Viscous and relaxation approximations to heteroclinic traveling waves of conservation laws with source terms

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In contrast to hyperbolic conservation laws, systems of hyperbolic balance laws

$$u_t + f(u)_x = g(u), \quad u \in \mathbb{R}^N, \quad x \in \mathbb{R} \tag{1}$$

can possess nontrivial continuous traveling wave solutions of the form $u(x,t) = U(\xi)$ where $\xi = x - st$ and s is the wave speed. These traveling waves satisfy the ordinary differential equation

$$-sU' + \partial_u f(U)U' = g(U) \tag{2}$$

where the prime denotes differentiation with respect to ξ.
Although there are typically many discontinuous traveling wave solutions, too, this paper deals exclusively with continuous traveling waves.

Definition 0.1 *A traveling wave U is said to be a* **heteroclinic wave** *if both* $\lim_{\xi\to-\infty} U(\xi)$ *and* $\lim_{\xi\to+\infty} U(\xi)$ *exist.*

From (2) one can easily conclude that the source term g has to vanish at both asymptotic states of a heteroclinic wave U. We will say that the heteroclinic wave connects the asymptotic state at $-\infty$ to the asymptotic state at $+\infty$.
For the case of a scalar equation with a strictly convex flux f, Mascia [Mas97] found some traveling wave solutions which exist only at isolated values of the wave speed. They are of special interest since one can show that similar waves exist for the viscous equation and that they are stable. In the present paper we will show that in the p-system with a nonlinear source term similar waves can exists for a whole range of wave speeds. Furthermore, we discuss two viscous regularizations of this hyperbolic system and show that both of them possess heteroclinic traveling waves that correspond to the heteroclinic traveling waves of the hyperbolic system.

1 Scalar balance laws

The case $N = 1$ has been studied most. The next lemma is a slight generalization Mascias result for a scalar balance law and allows f to be non-convex.

Lemma 1.1 *Assume that $u_1 < u_2 < u_3$ are three consecutive zeroes of g with $g'(u_1) < 0$, $g'(u_2) > 0$ and $g'(u_3) < 0$. Let furthermore $s_0 := f'(u_2)$. If*

$$f''(u_2) > 0 \;\; \textit{and} \;\; \begin{cases} f'(u) - s_0 < 0 & \textit{for} \quad u_1 \le u < u_2 \\ f'(u) - s_0 > 0 & \textit{for} \quad u_2 < u \le u_3 \end{cases}$$

then there exists a continuous strictly monotone heteroclinic traveling wave U_0 with wave speed s_0 connecting u_1 to u_3.

Proof: This follows from the fact that the traveling wave equation

$$U' = \frac{g(U)}{f'(U) - s_0}$$

has a removable singularity at $U = u_2$.

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Remark 1.2 *There might be many discontinuous, nonmonotone heteroclinic waves from u_1 to u_3.*

Together with (1) one often studies the *viscous* balance law

$$u_t + f(u)_x = \varepsilon u_{xx} + g(u), \tag{3}$$

where $\varepsilon > 0$ is a small parameter. A classical result by Kruzhkov states that for finite time intervals the solutions of (3) with the same initial condition converge to the unique entropy solution of (1) almost everywhere as ε tends to zero. Dealing with traveling waves we ask a slightly different question: Can we find solutions for (3) that are traveling waves with a profile which is close to the profile of the traveling wave found in lemma 1.1 ? Thus, we concentrate on identical qualitative properties (namely, that the solution does not change its shape) and infinite time intervals, but do not require the same initial data for $\varepsilon > 0$ and $\varepsilon = 0$. Moreover we will find that the wave speeds differ slightly, and, although the profiles themselves will be L^1-close, the waves will move apart from each other. Nevertheless, since hyperbolic conservation laws are often simplified models for viscous equations it is useful to establish existence of traveling waves for the viscous equations, too, and to describe the influence of the viscosity on the wave speed. In [Här99] the following result was proved:

Proposition 1.3 *Assume that $f \in C^3$ is convex and $g \in C^2$. Then for any ε sufficiently small, there exists a unique wave speed $s = s(\varepsilon)$ with $|s(\varepsilon) - s_0| = \mathcal{O}(\varepsilon)$ and a unique heteroclinic traveling wave U_ε of (3) such that*

$$||U_\varepsilon - U_0||_{L^1(\mathbb{R})} \longrightarrow 0 \ \textit{ as } \ \varepsilon \to 0.$$

We give an outline of the proof, since this will guide us later to prove the existence in the viscous p-system with source.

First the viscous traveling wave equation is written as a first order system

$$\begin{aligned} \varepsilon U' &= V + f(U) - sU \\ V' &= -g(U) \end{aligned}$$

and standard singular perturbation theory states that the curve $V + f(U) - sU = 0$ is a slow manifold for $\varepsilon = 0$. By normal hyperbolicity, a piece of it with $u_1 < U < u_2 - d$ will

survive as an invariant manifold. Since this manifold contains the saddle equilibrium (u_1, v_1), it must be the unstable manifold of (u_1, v_1). Similarly, another part of the slow manifold with $u_2 + d < U < u_3$ persists and is in fact the stable manifold of (u_3, v_3). A heteroclinic wave exists iff the two manifolds intersect and (since the are both one-dimensional) coincide near u_2. Since at the point (u_2, v_2) normal hyperbolicity of the slow manifold fails, the analysis along the lines of [Fen79] is not possible but using the method of rotated vector fields [Duf53, Per93] it is possible to show that for ε small, there is a unique value $s(\varepsilon)$ such that a heteroclinic orbit from u_1 to u_3 exists. A basic phase-plane analysis or a version of the blow-up method of Krupa & Szmolyan [KS99] shows that $|s(\varepsilon) - s_0| = \mathcal{O}(\varepsilon)$.
For the L^1-convergence of the profiles it is most convenient to show that the stable manifold of (u_3, v_3) lies in a narrow strip

$$P_+ = \{(U,V)\ ;\ u_1 \leq U \leq u_2 - \delta\sqrt{\varepsilon},\ \left| V + f(U) - sU - \varepsilon \frac{g(U)}{f'(U) - s_0} \right| \leq k\varepsilon^{3/2} \frac{g(U)}{U - u_2}\}$$

if $|s - s_0| = \mathcal{O}(\varepsilon)$. Since U_ε lies in this strip we have

$$U_\varepsilon' = \frac{1}{\varepsilon}(V_\varepsilon + f(U_\varepsilon) - sU_\varepsilon) = \frac{g(U_\varepsilon)}{f'(U_\varepsilon) - s_0} + \mathcal{O}(\varepsilon^{1/2})$$

as long as $u_1 \leq U_\varepsilon \leq u_2 - \delta\sqrt{\varepsilon}$. An analogous estimate holds for $u_2 + \delta\sqrt{\varepsilon} \leq u_\varepsilon \leq u_3$. To prove the statement one now has to combine a Gronwall-type argument for ξ in some bounded intervals with exponential decay estimates for $\xi \to \pm\infty$. The details are contained in [Där99].
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1.1 Relaxation approximation

In this section, we will discuss the relaxation system

$$\begin{aligned} u_t + w_x &= g(u) \\ w_t + a^2 u_x &= \frac{1}{\varepsilon}(f(u) - w) \end{aligned} \tag{4}$$

that is associated with the hyperbolic balance law (1). The only assumption about the variable a is the *subcharacteristic condition*

$$a > f'(u)\ \ \forall u \in [u_1, u_3].$$

In particular, this implies that $a^2 - s_0^2 > 0$. We will prove the following theorem

Theorem 1.1 *For any ε sufficiently small, there exists a heteroclinic traveling wave $(u_\varepsilon, v_\varepsilon)$ of (4) with wave speed $s = s(\varepsilon)$ such that*

$$\|u_\varepsilon - u_0\|_{L^1(\mathbb{R})} \longrightarrow 0\ \ \textit{as}\ \ \varepsilon \to 0.$$

The wave speed $s(\varepsilon)$ is $\mathcal{O}(\varepsilon)$-close to s_0.

Proof: The subcharacteristic condition allows to write the traveling wave equation in the form

$$\begin{aligned}(a^2-s^2)U' &= sg(U)+\frac{f(U)-V}{\varepsilon}\\ (a^2-s^2)V' &= a^2g(U)+\frac{s(f(U)-V)}{\varepsilon}.\end{aligned}$$

It is however more convenient to discuss the traveling wave equation in the following equivalent form with time rescaled by a factor (a^2-s^2):

$$\begin{aligned}\varepsilon U' &= \tilde{V}+f(U)-sU+\varepsilon sg(U)\\ \tilde{V}' &= -(a^2-s^2)g(U).\end{aligned}$$

Only slight modifications are necessary to deal with this system. Firstly, it turns out that the method of rotated vector fields does not work any more. However the existence of a heteroclinic can be established by a shooting argument using the following lemma:

Lemma 1.4 *There exists $\sigma>0$ and $\varepsilon_1>0$ such that for all $0<\varepsilon\le\varepsilon_1$ and $s=s_0-\sigma\varepsilon$ the unstable manifold $W^u(u_1)$ of u_1 intersects the line $U=u_2$ below the stable manifold $W^s(u_3)$ of u_3 while for $s=s_0+\sigma\varepsilon$ the unstable manifold $W^u(u_1)$ intersects $U=u_2$ above $W^s(u_3)$.*

Proof : We treat the case $s=s_0-\sigma\varepsilon$ only since the proof for $s=s_0+\sigma\varepsilon$ is completely analogous. Calculating the eigenvalues of the linearization at $U=u_2$ one finds that this equilibrium has a pair of complex eigenvalues if $s=s_0\pm\sigma\varepsilon$ and ε is small. Therefore no trajectory can approach this equilibrium in forward or backward time without intersecting the line $U=u_2$. Since the vector field is vertical along the line $\tilde{V}+f(U)-sU+\varepsilon sg(U)$ both $W^u(u_1)$ and $W^s(u_3)$ must intersect the line $U=u_2$. To check that the first intersection of $W^u(u_1)$ actually lies below that of $W^s(u_3)$ one can look at the vector field along the line

$$\tilde{V}=\gamma(U):=-f(U)+sU-\varepsilon sg(U)-\varepsilon\frac{(a^2-s^2)g(U)}{f'(U)-s_0}.$$

The slope of this curve is

$$\frac{d\gamma(U)}{dU}=-f'(U)+s-\varepsilon sg'(U)-\varepsilon\frac{(a^2-s^2)(f'(U)-s_0)g'(U)-f''(U)g(U)}{(f'(U)-s_0)^2}$$

while the slope of the vector field along $\gamma(U)$ is

$$\frac{\tilde{V}'}{U'}=-f'(U)+s_0.$$

Since

$$\lim_{U\to u_2}\frac{(f'(U)-s_0)g'(U)-f''(U)g(U)}{(f'(U)-s_0)^2}=\frac{f''(u_2)g''(u_2)-f'''(u_2)g'(u_2)}{2f''(u_2)^2}$$

exists one can always achieve that $\frac{d\gamma(U)}{dU} > \frac{\tilde{V}'}{U'}$ by choosing σ large. This implies that trajectories can cross γ only from above. A standard calculation comparing the tangent vector of γ at $U = u_1$ with the eigenvector corresponding to the negative eigenvalue shows that $W^u(u_1)$ lies below γ while a simlar calculation at $U = u_3$ shows that $W^s(u_3)$ lies above γ. Therefore the intersection of $W^u(u_1)$ with the line $U = u_2$ lies below that of $W^s(u_3)$ with $U = u_2$.

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Proof of theorem 1.1: Since the points of intersection between $W^u(u_1)$ resp. $W^s(u_3)$ and the line $U = u_2$ depend continuously on s there must exist a wave speed $s(\varepsilon) \in [s_0 - \sigma\varepsilon, s_0 + \sigma\varepsilon]$ such that a heteroclinic wave from u_1 to u_3 exists. Monotonicity follows from the fact that the trajectory lies above the curve $V + f(U) - sU + \varepsilon s g(U)$.

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Remark 1.5 *To establish the L^1-closeness of the traveling wave profiles one has to do some more work and locate the heteroclinic wave in the Liénard plane accurately.*

2 The p-system with source

In this part of the paper we prove that for a simple system continuous heteroclinic traveling waves can occur over a whole range of wave speeds and that they persist when a small physical viscosity is present.

2.1 Entropy traveling waves for the hyperbolic equation

We consider the p-system of isentropic gas dynamics with a general source term g:

$$\begin{aligned} u_t + v_x &= 0 \\ v_t - p(u)_x &= g(u,v). \end{aligned} \tag{5}$$

We assume that $p \in C^3$ with $p'(u) < 0$ and the source term $g \in C^2$ is a Morse function for which 0 is a regular value. For definiteness, we also assume $p''(u) > 0$.
Looking for traveling waves $(U(x - st), V(x - st))$ of the hyperbolic equation (5) leads to the study of the system of ordinary differential equations

$$\begin{aligned} (-s^2 - p'(U))U' &= g(U,V) \\ (-s^2 - p'(U))V' &= sg(U,V) \end{aligned}$$

where the prime denotes differentiation with respect to $\xi = x - st$. Note that trajectories of this vector field are restricted to lines where $V - sU$ is constant and therefore all solutions can be found by looking at the one-parameter family of problems that arises by setting $V - sU =: C \in \mathbb{R}$.

Proposition 2.1 *Fix the wave speed* s_0*. Assume that there exists* $u_1 < u_2 < u_3$ *and* $v_1,\ v_2,\ v_3$ *such that:*
(i) $p'(u_2) = -s_0^2$*,*
(ii) $g(u_i, v_i) = 0$ *for* $i = 1, 2, 3$*,*
(iii) $v_1 - s_0 u_1 = v_2 - s_0 u_2 = v_3 - s_0 u_3$ *,*
(iv) $g(u, v_2 + s_0(u - u_2))p'(u) > 0$ *for* $u \in (u_1, u_3) \setminus \{u_2\}$ *and*
(v) $\partial_u g(u_2, v_2) + s_0 \partial_v g(u_2, v_2) < 0$*.*
Then there exists a continuous monotone heteroclinic wave from (u_1, v_1) *to* (u_3, v_3) *with speed* s_0*.*

Proof: The statement follows from the study of the one-dimensional vector field on the invariant line $V - v_2 = s_0(U - u_2)$.

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Unlike in the scalar case, this heteroclinic wave is robust under small perturbations if certain non-degeneracy conditions hold.

Proposition 2.2 *Assume the following transversality conditions:*
(H) $(-1)^i\,(\partial_u g(u_i, v_i) + s_0 \partial_v g(u_i, v_i)) < 0$ *for* $i = 1, 2, 3$ *.*
Then for $|s - s_0|$ *small enough, there exists a unique heteroclinic wave with speed* s *that connects some state* $(u_1(s), v_1(s))$ *near* (u_1, v_1) *to a state* $(u_3(s), v_3(s))$ *near* (u_3, v_3)*.*

Proof: We have to solve the system of six equations

$$\begin{aligned} -s^2 - p'(u_2) &= 0 \\ g(u_i, v_i) &= 0, \quad i = 1, 2, 3, \\ v_i - s u_i - v_2 + s u_2 &= 0, \quad i = 1, 3 \end{aligned}$$

to satisfy assumptions (i)-(iii) from the previous lemma. By the implicit function theorem we find a unique solution $(u_1(s), v_1(s), u_2(s), v_2(s), u_3(s), v_3(s))$ near the known solution $(u_1, v_1, u_2, v_2, u_3, v_3,$ if **(H)** is satisfied since the Jacobian at the known solution is

$$\begin{pmatrix} 0 & -p''(u_2) & 0 & 0 & 0 & 0 \\ \partial_u g(u_1, v_1) & 0 & 0 & \partial_v g(u_1, v_1) & 0 & 0 \\ 0 & \partial_u g(u_2, v_2) & 0 & 0 & \partial_v g(u_2, v_2) & 0 \\ 0 & 0 & \partial_u g(u_3, v_3) & 0 & 0 & \partial_v g(u_3, v_3) \\ -s & s & 0 & 1 & -1 & 0 \\ 0 & s & -s & 0 & -1 & 1 \end{pmatrix}.$$

Assumptions (iv) and (v) will then automatically hold and condition **(H)** for $i = 2$ makes sure that there are no other points of intersection between the zero set of g and the line $V - sU = 0$ near (u_2, v_2).

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This shows that monotone heteroclinic waves which pass through the zero set of g in general exist for an open set of wave speeds. Under the non-degeneracy condition **(H)** they are contained in one-parameter families of heteroclinic waves parametrized by the velocity s.

2.2 The effect of physical viscosity

As in [MM99], we assume that $u > 0$ and add a small viscosity term to the second equation to get

$$\begin{aligned} u_t + v_x &= 0 \\ v_t - p(u)_x &= \left(\frac{\varepsilon v_x}{u}\right)_x + g(u, v). \end{aligned}$$

Since the first equation describes the conservation of mass, it remains unchanged. The traveling wave equation for this system is

$$\begin{aligned} -sU' + V' &= 0 \\ -sV' + p'(U)U' &= \left(\frac{\varepsilon V'}{U}\right)' + g(U, V) \end{aligned}$$

Again, the first equation can be integrated and after setting $C := V - sU$ for some constant C near $C_0 := v_2 - s_0 u_2$ the second equation can be written in Liénard coordinates after a rescaling of time as

$$\left.\begin{aligned} \varepsilon U' &= Z - sC - s^2 U - p(U) \\ Z' &= -\tfrac{s}{U} g(U, C + sU) =: -G(U, s). \end{aligned}\right\} \tag{6}$$

For $\varepsilon = 0$, we will assume that the conditions of proposition 2.1 are satisfied for some speed s_0 and some (u_i, v_i), $i = 1, 2, 3$, together with the transversality condition **(H)**. By condition **(H2)**, for any s near s_0 there exists $u_2(s)$ with $p'(u_2(s)) = -s^2$ and $C(s)$ near C_0 such that $g(u_2(s), C(s) + su_2 s)) = 0$. Moreover, there exist unique equilibria $E_i := (u_i(s), z_i(s))$ near $(u_i, p(u_i) + s_0^2 u_i)$ of (6). We are looking for values of s near s_0 and $\varepsilon > 0$ such that there is a heteroclinic orbit from $E_1 = (u_1(s), z_1(s))$ to $E_3 = (u_3(s, C), z_3(s, C))$ close to the heteroclinic traveling wave for the hyperbolic equation described in proposition 2.1.

Theorem 2.1 *There exists ε_0 such that for $0 < \varepsilon \le \varepsilon_0$ there exists a $s(\varepsilon)$ with $|s(\varepsilon) - s_0| = \mathcal{O}(\varepsilon)$ such that the viscous p-system (6) has a monotone continuous traveling wave connecting E_1 to E_3.*

Proof: The proof is similar to the one for the relaxation approximation of a scalar balance laws.
The linearization of (6) at E_i has the eigenvalues

$$\lambda_\pm = \frac{-(s^2 + p'(u_i)) \pm \sqrt{(s^2 + p'(u_i))^2 - 4\varepsilon \partial_U G(u_i, s)}}{2\varepsilon}.$$

Since $\partial_U G(u_2, s) > 0$ and $\partial_U G(u_i, s) \leq 0$ for $i = 1$ and $i = 3$, this implies that for ε small E_1 and E_3 are of saddle-type with a one-dimensional stable and one-dimensional unstable manifold. Moreover for $s \in [s_0 - \sigma\varepsilon, s_0 + \sigma\varepsilon]$ we have $s^2 + p'(u_2) = s^2 - s_0^2 = \mathcal{O}(\varepsilon)$ and hence for ε sufficiently small the eigenvalues at E_2 are complex.
Again one needs to find some wave speed such that the one-dimensional unstable manifold $W^u(E_1)$ and the one-dimensional stable manifold $W^s(E_3)$ intersect each other.
This is done by a shooting argument: For $s = s_0 - \sigma\varepsilon$ the unstable manifold $W^u(E_1)$ lies between the curves $Z - s^2 U - p(U) = 0$ and $\gamma = \{Z - s^2 U - p(U) - \varepsilon \frac{G(U,s)}{s_0^2 + p'(U)} = 0\}$ as long as $u_1 \leq U \leq u_2$ while $W^s(E_3)$ lies above the curve γ for $u_2 \leq U \leq u_3$. Therefore $W^u(E_1)$ intersects the line $U = u_2$ below $W^s(E_3)$. Since for $s = s_0 + \sigma\varepsilon$ the situation is vice versa there must be a wave speed $s(\varepsilon)$ for which a heteroclinic connection exists.

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To model numerical viscosity one would like to add a small viscous perturbation to both equations of the p-system. This leads to a singularly perturbed equation with two slow and one fast variable containing three parameters. Therefore the situation is far more complicated and cannot be treated in the framework of this article.

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