) - u_2||_{L^p} &\le ||U(x,t) - u_2||_{L^\infty(S^1)} \\
&= ||U(x,t) - u_2||_{L^\infty(\cup_{j\in\mathcal{J}_0}(a_j,b_j))} + ||U(x,t) - u_2||_{L^\infty(S^1\setminus\cup_{j\in\mathcal{J}_0}(a_j,b_j))} \\
&\le \sum_{j\in\mathcal{J}_0} (b_j - a_j)\cdot \max(u_3 - u_2, u_2 - u_1) + \frac{\varepsilon}{2} \\
&\le \varepsilon.
\end{aligned}$$

Convergence for $t \to +\infty$ to the rotating wave $\Psi_{+\infty}$ follows immediately from the paper of Sinestrari [12] who showed that $u$ converges to a rotating wave with

$$\mathcal{Z} = \cap_{t>0}\mathcal{Z}(u(\cdot, t))$$

if the latter set is nonempty. Alternately, one can prove convergence similar as for $t \to -\infty$ by direct use of the generalized characteristics.

**Second method:**
We describe this method for a single interval $(a_j, b_j)$. For $t > 0$ we fix one stationary shock at $\bar{x}_j$ with left state $\phi(\bar{x}_j - a_j)$ and right state $\phi(\bar{x}_j - b_j)$. At $t = 0$ we split this shock into two shock curves $x = y_{1/4}(t)$ and $x = y_{3/4}(t)$ and prescribe the left and right states of our desired solution $U$ along these curves as

$$\begin{aligned} U(y_{1/4}(t)-, t) &= \phi(y_{1/4}(t) - a_j), \quad U(y_{1/4}(t)+, t) = \phi(y_{1/4}(t) - \bar{x}_j) \\ U(y_{3/4}(t)-, t) &= \phi(y_{3/4}(t) - \bar{x}_j), \quad U(y_{3/4}(t)+, t) = \phi(y_{3/4}(t) - b_j). \end{aligned}$$

The reason for choosing rational indices is the fact that we want to construct a countable number of shock curves via infinitely many bifurcations of shock curves. Let $\hat{x}_{1/4}$ and $\hat{x}_{3/4}$ be the positions of the two shock curves $y_{1/4}$ and $y_{3/4}$ at $t = -1$. At $t = -1$ we split each of the two shock curves again: $y_{1/4}$ into $y_{1/8}$ and $y_{3/8}$ and $y_{3/4}$ into $y_{5/8}$ and $y_{7/8}$. The left and right states along these new shock curves will be given below. We then follow these for shock curves backward to $t = -2$ where each of them is split into two curves again. We continue this process inductively. At $t = -N$ there will be $2^N$ shock curves $y_{(2i-1)/2^{N+1}}, 1 \le i \le 2^N$. We set

$$\hat{x}_{(2i-1)/2^{N+1}} := y_{(2i-1)/2^{N+1}}(-N).$$

Then, at $t = -N$, we split the shock curve $y_{(2i-1)/2^{N+1}}$ into the two curves $y_{(4i-3)/2^{N+2}}$ and $y_{(4i-1)/2^{N+2}}$ with the left and right states of $U$ according to

$$\begin{aligned} U(y_{k/2^{N+2}}(t), t) &:= \phi(y_{k/2^{N+2}}(t) - \hat{x}_{(k-1)/2^{N+2}}), \\ U(y_{k/2^{N+2}}(t)+, t) &:= \phi(y_{k/2^{N+2}}(t) - \hat{x}_{(k+1)/2^{N+2}}) \end{aligned}$$

for any odd number $k$, where $\hat{x}_0 := a_j$, $\hat{x}_{\frac{1}{2}} := x_0(0,1)$ and $\hat{x}_1 := b_j$. The left and right states along the shock curve $y_{k/2^{N+2}}$ imply that

$$\hat{x}_{(k-1)/2^{N+2}} \le y_{k/2^{N+2}}(t) \le \hat{x}_{(k+1)/2^{N+2}}.$$

For $t > -N$ both inequalities are strict and in particular (for $t = -N - 1$)

$$\hat{x}_{(k-1)/2^{N+2}} < \hat{x}_{k/2^{N+2}} < \hat{x}_{(k+1)/2^{N+2}},$$

i.e. the $\hat{x}_{k/2^N}$ are in the same order as their indices. Note that the the half-lines $\{(x,t);\ x = \hat{x}_{(2i-1)/2^{N+1}}, t \le -N\}$ are genuine backward characteristic and hence

$$\hat{x}_{(2i-1)/2^{N+1}} \in \mathcal{Z}(U(\cdot, t)) \ \ \forall t \le -N.$$

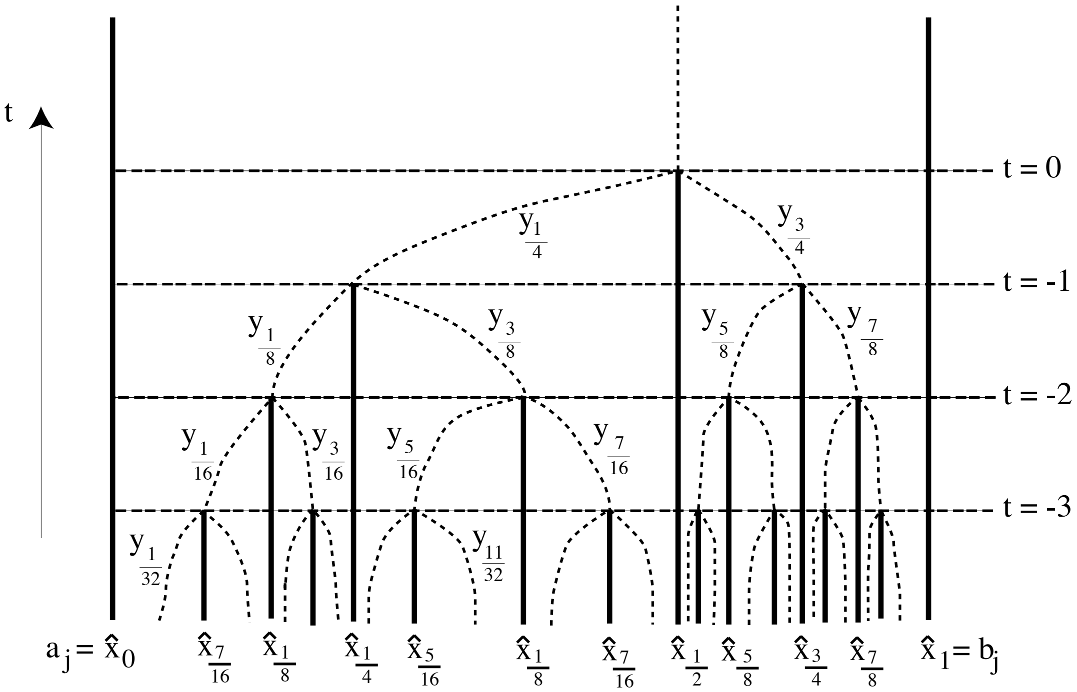


Figure 1: A schematic picture of the shock curves for a heteroclinic connection from a homogenous state to a rotating wave, dotted lines correspond to shock curves, bold lines to $U(x,t) = u_2$

**Lemma 3.6** *The genuine minimal and maximal backward characteristics emanating from the union of all shock curves, determine a global solution $U$.*

**Proof:** Given $(x_0, t_0) \in (a_j, b_j) \times \mathbb{R}$ not on one of the shock curves, we have to show that it lies on one of the genuine backwards characteristics. Let $t_0 \in (-N, -(N+1))$ and $x_0 \in [\hat{x}_{i/2^{N+1}}, \hat{x}_{(i+1)/2^{N+1}})$. In the same interval there is also the shock curve $y_{(2i+1)/2^{N+2}}$. Depending on whether $x_0$ lies to the left or to the right of the shock curve, we set

$$U(x_0, t_0) := \begin{cases} \phi(x_0 - \hat{x}_{i/2^{N+1}}) & \text{if } x_0 \leq y_{(2i+1)/2^{N+2}}(t_0) \\ \phi(x_0 - \hat{x}_{(i+1)/2^{N+1}}) & \text{if } x_0 > y_{(2i+1)/2^{N+2}}(t_0) \end{cases}$$

For definiteness, we assume that $U(x_0, t_0) = \phi(x_0 - \hat{x}_{i/2^{N+1}})$. Consider the forward characteristic $\zeta$ from $(x_0, t_0)$. Along this characteristic, by lemma 2.8, the value of $U$ is $\phi(\zeta(t) - \hat{x}_{i/2^{N+1}})$ up to some point where the characteristic meets a shock curve. However, the left state at any of the shock curves $\zeta$ might hit, is always $\phi(\cdot - \hat{x}_{i/2^{N+1}})$. Then we can argue as in the proof of theorem A: The forward characteristic $\zeta$ is in fact the minimal backward characteristic emanating from some point on a shock curve and hence our definition of $U(x_0, t_0)$ agrees with the value one gets along this minimal backward characteristic.

□

**Lemma 3.7** *The set*

$$Z := \{\hat{x}_{i/2^N}; N \in \mathbb{N}, 0 \le i \le 2^N\}$$

*is dense in* $(a_j, b_j)$.

**Proof:** We only sketch the proof since it involves only arguments we have used before. If the statement is not true then there exists some $x_0 \in (a_j, b_j)$ such that

$$\begin{aligned} \hat{x}_- := \inf\{x \in Z, x > x_0\} &> x_0 \\ \hat{x}_+ := \sup\{x \in Z, x < x_0\} &< x_0. \end{aligned}$$

We can then find a positive lower bound for the speeds of all shock curves near $\hat{x}_+$ and a negative upper bound for the speed of shock curves near $\hat{x}_-$. From this we can conclude that there is some shock curve $y_{i/2^N}$ which enters the interval $(\hat{x}_-, \hat{x}_+)$ after a finite time. This implies $\hat{x}_{i/2^N} \in (\hat{x}_-, \hat{x}_+)$, a contradiction.

□

**Lemma 3.8** *$U$ is a heteroclinic solution connecting $u \equiv u_2$ to $\Psi_{+\infty}$.*

**Proof:** Convergence for $t \to +\infty$ is easy to establish since by lemma 2.8 for $t > 0$ we have $U \equiv \Psi_{+\infty}$.
For $t \to -\infty$, convergence is a consequence of the preceding lemma. Given $\varepsilon > 0$, by continuity of $\phi$, we find $\delta > 0$ with

$$|x - \hat{x}_{i/2^N}| < \delta \Rightarrow |\phi(x - \hat{x}_{i/2^N}) - u_2| < \varepsilon.$$

There exists $N_0 \in \mathbb{N}$ such that

$$(\hat{x}_{i/2^{N_0+1}} - \hat{x}_{(i-1)/2^{N_0+1}}) < \delta \qquad \forall 1 \le i \le 2^{N_0}.$$

Obviously, this inequality holds also for all $N \ge N_0$.
For $t > -N_0$, the solution $U(x,t)$ takes only values $\phi(x - \hat{x}_{i/2^N})$ with $|x - \hat{x}_{i/2^N}| < \delta$. This proves uniform convergence of $U$ to $u \equiv u_2$.

□

**Third method:** Again we can restrict ourselves to one of the intervals $(a_j, b_j)$. For $t \geq 0$ we choose as before a stationary shock at $\bar{x}_j$ with left state $\phi(\bar{x}_j - a_j)$ and right state $\phi(\bar{x}_j - b_j)$. At $t = 0$ we consider *all* backward characteristics that emanate from the point $(\bar{x}_j, 0)$. These are exactly the backward solutions of the characteristic equation (5) with terminal condition $v(0) \in (\phi(\bar{x}_j - b_j), \phi(\bar{x}_j - a_j))$. Together these solutions form a funnel of backward characteristic curves. In other words, the shock at $\bar{x}_j$ is generated by a centered compression wave. Together with the backward characteristics from the shock curve these characteristics fill the whole region $(a_j, b_j) \times \mathbb{R}$ and $U$ can be defined by solving the characteristic equation along the backward characteristics. Convergence for $t \to \pm\infty$ is rather clear for this method since for $t > 0$ we have $U(x, t) \equiv \Psi_{+\infty}(x)$ while for $t \to -\infty$ we can use convergence of $U$ to $u_2$ along every backward characteristic.

$\square$

**Remark:** It is obvious that the three methods can be combined to yield more heteroclinic orbits. The simplest possibility consists of choosing different methods for some of the intervals $(a_j, b_j)$ since these intervals do not influence each other.

## 3.3 Heteroclinic orbits between different rotating waves

We show first that without restriction only the case $\mathcal{Z}_{+\infty} = \{0\}$ has to be discussed. In that case $S^1 \setminus \mathcal{Z}_{+\infty}$ consist of one interval $(0, 1)$. In general, $S^1 \setminus \mathcal{Z}_{+\infty}$ is the union of several open intervals but then we find a global solution by setting $u(x, t) = u_2$ for $x \in \mathcal{Z}_{+\infty}$ and performing the same construction in each of the open intervals. As so often before, we decompose $S^1 \setminus \mathcal{Z}_{-\infty} = \cup_{j \in \mathcal{J}}(a_j, b_j)$.
The main strategy to construct a global solution $U$ is as in the proof of the previous theorems: We describe the set of shock curves together with their left and right states such that:

(i) they satisfy the Rankine-Hugoniot condition and

(ii) the minimal and maximal backward characteristics emanating from these shock curves determine a global solution $U$.

Then we show that $U$ converges to $\Psi_{\pm\infty}$ as $t \to \pm\infty$.
For $t \geq 0$ we want to have a single stationary shock at the position $x \equiv \bar{x}$. As before, we proceed backwards in time and split, at $t = 0$ this shock curve into two curves $y_-$ and $y_+$. At each point where one of these curves crosses a line $x = \bar{x}_j$ for some $j \in \mathcal{J}$ a new stationary shock curve bifurcates from there. More precisely, if $y_-(t_j) = \bar{x}_j$ or $y_+(t_j) = \bar{x}_j$ for some time $t_j$ then we introduce a secondary shock

curve $\{(x,t);\ x = \bar{x}_j, t \le t_j\}$ with left state $\phi(\bar{x}_j - a_j)$ and right state $\phi(\bar{x}_j - b_j)$. By lemma 3.1 these shock curves satisfy the Rankine-Hugoniot condition.
For the primary shock curve $y_-(t)$, the left state will always be $\phi(y_-(t))$ while our choice for the right state is

$$U(y_-(t)+,t) := \begin{cases} \phi(y_-(t) - a_j) \text{ if } y_-(t) \in \bigcup_{j\in\mathcal{J}}(a_j, \bar{x}_j] \\ \phi(y_-(t) - b_j) \text{ if } y_-(t) \in \bigcup_{j\in\mathcal{J}}(\bar{x}_j, b_j) \\ u_2 \text{ if } y_-(t) \in \mathcal{Z}_{-\infty} \end{cases}$$

similarly, we set $U(y_+(t)+,t) := \phi(y_+(t) - 1)$ and

$$U(y_+(t)-,t) := \begin{cases} \phi(y_+(t) - a_j) \text{ if } y_+(t) \in \bigcup_{j\in\mathcal{J}}(a_j, \bar{x}_j] \\ \phi(y_+(t) - b_j) \text{ if } y_+(t) \in \bigcup_{j\in\mathcal{J}}(\bar{x}_j, b_j) \\ u_2 \text{ if } y_+(t) \in \mathcal{Z}_{-\infty} \end{cases}$$

We then get $y_-$ and $y_+$ by solving an ordinary differential equation with Lipschitz right hand sides which is given by the Rankine-Hugoniot condition.

**Lemma 3.9** *The minimal and maximal backward characteristics emanating from the union of all these shock curves with the line $t = 0$ determine a global solution $U$.*

**Proof:** The proof is very similar to the proof of lemma 3.6, so we omit the details. Again one can "guess" the solution $U$:

$$U(x,t) := \begin{cases} \phi(x) \ \text{ if } x < y_-(t) \\ \phi(x-1) \ \text{ if } x > y_+(t) \\ \phi(x - a_j) \ \text{ if } x \in (y_-(t), y_+(t)) \cap (\bigcup_{j\in\mathcal{J}}(a_j, \bar{x}_j]) \\ \phi(x - b_j) \ \text{ if } x \in (y_-(t), y_+(t)) \cap (\bigcup_{j\in\mathcal{J}}(\bar{x}_j, b_j)) \\ u_2 \ \text{ if } x \in (y_-(t), y_+(t)) \cap \mathcal{Z}_{-\infty} \end{cases}$$

Then one follows the forward characteristic up to a shock curve and verifies that this forward characteristic is in fact a minimal or maximal backward characteristic.
□

**Lemma 3.10**

*(i)* $y_{-\infty} = \lim_{t\to-\infty} y_-(t) = \bar{x}_{j_0}$ *if* $0 = a_{j_0}$ *for some* $j_0 \in \mathcal{J}$. *Otherwise* $y_{-\infty} = 0$.

*(ii) Similarly,* $y_{+\infty} = \lim_{t\to-\infty} y_+(t) = \bar{x}_{j_1}$ *if* $1 = b_{j1}$ *for some* $j_1 \in \mathcal{J}$. *Otherwise* $y_{+\infty} = 0$.

**Proof:** We sketch only part (i) since (ii) is completely analogous. $y_-$ is monotone and bounded from below by 0. Therefore it must converge to some point $y_{-\infty}$ If $0 = a_{j_0}$ and $y_-$ is already in the interval $(a_{j_0}, b_{j_0})$ then convergence to $\bar{x}_{j_0}$ follows from lemma 3.1. On the other hand, as long as $y_-$ is strictly bigger than $b_{j_0}$ there is a positive lower bound on $\dot{y}_-$ and hence, since we go backward in time, $y_-$ will enter the interval $(a_{j_0}, b_{j_0})$ after a finite time.
In the other case, since the shock speed $\dot{y}$ at $y_{-\infty}$ has to be zero, we get immediately a contradiction if $y_{-\infty} > 0$.

$\square$

**Lemma 3.11** *$U$ is a heteroclinic orbit from $\Psi_{-\infty}$ to $\Psi_{+\infty}$*

**Proof:** For $t \to +\infty$ nothing has to be proved since for $t > 0$ we have $U(x,t) \equiv \Psi_{+\infty}(x)$.
For $t \to -\infty$, observe that $U$ coincides with $\Psi_{-\infty}$ outside the intervals $(y_{-\infty}, y_-(t))$ and $(y_+(t), y_{+\infty})$ (with interval boundaries reversed if necessary). Since both $U$ and $\Psi_{-\infty}$ are bounded and the length of these intervals tends to zero for $t \to -\infty$ we have convergence of $U$ to $\Psi_{-\infty}$ in $L^p$ for any $1 \leq p < \infty$.

$\square$

**Remark:** There is a lot of freedom in this construction. The secondary shock curves could as well bifurcate from points different from $\bar{x}_j$ and would then for $t \to -\infty$ converge to the position $\bar{x}_j$. What we have described is just the simplest possibility.

# 4 An example

## 4.1 Stationary shocks

In this section we are going to study the simple polynomial example with $f(u) = u^2$ and an asymmetric cubic source term $g(u) = -u(u+1)(u-2)$. In this case it is possible to derive an explicit formula (7) for $\phi$ which helps to illustrate the previous results. Recall that $\phi$ satisfies

$$\int_0^{\phi(\xi)} -\frac{2\nu \, d\nu}{\nu(\nu+1)(\nu-2)} = \xi.$$

Since

$$\int -\frac{d\nu}{(\nu+1)(\nu-2)} = \frac{1}{3} \ln\left(\frac{\nu+1}{\nu-2}\right)$$

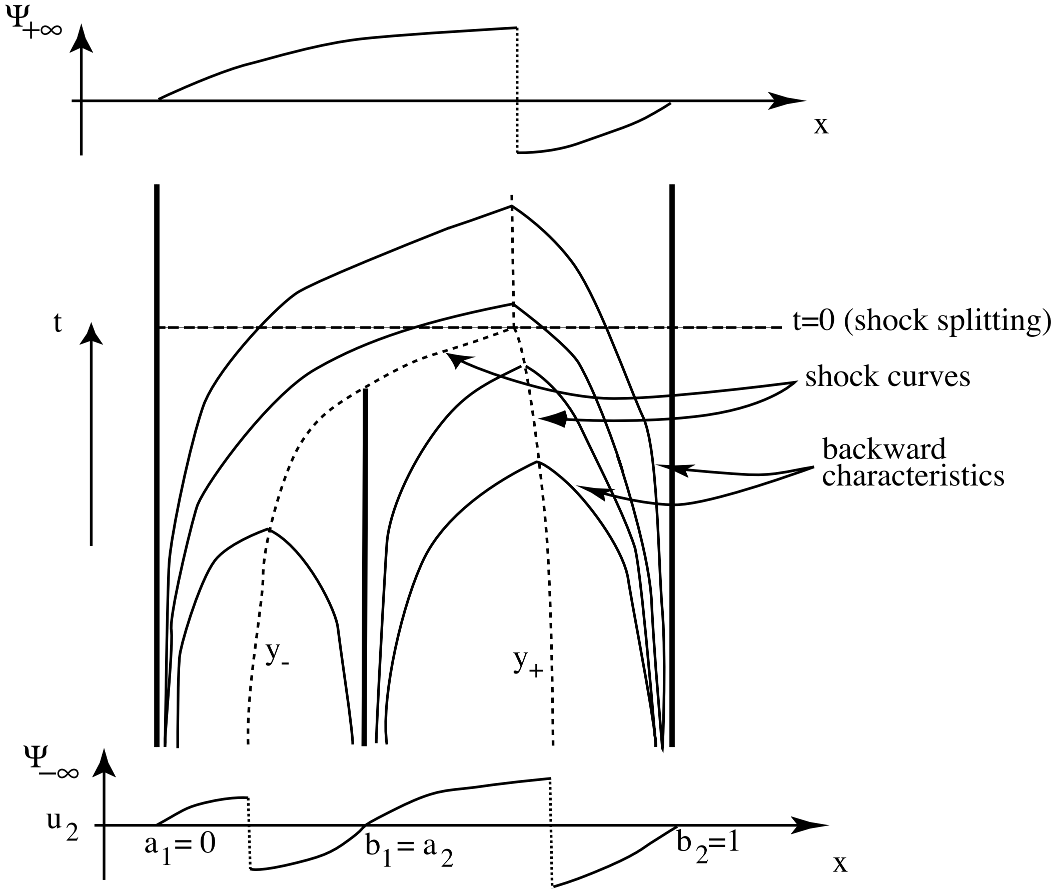


Figure 2: A schematic picture of the heteroclinic connection between two different rotating waves

we arrive at

$$\frac{1+\phi(\xi)}{2-\phi(\xi)} = \frac{1}{2}e^{3\xi/2}$$

and after another rearrangement we get

$$\phi(\xi) = \frac{2e^{3\xi/2}-2}{e^{3\xi/2}+2}. \tag{14}$$

Given $a_j$ and $b_j$ we can now determine the location $\bar{x}_j$ of the jump. As we have demonstrated in lemma 3.1, the condition

$$f(\phi(\bar{x}_j - a_j)) = f(\phi(\bar{x}_j - b_j)) \tag{15}$$

is satisfied for exactly one $\bar{x}_j \in (a_j, b_j)$. In our simple case and since $\phi$ is a monotone function, (15) amounts to the condition

$$\phi(\bar{x}_j - a_j) = -\phi(\bar{x}_j - b_j).$$

With (14), this yields

$$\frac{2\exp(3(\bar{x}_j - a_j)/2) - 2}{\exp(3(\bar{x}_j - a_j)/2) + 2} = -\frac{2\exp(3(\bar{x}_j - b_j)/2) - 2}{\exp(3(\bar{x}_j - b_j)/2) + 2}$$

$$\Longleftrightarrow 4e^{-\frac{3}{2}(a_j+b_j)}(e^{\frac{3}{2}\bar{x}_j})^2 + 2(e^{-\frac{3}{2}a} + e^{-\frac{3}{2}b})e^{\frac{3}{2}\bar{x}_j} - 8 = 0.$$

We can solve this equation for $\bar{y}_j := e^{\frac{3}{2}\bar{x}_j}$ and, observing that $\bar{y}_j > 0$, we arrive at

$$\bar{y}_j = \frac{\sqrt{(e^{-\frac{3}{2}a_j} + e^{-\frac{3}{2}b_j})^2 + 32e^{\frac{3}{2}(a_j+b_j)}} - e^{-\frac{3}{2}a_j} - e^{-\frac{3}{2}b_j}}{4}.$$

Translating this back to $\bar{x}_j$ gives

$$\bar{x}_j = a_j + \frac{2}{3}\ln\left(\sqrt{(1+e^{\frac{3}{2}(b_j-a_j)})^2 + 32e^{\frac{3}{2}(b_j-a_j)}} - (1+e^{\frac{3}{2}(b_j-a_j)})\right) - \frac{2}{3}\ln 4. \tag{16}$$

## 4.2 A heteroclinic orbit connecting two rotating waves

To be even more specific we will now calculate the shock fronts for a heteroclinic solution that connects a rotating wave $\Psi_{-\infty}$ with $\mathcal{Z}_{-\infty} = \{0, b_1\}$ to another rotating wave $\Psi_{+\infty}$ with $\mathcal{Z}_{+\infty} = \{0\}$.
We have $S^1 \setminus \mathcal{Z}_{-\infty} = (a_1, b_1) \cup (a_2, b_2)$ with $a_1 = 0$, $a_2 = b_1$ and $b_2 = 1$.

By (16) we can calculate explicitly the locations $\bar{x}_i$ of the stationary shocks in the intervals $(a_i, b_i)$ as well as the location $\bar{x}$ of the stationary shock for the interval $(0,1)$.
Like in section 3 the global solution possesses for $t < 0$ two shock curves $y_-(t)$ and $y_+(t)$ which collide for $t = 0$ at $\bar{x}$ to form a stationary shock there. To get an expression for $y_-$ we need only solve the ordinary differential equation

$$\begin{aligned} \dot{y}_-(t) &= \frac{f(\phi(t)) - f(\phi(t-b_1))}{\phi(t) - \phi(t-b_1)} \\ &= \phi(t) + \phi(t-b_1) \\ &= \frac{2e^{\frac{3}{2}t} - 2}{e^{\frac{3}{2}t} + 2} + \frac{2e^{\frac{3}{2}(t-b_1)} - 2}{e^{\frac{3}{2}(t-b_1)} + 2} \end{aligned}$$

backward in time with the terminal condition $y_-(0) = x_0$.
The differential equation for $y_+$ is

$$\dot{y}_+(t) = \phi(t - a_j) + \phi(t)$$

with the same terminal condition $y_+(0) = x_0$.
From lemma 2.8 we know that the solution has the following form:

$$U(x,t) = \begin{cases} \phi(x;0) & \text{for } t < 0,\ x \le y_-(t) \\ \phi(x-b_1) = \phi(x-a_2) & \text{for } t < 0,\ y_-(t) < x < y_+(t) \\ \phi(x;1) & \text{for } t < 0,\ y_+(t) < x \le 1 \\ \phi(x;0) & \text{for } t \ge 0,\ x \le x_0 \\ \phi(x;1) & \text{for } t \ge 0,\ x > x_0 \end{cases}$$

A schematic picture of this solution is given in figure 2.

# 5 Discussion

In this paper we have provided another step towards the geometric description of the (infinite-dimensional) global attractors for scalar hyperbolic balance laws. We have solved the connection problem by showing that $\mathcal{Z}_{+\infty} \subseteq \mathcal{Z}_{-\infty}$ is a necessary and sufficient condition for two rotating waves $\Psi_{-\infty}$ and $\Psi_{+\infty}$ to be connected by a heteroclinic orbit.
We have not tried to find *all* heteroclinic orbits between two given limiting profiles but we expect that the structure of all occuring heteroclinic orbits is similar to the

ones described in this paper. In particular the solutions constructed above share the same feature that the profile does only change near shocks, while it remains constant in regions where no shocks occur. In some sense, along the shock curves there is a "phase transition" from $\phi(x-a_j)$ to $\phi(x-b_j)$.
Another important issue is the question of viscosity limits. The hyperbolic balance law can be considered as the singular limit of parabolic equations

$$u_t = \varepsilon u_{xx} - f(u)_x + g(u)$$

when the "viscosity" $\varepsilon$ tends to 0. Under the dissipativeness condition (D) on $g$ this equation possesses a global attractor $\mathcal{A}_\varepsilon$. Such a limit was studied by the author in [6, 5] for $x \in [0,1]$ with Neumann boundary conditions. In this case the dimension of the global attractors $\mathcal{A}_\varepsilon$ remains bounded when the viscosity tends to zero. However, there is no real limiting equation since the hyperbolic problem with Neumann boundary conditions is not well posed.
On the other hand, for $x \in S^1$ the results on the parabolic side are less detailed than for $x$ in the interval with separated boundary conditions. A Poincaré-Bendixson theorem has been proved by Angenent and Fiedler [1] and recently by Matano and Nakamura [11], but the connection problem is open in this case. In particular, the important question remains whether the global attractor $\mathcal{A}_0$ of the hyperbolic balance law can be obtained as the limit of global attractors $\mathcal{A}_\varepsilon$ for $\varepsilon \searrow 0$.
Apart from these rather theoretical aspects, our paper could also be useful to find some solutions to specific test equations by integration of some ordinary differential equations alone. For instance, our example from the preceding section with $t$ sufficiently negative provides a simple initial condition with two shocks whose behaviour we know exactly. Therefore, using this initial condition, numerical methods can be easily compared to the correct analytical solution. Of course, more complicated examples can be constructed in a similar way, including ones with centered compression waves or multiple shock interactions.

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