Scalar conservation laws with a degenerate source: Traveling waves, large-time behavior and zero relaxation limit

Haitao Fan, Department of Mathematics, Georgetown University , Washington, DC 20057, U.S.A., email:fan@math.georgetown.edu

Jörg Härterich, Freie Universität Berlin, Arnimalleee 2-6, 14195 Berlin, Germany, email:haerter@math.fu-berlin.de

Abstract

We study a scalar hyperbolic conservation law with a bistable source term whose zeroes are degenerate. It turns out that the structure of the set of traveling waves as well as the long-time behavior is different from the case where the zeroes are non-degenerate. In particular, for $x \in S^1$, there exist initial data for which the solution converges pointwise to the unstable zero of the source term.

We also generalize recent results of Fan, Jin and Teng concerning the zero reaction time limit to the case of source terms with degenerate zeroes. This shows that in general the long time limit and the zero reaction time limit cannot be interchanged.

1 Introduction

If a nonlinear source term is added to a hyperbolic conservation law both the short and long time behaviour may change drastically. In this paper, we demonstrate how the nature of the source term affects the behaviour. Motivated by nonlinearities that occur in equations describing liquid-vapor phase transitions and in reactive flow models, we study the equation

$$u_t + f(u)_x = \frac{g(u)}{\epsilon} \qquad u \in \mathbb{R}, \qquad \alpha, \beta \geq 1, \tag{1}$$

with

$$g(u) = u|u|^{\alpha-1}(1-u^2)|1-u^2|^{\beta-1},$$

where the flux is convex and the space variable x is either considered on the whole real line or on the circle S^1. Many of our results do not depend on the parameter ε so we will set $\varepsilon = 1$ in chapters 2 through 4. Only in chapter 5 when the zero reaction time limit is considered, ε will play a role.

The source term in above equation is of "bistable" type, i.e. there are two states $u = +1$ and $u = -1$ which are stable equilibria of the pure reaction dynamics $u_t = g(u)$. In between there is an unstable state $u = 0$ which is weakly unstable if $\alpha > 1$, i.e. the instability is not visible on the linear level.

Before we give precise statements and proofs, let us comment on some related work. The case $\alpha = \beta = 1$ where all zeroes of the source term are non-degenerate, has been studied intensively by several authors. Most of the work also assumes a convex flux so that generalized Characteristics can be used as a tool. Mascia and Sinestrari [11] studied the perturbed Riemann problem for convex flux when the initial data coincides with that of the Riemann problem outside a compact set. If both asymptotic states are zeroes of g, then the solution for large time can be described by a sequence of shocks and possibly discontinuous traveling waves.

A related situation has been studied by Mascia [8]. He considers a source term with only two zeroes $g(0) = g(1) = 0$ and $g(u) > 0$ between 0 and 1 but with a possibly non-convex flux. The existence of traveling waves between 0 and 1 and the solution of a corresponding Riemann problem with states $u_- = 0$ and $u_+ = 1$ is studied. In contrast to the situation treated below, the source term has a definite sign and all the technical difficulties come from the non-convexity of the flux f.

Another line of research deals with $x \in S^1$ when the flux is convex and g has simple zeroes. Lyberopoulos [7], Fan and Hale [2] and Sinestrari [12] could prove independently that all solutions converge either to a spatially homogenous state or to a rotating wave. As Sinestrari showed, the source term typically dominates the large-time behaviour and almost all solutions converge pointwise to one of the stable equilibria of the pure reaction dynamics. Fan and Hale [3] proved that there exists a global attractor for the semiflow which is generated in $BV \cap L^p$ for $1 \leq p < \infty$ and that this attractor consists of the steady states, infinitely many rotating waves and heteroclinic orbits between these objects. A geometric description of the heteroclinic orbits has been given by Härterich [5].

In the case where the source term has simple zeroes the rotating waves represent the solutions for which the effect of the reaction term and the convection balance each other. As we will see, the situation in the case of a more degenerate source term is different: Firstly, there are no rotating waves in the degenerate case. Furthermore, instead of converging to rotating waves, there are solutions which decay pointwise to zero although 0 is unstable for the reaction dynamics. However it seems a difficult issue to predict the ultimate fate of a given initial condition.

The relaxation limit has been studied in a similar situation by Fan, Jin and Teng [4]. They considered a bistable source term with three simple zeroes ± 1 and 0. The initial condition u_0 was supposed to have a change of sign at $x = 0$. There are two different possibilities: If $u_0(x) \cdot x < 0$ for $x \neq 0$ then the solution converges to $+1$ for $x < \gamma t$ and to -1 for $x > \gamma t$ where γ comes from the Rankine-Hugoniot shock condition. If however $u_0(x) \cdot x > 0$ for $x \neq 0$ then the limiting solution has a "non-shock" discontinuity as it converges to -1 for $x < \gamma t$ and to $+1$ for $x > \gamma t$ with $\gamma = f'(0)$.

From these results, we see that the behavior large time limit and relaxation limit of the solution can be quite different.

1.1 The main results

The existence and uniqueness theory for scalar conservation laws with source terms differs only slightly from the theory without source term. In particular, there are no smooth solutions even for smooth initial data. However the Cauchy problem

$$u_t + f(u)_x = g(u), \quad u(0,\cdot) = u_0(\cdot) \tag{2}$$

has a (local) solution for $u_0 \in BV(\mathbb{R}) \cap L^1_{loc}(\mathbb{R})$ if one considers as a solution all weak solutions which satisfy the Kruzhkov entropy condition, see [6]. This solution is also unique and a comparison principle holds:

Proposition 1.1 (Kruzhkov) *Consider solutions u and v of the Cauchy problem (2) with initial data u_0 and v_0. If u_0 and v_0 are ordered, i.e.*

$$u_0(x) \le v_0(x) \textit{ for almost all } x \in \mathbb{R}$$

then the same inequality holds for later times:

$$u(t,x) \le v(t,x) \textit{ for almost all } x \in \mathbb{R}.$$

Although existence, uniqueness and the comparison principle hold in broader generality, we will take

$$g(u) = \frac{1}{\varepsilon}\left(u|u|^{\alpha-1}(1-u^2)|1-u^2|^{\beta-1}\right)$$

with α, $\beta \ge 1$. Our proofs for for the part $u(x,t) \le 0$ is almost the same as that for the $u(x,t) \ge 0$ part. In fact, a change of variable $u \mapsto -u$ in (1) can convert the proofs for one part to the other. For this reason, and for simplicity of presentation, we use the simplified notation

$$g(u) = u^\alpha(1-u^2)^\beta.$$

We consider the Cauchy problem

$$u_t + f(u)_x = \frac{g(u)}{\varepsilon}, \qquad u(0,\cdot) = u_0(\cdot) \tag{3}$$

and assume the following on f:

(F1) f is C^2,

(F2) f is strictly convex, i.e. $f'' > 0$,

(F3) $f(0) = f'(0) = 0$.

Note that **(F3)** is no real restriction since we can first add a constant to achieve $f(0) = 0$. If $f'(0) \ne 0$ then we consider the equation in a moving coordinate system, i.e. we replace x by $\tilde{x} = x - f'(0)t$ such that $\tilde{u}$ defined by $\tilde{u}(\tilde{x},t) := u(x,t)$ satisfies

$$\tilde{u}_t + (f'(\tilde{u}) - f'(0))\tilde{u}_{\tilde{x}} = \frac{g(\tilde{u})}{\varepsilon}.$$

Remark 1.2 *The special choice of the source term was made to facilitate some calculations and make the presentation easier. Similar results hold for more general source terms $g(u)$. Instead of assuming $\alpha \geq 2$ in theorems 1.2 and 1.3 below one then requires that $\int_{-\infty}^{0} f'(U(\tau))\ d\tau$ does not exist where $U(\tau)$ solves the terminal value problem*

$$\frac{dU}{dt} = g(U), \quad U(0) = 1.$$

This is equivalent to the condition

$$f'/g \notin L^1((-1,1), \mathbb{R})$$

which also appears as a condition in [10] in the case of a non-convex flux f where generalized characteristics cannot be used.

For convenience, we will always assume that the solution is continuous from the left at any point, i.e.

$$u(x,t) = u(x-,t).$$

By the comparison principle, any solution to bounded initial data will exist globally in time and hence we restrict our attention initial data in $L^1_{loc}(\mathbb{R}) \cap BV_{loc}(\mathbb{R})$.
It seems that the behavior for $1 \leq \alpha < 2$ is very similar to the case $\alpha = 1$ which has been studied previously [2, 3, 9, 5] In fact, many of the proofs carry over to the case $1 \leq \alpha < 2$ without modification. Therefore, we focus mainly on $\alpha \geq 2$ where the behavior is quite different.
The following result on the existence of entropy traveling waves is proved below:

Theorem 1.1 (Traveling waves) *Fix $\varepsilon = 1$ and $\alpha \geq 2$. Then the following holds:*

(i) A continuous entropy traveling wave from -1 to 0 exists iff $c \geq f'(0)$. A continuous entropy traveling wave from 0 to -1 exists iff $c \leq f'(-1)$.

Similarly, continuous entropy traveling wave from 0 to $+1$ exist for $c \leq f'(0)$ and from $+1$ to 0 for $c \geq f'(1)$.

(ii) There is a unique wave speed $c_{RH} = \frac{f(1)-f(-1)}{2}$ determined by the Rankine-Hugoniot condition for which a travelingshock wave from $+1$ to -1 exists.

(iii) If $c_{RH} < c < f(0)$ then there is a discontinuous traveling wave from 0 to -1. Similarly, discontinuous traveling waves from $+1$ to 0 exist for wave speeds c with $f(-1) < c < c_{RH}$.

(iv) There exist no other bounded traveling waves.

Our next theorem concerns the long-time behavior of solutions to the Cauchy problem with periodic initial data for $\alpha \geq 2$. Since the periodicity ispreserved it is equivalent to consider $x \in S^1 \sim \mathbb{R}/\mathbb{Z}$. In thecase $\alpha = \beta = 1$ of a source term with simple zeroes it is known that any solutioneither converges in $L^\infty(S^1)$ to $+1$ or -1 or that it converges in $L^1(S^1)$ to a discontinuous rotating wave solution. In the case $\alpha \geq 2$ these rotating waves do not exist due to theorem 1.1(iv). Instead we have:

Theorem 1.2 (Convergence) *Fix $\varepsilon = 1$ and let $\alpha \geq 2$. Then for any initial data $u_0 \in BV(S^1)$ there exists a number $\omega = \omega(u_0) \in \{-1, 0, +1\}$ such that the solution $u(x,t)$ of the Cauchy problem (3) satisfies*

$$\|u(x,t) - \omega(u_0)\|_{L^\infty} \to 0 \quad as \quad t \to \infty.$$

It is not suprising that a solution of (2) can possiblely converges to one of the stable equilibria $u = \pm 1$. A trivial solution of (2) that converges to 0 is $u \equiv 0$. There are many non-trivial solutions of (2) which converge to 0. For example, if $f(u)$ is symetric about $u = 0$ and the initial data $u_0(x)$ is anti-symmetric about $x = 0$, then the solution of (2) will converge to $u = 0$ as $t \to \infty$. The following theorem indicates that most solutions converge to one of the stable zeroes ± 1:

Theorem 1.3 (Instability) *Fix $\varepsilon = 1$ and consider again the Cauchy problem (3) with $x \in S^1$ and $\alpha \geq 2$. Let $u_0 \in BV(S^1)$ be an initial condition such that the corresponding solution $u(x,t)$ converges to zero as $t \to \infty$. Then for any $\delta > 0$, there exists some $v_0 \in BV(S^1)$, with $\|u_0 - v_0\| \leq \delta$ such that the solution of (3) with initial data v_0 will converge to $+1$. Similarly, there exists an initial condition $\tilde{v}_0$ close to u_0 such that the solution converges to -1.*

It is interesting to compare the long-time behavior $t \to \infty$ with the relaxation limit $\varepsilon \to 0$. To this end, we consider initial data on $\mathbb{R}$ with simple, isolated zeroes:

$$u_0(x) \in BC^1(\mathbb{R}) \quad \text{has zeros} \quad a_j \quad \text{where} \quad j \in \mathcal{J} \subseteq \mathbb{Z} \tag{4}$$

with

$$u_0'(a_j) \neq 0 \quad \text{and } \rho := \inf_{j_1, j_2 \in \mathcal{J}, j_1 \neq j_2} |a_{j_1} - a_{j_2}| > 0.$$

Under this assumption it can be proved that (in contrast to theorem 1.2) only the stable states $+1$ and -1 occur in the limiting solution.

Theorem 1.4 (Relaxation) *Let the initial value $u_0(x)$ satisfy (4), and u_ε be the admissible solution of (3). Then the limit*

$$u(x,t) := \lim_{\varepsilon \to 0} u_\varepsilon(x,t)$$

exists for almost all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. The function $u(x,t)$ is piecewise constant with the constants being ± 1. Constant pieces of $u(x,t)$ are separated by Lipschitz continuous curves $x = z_j(t)$ defined on $[0, T_j)$, $j \in \mathcal{J}$. Moreover, the following hold for these curves $x = z_j(t)$, $j \in \mathcal{J}$:

(i) $z_j(0) = a_j$.

(ii) If $\lim_{x\to a_j-} sign(u_0(x)) = 1$*, then* $z_j(t) = a_j + \frac{f(1)-f(-1)}{2}t$.

(iii) If $\lim_{x\to a_j-} sign(u_0(x)) = -1$*, then* $z_j(t) = a_j + f'(0)t$.

(iv) Curves $x = z_j(t)$ *do not intersect each other except at* $t = T_j$*, the end points of their domain of definition.*

(v) At $t = T_j < \infty$*, the curve* $x = z_j(t)$ *must intersect with another curve* $x = z_{j'}(t)$.

The rest of the paper is organized as follows. In chapter 2 we study elementary solutions such as entropy traveling waves and solutions of the Riemann problem and prove theorem 1.1. Chapter 3,4 and 5 contain the the proofs of theorems 1.2, 1.3 and 1.4, respectively.

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2 Elementary Solutions

2.1 Generalized characteristics

Since generalized characteristics are our main tool in this paper, we recall some of their properties in short.

Definition 2.1 *Given a solution* u *of the Cauchy problem (3), a Lipschitz-continuous curve* $\zeta = \zeta(t)$ *with* $t \in [a, b]$ *is called a* **generalized characteristic** *if for almost all* $t \in [a, b]$ *the differential inequality*

$$\frac{d\zeta}{dt} \in [f'(u(\zeta(t)+, t), u(\zeta(t)-, t)]$$

holds.
A characteristic is called **genuine** *if* $u(\zeta-) = u(\zeta+)$ *along the whole characteristic.*

Dafermos [1] has shown that generalized characteristics are either genuine characteristics or shock curves, more precisely:

Proposition 2.2 (Dafermos) *From any point* $(\bar{x}, \bar{t})$ *with* $\bar{t} > 0$ *there is a unique forward characteristic. This forward characteristic satisfies*

$$\frac{d\zeta}{dt} = \begin{cases} f'(u(\zeta(t), t) & \text{if } u(\zeta(t)-, t) = u(\zeta(t)+, t) \\[2ex] \dfrac{f(u(\zeta(t)-, t)) - f(u(\zeta(t)+, t))}{u(\zeta(t)-, t) - u(\zeta(t)+, t)} & \text{if } u(\zeta(t)-, t) > u(\zeta(t)+, t) \end{cases}$$

with initial value $\zeta(\bar{t}) = \bar{x}$.

For genuine characteristics, this implies that $\zeta(t)$ and $U(t) := u(\zeta(t), t)$ satisfy the characteristic equation

$$\left.\begin{array}{rcl} U'(t) & = & \frac{1}{\varepsilon} U^\alpha (1 - U^2)^\beta \\ \\ \zeta'(t) & = & f'(U(t)) \end{array}\right\} \qquad (5)$$

Definition 2.3 *Given a solution u of the Cauchy problem (3), a* **backward characteristic** *is a solution of the characteristic equation (5) with terminal condition*

$$\begin{array}{rcl} \zeta(T) & = & \zeta_{end}, \\ U(T) & = & U_{end} \in [u(\zeta(T)+, T), u(\zeta(T)-, T)]. \end{array}$$

All backward characteristics are confined between the **maximal and minimal backward characteristics**. *These extremal backward characteristics are the backward characteristic curves with $U(T) = u(\zeta(T)-, T)$ and $U(T) = u(\zeta(T)+, T)$.*

The following properties make generalized characteristics an important tool:

Proposition 2.4 (Dafermos)

(i) Genuine characteristics can intersect only at their endpoints.

(ii) The maximal and minimal backward characteristic are defined on $[0, T]$ and are genuine characteristics.

Note that this implies the following: If we can construct a solution of the Cauchy problem by filling a region $S^1 \times [0, T]$ in (x, t)-space completely with non-intersecting characteristics such that all maximal backward characteristics extend to $t = 0$, then this must actually be the solution of the Cauchy problem up to time T.

2.2 The Riemann problem

We fix $\varepsilon = 1$ and collect some basic properties of solutions to the Riemann problem. Given two states u_-, $u_+ \in [-1, 1]$, we study solutions of (3) with initial condition

$$u_0(x) = \begin{cases} u_- & \text{for } x < 0 \\ u_+ & \text{for } x > 0 \end{cases}$$

As in the conservation law case we distinguish the two cases $u_- > u_+$ and $u_- < u_+$. In the first situation, there will be a shock curve emanating from $x = 0$. This shock is not a straight line but a curve $x = \xi_s(t)$ since the left and right states $u_-(t)$ and $u_+(t)$ evolve according to the reaction dynamics

$$\frac{du_\pm}{dt} = u_\pm^\alpha (1 - u_\pm^2)^\beta. \qquad (6)$$

From the Rankine-Hugoniot condition we conclude that the shock location ξ_s is the solution of the initial value problem

$$\xi_s'(t) = \frac{f(u_+(t)) - f(u_-(t))}{u_+(t) - u_-(t)}$$

with initial condition $\xi_s(0) = 0$. Since all solutions of (6) converge to -1, 0 or $+1$, the shock speed converges to $f'(-1)$, $(f(1) - f(-1))/2$ or $f'(1)$ depending on the signs of u_- and u_+.
In the case $u_- < u_+$ the solution of the Riemann problem is similar to a rarefaction wave of a conservation law without source term. The solution is most easily obtained by solving the characteristic equations (5) with $\varepsilon = 1$

$$\left.\begin{array}{rcl} U'(t) & = & U^\alpha(1-U^2)^\beta \\ \zeta'(t) & = & f'(U(t)) \end{array}\right\} \tag{7}$$

for $U(t) = u(t, \zeta(t))$ with initial values

$$\begin{array}{ll} U(0) = u_-, & \zeta(0) \in (-\infty, 0] \\ U(0) \in [u_-, u_+], & \zeta(0) = 0 \qquad\qquad \text{and} \\ U(0) = u_+, & \zeta(0) \in [0, +\infty). \end{array}$$

By continuous dependence on initial data, the union of all these characteristics fills the whole (x,t)-plane. As no two of those characteristics intersect, the solution along these characteristics provides the correct solution to the Riemann problem.
The profile of these rarefaction waves at a fixed time $t > 0$ looks different from the profile when no source term is present. In particular, for any given $\delta < 1$, the measure of the set where $|u(x,t)| \leq \delta$ grows sublinearly with t, if $\alpha \geq 1$ and if the left and right state have a different sign.
Independent of the signs of u_- and u_+ we have the following monotonicity result:

Lemma 2.5 *Let u be the solution of the Riemann problem with $u_- < u_+$. Then for any fixed time $t \geq 0$ the profile $u(x,t)$ is monotone in x.*

Proof: Note first that $u(x,t) \equiv U_-(t)$ for $x \leq \zeta_-(t)$ where (U_-, ζ_-) solve the characteristic equation (7) with $U_-(0) = u_-$ and $\zeta_-(0) = 0$. Similarly, $u(x,t) \equiv U_+(t)$ for $x \geq \zeta_+(t)$ with (U_+, ζ_+) being the solution of (7) with $U_+(0) = u_+$ and $\zeta_+(0) = 0$. The characteristic curves $\zeta_\pm$ are the minimal and maximal forward characteristics emanating from the point $x = t = 0$. Therefore, if the profile at some time $T > 0$ is not monotone, we can find $\zeta_-(T) \leq x_1 < x_2 \leq \zeta_+(T)$ with

$$u(x_1, T) > u(x_2, T).$$

We consider the minimal backward characteristics $\zeta_1(t)$ emanating from (x_1, T) and $\zeta_2(t)$ emanating from (x_2, T) together with the corresponding functions $U_1(t) :=$

$u(\zeta_1(t),t)$ and $U_2(t) := u(\zeta_2(t),t)$. Since minimal backward characteristics are genuine by proposition 2.4 both (U_1,ζ_1) and (U_2,ζ_2) satisfy the characteristic equation (7). Therefore

$$U_1(t) > U_2(t) \ \text{ for all } \ 0 \le t \le T$$

and hence by convexity of f we conclude that $\zeta_2(t) - \zeta_1(t)$ is a decreasing function of t. In particular $\zeta_2(0) > \zeta_1(0)$ which contradicts the fact that for the Riemann problem all backward characteristics starting with $\zeta(T) \in [\zeta_-(T), \zeta_+(T)]$ must have $\zeta(0) = 0$. From this contradiction it follows that $u(x,T)$ is an increasing function of x.

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Next we show that in the case where the left and right state have different sign, the region where the solution of the Riemann problem is close to 0 is smaller than in the case when $g(u) \equiv 0$.Moreover, the growth rate of such a region depends on the size of α. For $1 < \alpha < 2$ the behavior is as in the previously studied case $\alpha = 1$, but for $\alpha \ge 2$the rates are different.

Lemma 2.6 *Fix $\varepsilon = 1$ and assume $u_- < 0 < u_+$ and $\alpha > 1$. Then for any given $0 < \delta < min(-u_-, u_+)$ there exists a constant $M = M(\alpha,\delta) > 0$ independent of $u_\pm$ and a time t_* such that*

$$|u(x,t)| \ge \delta \ \ for \ \ t \ge t_* \ \ and \ \left\{ \begin{array}{ll} |x| \ge M & if \ 1 < \alpha < 2 \\ |x| \ge M \log(1+\delta t) & if \ \alpha = 2 \\ |x| \ge M t^{\frac{\alpha-2}{\alpha-1}} & if \ \alpha > 2. \end{array} \right.$$

In particular

$$\lim_{t\to\infty} \frac{meas\{x \ ; \ |u(x,t)| \le \delta\}}{t} = 0$$

for all $\alpha \ge 1$ and any $\delta \in [0,1)$.

Proof: We restrict our considerations to $x \ge 0$ corresponding to characteristic curves with $U(0) \ge 0$. The situation for $x < 0$ can be handled similarly by changing signs appropriately.
It is our goal to provide lower and upper estimates for the solution U of the characteristic equation (7) and the corresponding characteristic curve ζ.
Let $U(t;U_0)$ be the solution to the initial value problem

$$\begin{aligned} U'(t) &= U^\alpha(1-U^2)^\beta \\ \zeta'(t) &= f'(U(t)) \\ U(0) = U_0, & \qquad \zeta(0) = 0. \end{aligned}$$

Given some time $T > 0$, we will first find estimates for the initial condition U_0 such that $U(T;U_0) = \delta$.

As long as $0 \le u \le \delta$

$$0 \le u^\alpha (1-\delta^2)^\beta \le u^\alpha (1-u^2)^\beta \le u^\alpha$$

holds. By elementary integration, we find that

$$U_0^{1-\alpha} - U(t;U_0)^{1-\alpha} \le (\alpha-1)t \le (1-\delta^2)^{-\beta}\left(U_0^{1-\alpha} - U(t;U_0)^{1-\alpha}\right), \qquad (8)$$

which implies that in order to have $U(T;U_0) = \delta$, the initial value U_0 has to satisfy the estimate

$$\left(\delta^{1-\alpha} + T(\alpha-1)\right)^{\frac{1}{1-\alpha}} \le U_0 \le \left(\delta^{1-\alpha} + T(\alpha-1)(1-\delta^2)^\beta\right)^{\frac{1}{1-\alpha}}. \qquad (9)$$

The characteristic curve $\zeta(t;U_0)$ satisfies

$$\begin{aligned} \zeta(t;U_0) &= \int_0^t f'(U(\tau))\, d\tau \\ &= \int_{U_0}^{U(t;U_0)} \frac{f'(v)}{v^\alpha(1-v^2)^\beta}\, dv \\ &\le \int_{U_0}^{U(t;U_0)} \frac{C_f}{v^{\alpha-1}(1-v^2)^\beta}\, dv \end{aligned}$$

for some constant $C_f > 0$. Since $U(T;U_0) = \delta$ we have

$$\zeta(T;U_0) \le \int_{U_0}^{\delta} \frac{C_f}{v^{\alpha-1}(1-\delta^2)^\beta}\, dv.$$

Now we have to distinguish three cases: For $1 < \alpha < 2$ the integration yields

$$\begin{aligned} \zeta(T;U_0) &\le \frac{C_f}{(2-\alpha)(1-\delta^2)^\beta}\left(\delta^{2-\alpha} - U_0^{2-\alpha}\right) \\ &\le \frac{C_f}{(2-\alpha)(1-\delta^2)^\beta}\left(\delta^{2-\alpha} - (\delta^{1-\alpha} + (\alpha-1)T)^{\frac{2-\alpha}{1-\alpha}}\right) \end{aligned}$$

using (9). Since $\frac{2-\alpha}{1-\alpha} < 0$ the right hand side converges to some limit as $T \to +\infty$ and therefore $\zeta(T;U_0) < M$ for some constant M. However, U_0 was chosen in a way that $U(T;U_0) = \delta$. Hence by the monotonicity proved in the previous lemma we know that $u(x,T) > \delta$ for $x > M > \zeta(T)$. Choosing $t_* = 0$ proves the lemma for $1 < \alpha < 2$.
For $\alpha = 2$ the same reasoning yields

$$\zeta(T;U_0) \le \frac{C_f}{(1-\delta^2)^\beta}\log(1+\delta T)$$

and again we may choose $t_* = 0$.

For $\alpha > 2$ the estimate is

$$\begin{aligned}\zeta(T;U_0) &\leq \frac{C_f}{(\alpha-2)(1-\delta^2)^\beta}\left(U_0^{2-\alpha}-\delta^{2-\alpha}\right)\\ &\leq \frac{C_f}{(\alpha-2)(1-\delta^2)^\beta}\left((\delta^{1-\alpha}+(\alpha-1)T)^{\frac{\alpha-2}{\alpha-1}}-\delta^{2-\alpha}\right)\\ &= \mathcal{O}(T^{\frac{\alpha-2}{\alpha-1}}).\end{aligned}$$

By choosing t_* and M large enough, this proves the estimate for $\alpha > 2$.

$\bowtie$

2.3 Traveling waves

In this chapter we classify all possible traveling wave solutions. Since we deal with possibly discontinuous entropy solutions of our equation, we have to adopt the notion of traveling waves to this case.

Definition 2.7 *An* **entropy traveling wave** *solution of the hyperbolic balance law is a piecewise C^1 solution of the form $u(x,t) = u(x-ct)$ that satisfies the ordinary differential equation $(f'(u)-c)u' = u^\alpha(1-u^2)^\beta$ at all points where u is differentiable. Moreover, at any point ξ where u is discontinuous the Rankine-Hugoniot condition*

$$f(u(\xi+)) - f(u(\xi-)) = c(u(\xi+) - u(\xi-))$$

and the entropy condition

$$u(\xi+) \leq u(\xi-)$$

for the one-sided limits $u(\xi-)$ and $u(\xi+)$ hold.

It turns out that the case $1 \leq \alpha < 2$ is very similar to the situation for $\alpha = 1$ and can be treated as in [9] while the case $\alpha \geq 2$ behaves quite differently. We therefore concentrate on the latter case.

Lemma 2.8 *For $\alpha \geq 2$, any entropy traveling wave can have at most one point of discontinuity.*

Proof: By the mean value theorem, $c = f'(u_c)$ for some u_c between $u(\xi+)$ and $u(\xi-)$. The entropy condition then yields $u(\xi-) > u_c > u(\xi+)$. Therefore a traveling wave solution would have to pass continuously through u_c between any two shocks. This is impossible since we have

$$u' = \frac{u^\alpha(1-u^2)^\beta}{f'(u)-c} = 0$$

at $u = u_c$ as the right hand side has a removable singularity at $u = u_c$ and is Lipschitz-continuous. It follows from the basic existence and uniqueness theorem for ordinary differential equations that any solution u that satisfies $u(\xi) = 0$ for some ξ vanishes identically. Consequently, there are no traveling waves with more than one shock.

$\bowtie$

Lemma 2.9 *For $\alpha \geq 2$ all bounded entropy traveling waves are heteroclinic traveling waves, i.e. they converge to some steady state for $\xi \to -\infty$ and to some steady state for $\xi \to +\infty$.*

Proof: Since there can be at most one point of discontinuity, the solution is a solution of the one-dimensional traveling wave o.d.e.

$$u' = \frac{u^\alpha (1-u^2)^\beta}{f'(u) - c} \tag{10}$$

for ξ sufficiently large. Therefore it can be bounded only if it converges to some equilibrium for $\xi \to +\infty$. The same argument applies for ξ sufficiently negative.

$\bowtie$

It is a simple matter to characterize the continuous traveling waves that connect the states ± 1 and 0 so we state the result without proof.

Proposition 2.10 (Continuous traveling waves) *Let $\alpha \geq 2$.*

(i) A continuous entropy traveling wave from -1 to 0 exists iff $c \geq f'(0)$. A continuous entropy traveling wave from 0 to -1 exists iff $c \leq f'(-1)$.

Similarly, continuous entropy traveling wave from 0 to $+1$ exist for $c \leq f'(0)$ and from $+1$ to 0 for $c \geq f'(1)$.

(ii) There exist no continuous traveling waves connecting -1 to $+1$ or vice versa.

The existence of discontinuous entropy traveling waves depends again on the wave speed. By condition **(F2)** and **(F3)** of f, given any c and u, there is a unique state which we call $h(u,c)$ such that u and $h(u,c)$ satisfy the Rankine-Hugoniot condition

$$f(u) - cu = f(h(u,c)) - ch(u,c)$$

with shock speed c.

Lemma 2.11 (Discontinuous traveling waves) *The following three types of discontinuous traveling waves are possible for $\alpha \geq 2$:*

(i) If h(-1,c)=1, i.e. $c = c_{RH}$ then there exists a traveling shock wave of the form

$$u(\xi) = \begin{cases} +1 & \text{for } \xi \leq 0 \\ -1 & \text{for } \xi > 0 \end{cases}$$

(ii) *If* $f'(-1) < c \leq 0$ *and* $h(-1, c) < 1$ *then there is an entropy traveling wave that converges to* 0 *as* $\xi \to -\infty$ *is monotone for* $\xi \leq 0$ *and satisfies* $u(\xi) \equiv 1$ *for* $\xi > 0$.

(iii) *If* $0 \leq c < f'(1)$ *and* $h(1, c) > -1$ *then there is an entropy traveling wave that converges to* 0 *as* $\xi \to +\infty$ *is monotone for* $\xi > 0$ *and satisfies* $u(\xi) \equiv -1$ *for* $\xi \leq 0$.

Proof: Without restriction we suppose that the heteroclinic entropy traveling waves possess exactly one discontinuity at $\xi = 0$. This implies that $c \in [f'(-1), f'(1)]$ because if $c > f'(1)$ then $u(0-) > (f')^{-1}(c) > 1$ by convexity of f and hence $u(\xi) > (f')^{-1}(c)$ for all $\xi < 0$. This however contradicts the fact that $u(\xi)$ must converge to an equilibrium as $\xi \to -\infty$. The same argument shows that it is impossible to have $c < f'(-1)$.
For the traveling wave equation (10) and $c \in [f'(-1), f'(1)]$ the equilibrium $u = 1$ is always stable while $u = -1$ is unstable. Therefore a connection from $+1$ to -1 can only exist if $+1$ and -1 satisfy the Rankine-Hugoniot condition. This is case (i).
For $f'(-1) < c < f'(0) = 0$ the equilibria $u = -1$ and $u = 0$ are both unstable for the traveling wave equation (10). A solution is therefore only possible if the point $h(-1, c)$ is between $(f')^{-1}(c)$ and 1. This is exactly the situation (ii). The same arguments apply if $0 = f'(0) \leq c < f'(1)$ and lead to case (iii).

$\bowtie$

From the previous lemmas we now obtain the
Proof of theorem 1.1:
(i) is the content of lemma 2.10,
(ii) and (iii) are proved in lemma 2.11,
(iv) follows immediately from the previous results together with lemma 2.9.

$\bowtie$

Remark 2.12 *We emphasize again that this situation is in contrast to the case* $1 \leq \alpha < 2$ *where an abundance of discontinuous traveling waves can be found.*

3 Convergence Results

In this section we proof that for $\alpha \geq 2$ all solutions of the Cauchy problem (3) with $x \in S^1 \cong \mathbb{R}/\mathbb{N}$ converge pointwise to one of the zeroes of the source term. Again, the case $1 < \alpha < 2$ is very similar to the case $\alpha = 1$ and will not be considered here. To decide whether a particular solution converges to $+1$, -1 or 0, the following definition will prove useful.

Definition 3.1 *Given the solution* $u(t, x)$ *of the Cauchy problem we define the* **range** *of the solution as*

$$\mathcal{R}(u(\cdot, t)) := \{u(x-, t) \; ; \; x \in S^1\} \cup \{u(x+, t) \; ; \; x \in S^1\}.$$

In [2] it was proved that for $t > 0$ the range has to be connected, since u can only jump downward at discontinuities by the entropy condition.

Proposition 3.2 ([2]) *For $x \in S^1$ and any $t > 0$, $\mathcal{R}(u(\cdot, t))$ is a closed interval.*

The important property that distinguishes the case of $\alpha \geq 2$ from the previously considered case $\alpha = 1$ is stated next.

Lemma 3.3 *Let u be a solution of (3) with $\alpha \geq 2$ and assume that for some time $T > 0$ we have $0 \in \mathcal{R}(u(\cdot, T))$. Then there exists some function $\eta = \eta(T)$ with $\lim_{T\to+\infty} \eta(T) = 0$ such that*

$$\mathcal{R}(u(\cdot, T)) \subseteq [-\eta(T), \eta(T)].$$

Proof: As $0 \in \mathcal{R}(u(\cdot, T))$ there exists some $x_0 \in S^1$ such that $u(x_0-, T) = 0$ or $u(x_0+, T) = 0$. In either case, $\zeta_0 = \{(x, t) \; ; \; x = x_0, 0 \leq t \leq T\}$ is an extremal backward characteristic and therefore must be a genuine characteristic. In particular,

$$u(x_0-, t) = 0 \;\text{ for all }\; 0 \leq t < T.$$

We will now show that if $u(x_1, T) > \eta(T)$ for some $\eta = \eta(T)$ and some $x_1 \in S^1$ then the minimal backward characteristic $\zeta_1(\cdot; x_1, T)$ satisfies

$$\zeta_1(T; x_1, T) - \zeta_1(0; x_1, T) > 1. \tag{11}$$

Since x is on the circle $S^1 \sim \mathbb{R}/\mathbb{N}$, ζ_1 must intersect the genuine backward characteristic ζ_0 at some positive time. This however contradicts proposition 2.4 and therefore proves that $u(x, T) < \eta(T)$ for all x. A similar argument shows that $u(x, T) > -\eta(T)$ for all x.
To show (11) we distinguish two cases:

(i) $\alpha = 2$:

In this case, we set $\eta(T) := 1/(CT)$ with a constant C that will be determined later and assume that $u(x_1, T) > \eta(T)$ for some $x_1 \in S^1$. Let (U_1, ζ_1) be the solution of the (backward) characteristic equation (7) with terminal values $U_1(T) = u(x_1, T)$ and $\zeta_1(T) = x_1$. By comparison with the equation

$$U' = U^\alpha$$

we find that

$$U_1(t) \geq (CT + (T - t))^{-1}$$

and hence

$$\zeta_1(T) - \zeta_1(0) \;\; = \;\; \int_0^T f'(U_1(t)) \; dt$$

$$
\begin{aligned}
&\geq f''(0)\int_0^T U_1(t)\; dt \\
&\geq f''(0)\int_0^T \frac{dt}{CT+(T-t)} \\
&= f''(0)\log\left(\frac{C+1}{C}\right) \\
&> 1
\end{aligned}
$$

if $C < (\exp(1/f''(0)) - 1)^{-1}$ which proves (11). Completely analogous arguments show that $u(x_1, T) < -\eta(T)$ leads to $\zeta_1(T) - \zeta_1(0) < -1$. Noting that $\eta(T) \to 0$ as $T \to \infty$ completes the proof for the case $n = 2$.

(ii) $\alpha > 2$:

In this case we choose η differently and let $\eta(T)$ be the solution at time T of the initial value problem

$$
\begin{aligned}
U'(t) &= U^\alpha(1-U^2)^\beta, \\
U(0) &= (f''(0)T)^{-1}.
\end{aligned}
$$

Again we assume that $u(x_1, T) > \eta(T)$ for some $x_1 \in S^1$ and denote with (U_1, ζ_1) the solution of the (backward) characteristic equation. Since U_1 solves the same differential equation as U we get that $U_1(t) > U(t) \geq U(0) = (f''(0)T)^{-1}$ for $0 \leq t \leq T$. Using the convexity of f, this implies that

$$
\begin{aligned}
\zeta(T) - \zeta(0) &= \int_0^T f'(U_1(t))\; dt \\
&\geq f''(0)\int_0^T U_1(t)\; dt \\
&> 1
\end{aligned}
$$

which contradicts proposition 2.4. Again, the assumption $u(x_1, t) < -\eta(T)$ leads to the same contradiction.

It remains to show that η converges to zero as $T \to +\infty$. This follows immediately by comparison with the initial value problem

$$
\bar{U}'(t) = \bar{U}^\alpha, \;\; \bar{U}(0) = \frac{1}{Tf''(0)}
$$

which can be solved explicitly and which yields

$$
\eta(T) \leq \bar{U}(T) = \left((f''(0)T)^{\alpha-1} - (\alpha-1)T\right)^{\frac{1}{1-\alpha}}.
$$

It is easy to check that the right hand side converges to 0 as $T \to +\infty$.

$\bowtie$

Now we can prove theorem 1.2 on the convergence of solutions as $t \to +\infty$.

Proposition 3.4 *For any initial data u_0 with $u_0(x) \in [-1, 1]$ for all $x \in S^1$ there exists a number $\omega = \omega(u_0) \in \{-1, 0, +1\}$ such that*

$$\|u(x,t) - \omega(u_0)\|_{L^\infty} \to 0$$

as $t \to \infty$.

Proof: We distinguish two cases:
1) $0 \notin \mathcal{R}(u(\cdot, T))$ for some time $T > 0$. Then either $\mathcal{R}(u(\cdot, T)) \in (0, +\infty)$ or $\mathcal{R}(u(\cdot, T)) \in (-\infty, 0)$. For definiteness, we assume the first possibility. Since the range is a closed interval, we can find $\kappa > 0$ such that $u(x,T) \geq \kappa$ for all $x \in S^1$. Applying the comparison principle to the solution u and the solution of the Cauchy problem with $u \equiv \kappa$ at time $t = T$ we can then conclude that $u(x,t) \to 1$ as $t \to +\infty$ for a.e. $x \in S^1$. Similarly, if $\mathcal{R}(u(\cdot, T)) \in (-\infty, 0)$ we have $u(x,t) \to -1$ for $t \to +\infty$.
2) $0 \in \mathcal{R}(u(\cdot, t))$ for all times $t > 0$. We prove that in this case the solution converges to zero pointwise. This is just a consequence of the previous lemma, since for any $\delta > 0$ we can find some $T > 0$ such that $\eta(t) < \delta$ for all $t \geq T$. Lemma 3.3 then implies that

$$|u(x,t)| = |u(x_-, t)| \leq \eta(t) \leq \delta \text{ for all } x \in S^1 \text{ and } t \geq T.$$

and hence

$$\|u(\cdot, t)\|_{L^\infty} \to 0 \text{ as } t \to +\infty.$$

$\bowtie$

3.1 An example

We illustrate by a simple example that convergence to the middle zero actually occurs. To this end, we choose $f(u) = u^2$ and consider the initial condition

$$u_0(x) = \begin{cases} +1 & \text{for } 0 < x \leq \frac{1}{2} \\ -1 & \text{for } \frac{1}{2} < x \leq 1. \end{cases}$$

By symmetry, if $(U(t), \zeta(t))$ solves the characteristic equation, so does $(-U(t), 1 - \zeta(t))$. These two characteristics meet at $x = \frac{1}{2}$ and by the Rankine-Hugoniot condition, a standing shock is formed at $x = \frac{1}{2}$. For $0 < t \leq 1$ the forward characteristics with value $U(t) = \pm 1$ meet at the shock, but for $t > 1$ the L^∞-norm of the solution decreases. Since all characteristics are symmetric with respect to $x = \frac{1}{2}$ the solution can neither converge to $+1$ nor to -1 and so it must tend to 0.

4 Instability under perturbations

As we have seen in the previous chapter, any solution tends to one of the zeroes of the source term as t tends to infinity. However, most of the solutions will tend to the stable zeroes $u = +1$ and $u = -1$ of the source term, while convergence to the weakly unstable zero $u = 0$ is very exceptional. In this chapter we study the basins of attraction for the three zeroes of the source term and prove theorem 1.3.

Definition 4.1 *Let $\mathcal{B}_{-1}$, $\mathcal{B}_0$ and $\mathcal{B}_1 \subseteq L^1(S^1, \mathbb{R})$ be the sets of those initial conditions for which the solution of the Cauchy problem (3) converges to -1, 0 and 1 pointwise as $t \to +\infty$.*

We will show that for any $u_0 \in \mathcal{B}_0$ we can find an arbitrarily small perturbation v_0 such that $u_0 + v_0 \in \mathcal{B}_1$. To construct a suitable perturbation, we first need some additional notation:

Definition 4.2 *For $u_0 \in L^1(S^1) \cap BV(S^1)$ and $\delta > 0$ we set*

$$u_0^\delta(x) := sup_{|x-y|\leq\delta}\, u_0(y) + \delta.$$

The function u_0^δ has the following properties:

Lemma 4.3

(i) $u_0^\delta > u_0$

(ii) $\|u_0^\delta - u_0\|_{L^1(S^1)} \leq \delta(1 + 3TV(u_0))$ *where TV is the total variation of u_0.*

(iii) For almost all $\delta > 0$ the functions u_0 and u_0^δ have no common point of discontinuity, i.e.

$$u_0(x-) \neq u_0(x+) \Rightarrow u_0^\delta(x-) = u_0^\delta(x+).$$

Proof: (i) follows immediately from the definition of u_0^δ.
(ii) Consider the partition $0 = x_1 < x_2 < \ldots < x_N = 1$ of S^1 with $x_{i+1} - x_i < \delta$. Note that by definition of u_0^δ for any $x \in [x_i, x_{i+1}]$

$$0 \leq u_0^\delta(x) - u_0(x) \leq \sup_{y\in[x-\delta,x+\delta]} u_0(y) + \delta - \inf_{y\in[x-\delta,x+\delta]} u_0(y)$$

and hence

$$\begin{aligned}\int_{x_i}^{x_{i+1}} |u_0^\delta(x) - u_0(x)|\, dx &\leq \delta TV(u_0, [x_i - \delta, x_{i+1} + \delta]) + \delta(x_{i+1} - x_i)\\ &\leq \delta\, TV(u_0, [x_{i-1}, x_{i+2}]) + \delta(x_{i+1} - x_i)\end{aligned}$$

where $TV(u_0, [a, b])$ is the total variation of u_0 on the interval $[a, b]$. Summation over i then yields the desired inequality.
(iii) Note first that if u_0^δ is discontinuous at some $x_0 \in S^1$, then u_0 is discontinuous

at $x_0-\delta$ or at $x_0+\delta$ because if u_0 was continuous both at $x_0-\delta$ and $x_0+\delta$ then the supremum of u_0 over $[x-\delta, x+\delta]$ also had to be continuous at x_0. As u_0 is in BV, it possesses at most countably many discontinuities, and the set of all differences between two of these discontinuities is countable, too. This implies that only for countably many δ the function u_0 is discontinuous simultanously at some x_0 and $x_0-\delta$. Similarly, there are only countably many δ such that u_0 is discontinuous at some x_0 and $x_0+\delta$. For all other values of δ there is no point where both functions u_0 and u_0^δ are discontinuous.

$\bowtie$

The importance of property (iii) is due to the following lemma:

Lemma 4.4 *Assume that $u_0, v_0 \in \mathcal{B}_0$ and $u_0 \le v_0$. Then for any $\eta_0 > 0$ the set*

$$\{x \in S^1; [u_0(x-), u_0(x+)] \cap [v_0(x-), v_0(x+)] \cap [-\eta_0, \eta_0] \ne \emptyset\}$$

is non-empty.

Proof: We consider the solutions u and v associated with the initial conditions u_0 and v_0.
For any solution w of (3) we define the spatial average

$$I(w(\cdot,t)) := \int_{S^1} w(x,t)\, dx.$$

From the weak formulation of the hyperbolic balance law, one can derive the ordinary differential equation

$$\frac{d}{dt} I(w) = \int_{S^1} w^\alpha (1-w^2)^\beta \, dx$$

for $I(w)$. In view of the previous section both $I(u(\cdot,t))$ and $I(v(\cdot,t))$ must converge to 0 as $t \to \infty$.
Let J be the maximal interval where the source term $u^\alpha(1-u^2)^\beta$ is monotone increasing. From lemma 3.3 we know that there exists some time $T > 0$ such that both $\mathcal{R}(u(\cdot,T)) \subseteq J$ and $\mathcal{R}(v(\cdot,T)) \subseteq J$.
If $I(v(\cdot,T)) > I(u(\cdot,T))$ then $I(v(\cdot,t))$ cannot converge to 0 because for all $t > T$ we have

$$\frac{d}{dt} I(v(\cdot,t)) \ge \frac{d}{dt} I(u(\cdot,t))$$

and hence $I(v(\cdot,t)) - I(u(\cdot,t)) \ge I(v(\cdot,T)) - I(u(\cdot,T)) > 0$.
This implies that at time $t = T$ we must have $I(v(\cdot,T)) \le I(u(\cdot,T))$. Since $u \le v$ by Kruzhkov's comparison principle, this implies that $I(v(\cdot,T)) = I(u(\cdot,T))$ and $u(x,T) = v(x,T)$ for almost every $x \in S^1$.
Consider now the extremal backward characteristics (U, ζ) evolving from some point with $t = T$. By proposition 2.4, the characteristic curve ζ extends back to $t = 0$ and is a genuine characteristic both for u and v. In particular, $u(\zeta(t),t) = U(t) = v(\zeta(t),t)$ for all $t \in (0,T]$. This implies that $u(\zeta(0)+,0) \le U(0) \le u(\zeta(0)-,0)$ and

$v(\zeta(0)+,0) \leq V(0) \leq u(\zeta(0)-,0)$. So, either $u_0(\zeta(0)-) = u_0(\zeta(0)+)$ or $u_0(\zeta(0)-) < u_0(\zeta(0)+)$ with $U(0) \in [u_0(\zeta(0)-), u_0(\zeta(0)+)]$.
Choosing T possibly larger and using lemma 3.3, we can make sure that $|U(0)| \leq \eta_0$ holds and hence both possibilities imply that

$$[-\eta_0, \eta_0] \cap [u_0(\zeta(0)-), u_0(\zeta(0)+)] \neq \emptyset.$$

Since the same holds for v_0 the lemma is proved.

$\bowtie$

We are now able to prove theorem 1.3.
Proof of theorem 1.3: Assume that $u_0 \in \mathcal{B}_0$. We will show that for δ sufficiently small u_0 and u_0^δ cannot satisfy the assumptions of the previous lemma. From lemma 4.3 we know that $u_0^\delta \geq u_0 + \delta$ and $u_0^\delta \to u_0$ in $L^1(S^1)$ as $\delta \to 0$. Therefore, $u_0^\delta \in \mathcal{B}_1$ for all δ small.
Given δ, assume now that $\eta_0 < \frac{\delta}{3}$ and $u_0^\delta \in \mathcal{B}_0$. The previous lemma states that there must exist at least one x_0 with

$$[u_0(x_0-), u_0(x_0+)] \cap [u_0^\delta(x_0-), u_0^\delta(x_0+)] \cap [-\eta_0, \eta_0] \neq \emptyset. \tag{12}$$

However by lemma 4.3 we know that u_0 and u_0^δ have no common point of discontinuity. So, there are three possibilities:
1) If $u_0(x_0-) = u_0(x_0+)$ and $u_0^\delta(x_0-) = u_0^\delta(x_0+)$ then (12) is not possible as $u_0^\delta - u_0 \geq \delta > 2\eta_0$.
2) Similarly, for $u_0(x_0-) < u_0(x_0+)$ and $u_0^\delta(x_0-) = u_0^\delta(x_0+)$ (12) is not possible as $u_0^\delta(x_0) \geq u_0(x_0+) + \delta$.
3) For $u_0(x_0-) = u_0(x_0+)$ and $u_0^\delta(x_0-) < u_0^\delta(x_0+)$ the same argument applies.
So, in all three possible cases, (12) cannot be satisfied and therefore our assumption that $u_0^\delta \in \mathcal{B}_0$ has led to a contradiction. This shows that $u_0^\delta \in \mathcal{B}_1$ for all small δ. To show that there are initial data near u_0 for which the solution converges to -1 one has to perform a similar construction as for u_0^δ to get a function which is strictly smaller than u_0 and has no common points of discontinuity with u_0. The construction of such a function is completely analogous to the construction of u_0^δ.

$\bowtie$

Sinestrari has proved in [13](theorem 4.7) that $\mathcal{B}_{-1}$ and $\mathcal{B}_1$ are open in $L^1(S^1)$. Since his proof only requires convergence to a stable zero of the source term in L^∞, it applies immediately to our situation. As the previous lemma states that $\mathcal{B}_0$ is nowhere dense we have the following corollary.

Corollary 4.5 *The set of initial conditions u_0 for which the solution of the Cauchy problem (3) converges to -1 or $+1$ is open and dense in $L^1(S^1)$.*

4.1 An example

To illustrate the convergence to 0 we study as an example the behaviour of solutions of the Cauchy problem for which the initial condition is a step-function

$$u_\eta(x) = \begin{cases} u_- & \text{for} \quad 0 < x \leq \eta \\ u_+ & \text{for} \quad \eta < x \leq 1 \end{cases}$$

where $u_- > 0 > u_+$ for definiteness and η is treated as a parameter.

Lemma 4.6 *There exists a unique value $\eta_0 = \eta_0(u_-, u_+) \in (0, 1]$ such that $u_{\eta_0} \in \mathcal{B}_0$. For $\eta_0 < \eta \leq 1$ we have $u_\eta \in \mathcal{B}_1$ and for $0 < \eta < \eta_0$ we have $u_\eta \in \mathcal{B}_{-1}$.*

Proof: The u_η are ordered:

$$\eta_1 \leq \eta_2 \Rightarrow u_{\eta_1} \leq u_{\eta_2}.$$

Therefore if $u_{\eta_1} \in \mathcal{B}_1$ for some η_1 we have automatically $u_\eta \in \mathcal{B}_1$ for all $\eta \in [\eta_1, 1]$ by the comparison principle. Similarly, if $u_{\eta_2} \in \mathcal{B}_{-1}$ for some η_2 we have $u_\eta \in \mathcal{B}_{-1}$ for all $\eta \in [0, \eta_2]$. Therefore we can find $\underline{\eta} := \sup\{\eta\ ;\ u_\eta \in \mathcal{B}_1\}$ and $\bar{\eta} := \inf\{\eta\ ;\ u_\eta \in \mathcal{B}_{-1}\}$. Since $\mathcal{B}_1$ and $\mathcal{B}_{-1}$ are open in L^1 we know that $u_\eta \in \mathcal{B}_0$ for $\eta \in [\underline{\eta}, \bar{\eta}]$. To show that $\underline{\eta} = \bar{\eta}$ we just note that $u_{\underline{\eta}}(0)$ and $u_{\bar{\eta}}(0)$ obviously do not satisfy the condition of lemma 12 if $\underline{\eta} < \bar{\eta}$ and hence cannot both belong to $\mathcal{B}_0$.

$\bowtie$

5 Convergence in the zero reaction time limit

In this section, we shall consider the limit $\varepsilon \to 0$ and prove theorem 1.4.
Recall that by (4) we require the initial condition u_0 to be differentiable with simple zeroes which are at least a distance of ρ apart from each other.
This includes in particular the case that u_0 is periodic with simple zeroes and allows therefore a comparison with the long-time limit results.
The following lemma on the structure of the solutions of (3) states that the number of sign changes of the solution is non-increasing in time.

Lemma 5.1 ([4], Lemma 3.2.1) *For fixed ε let $u_\varepsilon(x,t)$ be the solution of (3) with initial value $u_0(x)$ satisfying assumption (4). Then at each fixed $t \geq 0$, there are points $z_j^\varepsilon(t)$, $j \in \mathcal{J}' \subseteq \mathcal{J}$, such that changes of sign$(u_\varepsilon(x,t))$ occur and only occur when x crosses one of the curves $x = z_j^\varepsilon(t)$, $t < t_j$.*
Moreover, the $z_j^\varepsilon(t)$ are curves defined on $[0, T_j^\varepsilon)$, $j \in \mathcal{J}' \subseteq \mathcal{J}$.

Proof: This lemma is similar to Lemma 3.2.1 in [4] and the remark thereafter. Although the lemma in [4] is for the case where $\mathcal{J}$ is finite and the source term in (3) is $u(1-u^2)/\varepsilon$, corresponding to $\alpha = \beta = 1$, the proof of it only used the properties that two extremal backward characteristics of (3) do not intersect and that along extremal backward characteristics, the sign of u_ε does not change. Since these two properties still hold for (3), the lemma holds for arbitrary α, $\beta \geq 1$.

$\bowtie$

The following lemma, proved in [4] for $\alpha = \beta = 1$ still holds for (3) since the proof in [4] did not use anything related to the source term.

Lemma 5.2 *Let $u_0(x)$ satisfy the assumption (4). Then the curves $z_j^\varepsilon(t)$, $j \in \mathcal{J}$ given in Lemma 5.1 are Lipschitzian with Lipschitz constant $L \leq \max_{u \leq sup|u_0(x)|} |f'(u)|$ independent of ε.*
Moreover, if the domain of definition of $z_j^\varepsilon(t)$ is $[0, T_j^\varepsilon]$ with $T_j^\varepsilon < \infty$, then there is another curve $z_{j'}^\varepsilon(t)$ intersecting $z_j^\varepsilon(t)$ at $t = T_j^\varepsilon = T_{j'}^\varepsilon$.

Since, by (4), the zeroes of u_0 are isolated, the uniform Lipschitz estimate shows that the $T_j^\varepsilon > \frac{\rho}{2L} > 0$ for all j independent of ε.
We now consider the limit $\varepsilon \to 0$.

Lemma 5.3 *For any sequence $\{\varepsilon_n\}_{n=1}^\infty$ with $\varepsilon_n \to 0+$ as $n \to \infty$, there is a subsequence, also denoted by $\{\varepsilon_n\}$ for simplicity, such that the limit*

$$u(\bar{x}, \bar{t}) = \lim_{\varepsilon_n \to 0+} u_{\varepsilon_n}(\bar{x}, \bar{t}) \tag{13}$$

exists for almost all $(\bar{x}, \bar{t}) \in \mathbb{R} \times \mathbb{R}^+$.
The range of $u(\bar{x}\pm, \bar{t})$ is $\{-1, 1\}$.
Furthermore, there are uniformly Lipschitzian curves $z_j(t)$ defined on $[0, T_j)$, $j \in \mathcal{J}$ such that for each fixed $t > 0$, $u(x, t)$ is constant for all x between two adjacent curves $z_j(t)$.

Proof: Let $x = z_j^\varepsilon(t)$, $j \in \mathcal{J}$, be the curves provided by lemma 5.2. These curves $z_j^{\varepsilon_n}(t)$ defined on $[0, T_j^{\varepsilon_n}]$ are Lipschitzian uniformly in $\varepsilon_n > 0$ and j. Thus, by the Arzela-Ascoli theorem, there is a sequence $\{\varepsilon_n\}$ such that

$$z_j^0(t) := \lim_{n \to \infty} z_j^{\varepsilon_n}(t) \tag{14}$$

exists on $[0, z_j := \lim_{n \to \infty} T_j^{\varepsilon_n}]$. By the definition of $z_j^{\varepsilon_n}(t)$, for each fixed $t > 0$, $\lim_{n \to \infty} \operatorname{sign}(u_{\varepsilon_n}(x, t))$ is fixed for all x between two adjacent points among $z_j(t)$, $j \in \mathcal{J}$. To simplify the notation throughout the rest of this proof, we just ignore $z_j(t)$ if t is outside the domain of definition of $z_j(t)$.
Fix now some time $\bar{t} > 0$.
Any point $\bar{x} \in \mathbb{R} \setminus \cup_j z_j^0(\bar{t})$ must fall between some adjacent curves $x = z_j^0(\bar{t})$ and $x = z_j^0(\bar{t})$ where $j, j \in \mathcal{J} \cup \{-\infty, +\infty\}$ and we have set $z_{-\infty}^0(t) := -\infty$ and $z_{+\infty}^0(t) := \infty$. Let $z_j^0(\bar{t}) < z_{j'}^0(\bar{t})$ be two adjacent points at $t = \bar{t}$.

We have already seen that the limit $\lim_{n\to\infty} \operatorname{sign}(u_{\varepsilon_n}(\bar{x},\bar{t}))$ is a constant for all $z_j^0(\bar{t}) < \bar{x} < z_{j'}^0(\bar{t})$. For definiteness, we assume this constant is 1, i.e.

$$u_{\varepsilon_n}(\bar{x},\bar{t}) > 0$$

for all $z_j^0(\bar{t}) < \bar{x} < z_{j'}^0(\bar{t})$ and n sufficiently large.
The case $u_{\varepsilon_n}(\bar{x},\bar{t}) < 0$ can be handled in the same way.
We consider the minimal backward characteristic $\zeta_{\varepsilon_n}(t;\bar{x},\bar{t})$ through the point $(\bar{x},\bar{t})$ which satisfies

$$\left.\begin{aligned} \frac{d\zeta_{\varepsilon_n}}{dt} &= f'(U_{\varepsilon_n}(t)), \\ \frac{dU_{\varepsilon_n}}{dt} &= \frac{1}{\varepsilon_n} U_{\varepsilon_n}^{\alpha}(1-U_{\varepsilon_n}^2)^{\beta}, \\ (\zeta_{\varepsilon_n}(\bar{t}), U_{\varepsilon_n}(\bar{t})) &= (\bar{x}, u_{\varepsilon_n}(\bar{x}-,\bar{t})). \end{aligned}\right\} \tag{15}$$

According to Proposition 2.4, the solution of (15) is defined on $[0,\bar{t}]$ and

$$U_{\varepsilon_n}(t) = u_{\varepsilon_n}(\zeta_{\varepsilon_n}(t;\bar{x},\bar{t})-,t) = u_{\varepsilon_n}(\zeta_{\varepsilon_n}(t;\bar{x},\bar{t})+,t) \text{for almost all} t \in [0,\bar{t}\,].$$

If $u_{\varepsilon_n}(\bar{x},\bar{t}) = 1$, then $U_{\varepsilon_n}(t) \equiv 1$.
Hence the limit is $u(\bar{x},\bar{t}) = 1$ if $u_{\varepsilon_n}(\bar{x},\bar{t}) = 1$ for all sufficiently large n.
If $u_{\varepsilon_n}(\bar{x},\bar{t}) \neq 1$ for infinitely many n, then only two possibilities exist: Either

$$\liminf_{n\to\infty} u_0(\zeta_{\varepsilon_n}(0;\bar{x},\bar{t})) > 0$$

in which case it follows easily from (15) that $\lim_{n\to\infty} u_{\varepsilon_n}(\bar{x},\bar{t}) = 1$ or

$$\liminf_{n\to\infty} u_0(\zeta_{\varepsilon_n}(0;\bar{x},\bar{t})) = 0.$$

By extracting a subsequence of $\{\varepsilon_n\}_{n=1}^{\infty}$, still denoted by $\{\varepsilon_n\}$, we have

$$\lim_{n\to\infty} u_0(\zeta_{\varepsilon_n}(0;\bar{x},\bar{t})) = 0.$$

Since $u_0(x)$ is continuous with isolated zeroes we can assume that

$$\zeta_{\varepsilon_n}(0;\bar{x},\bar{t}) \to a \tag{16}$$

with $u_0(a) = 0$, possibly after extracting a further subsequence if necessary.
Since $u_{\varepsilon_n}(\bar{x}-,\bar{t}) > 0$, it is necessary in view of (15) that $a < \bar{x}$. We claim that if for some $0 < \delta < 1$ and all n sufficiently large

$$0 < u_{\varepsilon_n}(\bar{x},\bar{t}) \leq \delta < 1, \tag{17}$$

then

$$|\bar{x} - a| = \begin{cases} \mathcal{O}(1)\ \varepsilon_n^{\frac{1}{\alpha-1}}, & \text{if} \quad \alpha > 1,\ \alpha \neq 2 \\ \mathcal{O}(1)\ |\varepsilon_n \ln \varepsilon_n|, & \text{if} \quad \alpha = 2. \end{cases} \tag{18}$$

To this end, we consider the system for the extremal backward characteristics (15) to derive

$$\frac{\bar{t}}{\varepsilon_n} = \int_{U_0}^{\bar{U}} \frac{du}{u^\alpha (1-u^2)^\beta} \tag{19}$$

where $U_0 := u_0(\zeta_{\varepsilon_n}(0; \bar{U}, \bar{t}))$.
We continue to estimate (19), using (17) as follows:

$$\frac{\bar{t}}{\varepsilon_n} = \mathcal{O}(1) \int_{U_0}^{\bar{U}} \frac{du}{u^\alpha} = \mathcal{O}(1) \left(\frac{1}{U_0^{\alpha-1}} - \frac{1}{\bar{U}^{\alpha-1}} \right).$$

Since $0 < U_0 < \bar{U}$ we have

$$U_0 = \mathcal{O}(1)\, \varepsilon_n^{\frac{1}{\alpha-1}}. \tag{20}$$

From (15), we can also derive, for the case $\alpha \neq 2$ that

$$\begin{aligned}
\bar{x} - \zeta_{\varepsilon_n}(0; \bar{x}, \bar{t}) &= \int_{U_0}^{\bar{U}} \frac{\varepsilon_n f'(u) du}{u^\alpha (1-u^2)^\beta} \\
&= \mathcal{O}(1)\, \varepsilon_n \int_{U_0}^{\bar{U}} \frac{du}{u^{\alpha-1}(1-u^2)^\beta} \\
&= \mathcal{O}(1)\, \varepsilon_n \left(\frac{1}{U_0^{\alpha-2}} - \mathcal{O}(1) \right) \\
&= \mathcal{O}(1)\, \varepsilon_n^{\frac{1}{\alpha-1}}.
\end{aligned}$$

Similarly, for the case $\alpha = 2$, we get by integration

$$\bar{x} - \zeta_{\varepsilon_n}(0; \bar{x}, \bar{t}) = \mathcal{O}(1) |\varepsilon_n \ln \varepsilon_n|.$$

From the last two equations, we have

$$|\bar{x} - \zeta_{\varepsilon_n}(0; \bar{x}, \bar{t})| = \begin{cases} \mathcal{O}(1)\, \varepsilon_n^{\frac{1}{\alpha-1}}, & \text{if} \quad \alpha > 1,\ \alpha \neq 2 \\ \mathcal{O}(1) |\, \varepsilon_n \ln \varepsilon_n|, & \text{if} \quad \alpha = 2. \end{cases}$$

Then the claim (18) follows easily.
Based on (18), we can see that if $\bar{x} \neq a$, then $\lim_{\varepsilon_n \to 0} u_{\varepsilon_n}(\bar{x}, \bar{t}) = 1$ if $\bar{x}$ is between two adjacent points among $z_j(\bar{t})$, $j \in \mathcal{J}$ with $\lim_{\varepsilon_n \to 0} \operatorname{sign}(u_{\varepsilon_n}(\bar{x}, \bar{t})) = 1$. Similarly one shows $\lim_{\varepsilon_n \to 0} u_{\varepsilon_n}(\bar{x}, \bar{t}) = -1$ for the case $\lim_{\varepsilon_n \to 0} \operatorname{sign}(u_{\varepsilon_n}(\bar{x}, \bar{t})) = -1$. From the above arguments, we see that the limit $u(x,t)$ is a piecewise constant function with constants being ± 1 which are separated by the Lipschitzian curves $z_j(t)$, $j \in \mathcal{J}$. These curves intersect each other only at the end points of their domain of definition.

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The next lemma completes the proof of theorem 1.4.

Lemma 5.4 *Let u_0 satisfy the assumption (4) and u_ε be the solution of (3). Then the limit*

$$u(\bar{x},\bar{t}) = \lim_{\varepsilon\to 0+} u_\varepsilon(\bar{x},\bar{t}) \tag{21}$$

exists for almost all $(\bar{x},\bar{t}) \in \mathbb{R}\times\mathbb{R}^+$.
The value of $u(\bar{x}\pm,\bar{t})$ is either 1, or -1.
Furthermore, there exist curves $z_j^0(t)$, $j\in\mathcal{J}$ defined on $[0,T_j)$, respectively, such that:

(i) if $u(z_j^0(t)-,t) > u(z_j^0(t)+,t)$, then $x = z_j^0(t)$ satisfies the Rankine-Hugoniot condition

$$\frac{dz_j^0}{dt} = \frac{f(u(z_j^0(t)-,t)) - f(u(z_j^0(t)+,t))}{u(z_j^0(t)-,t) - u(z_j^0(t)+,t)} = \frac{f(1)-f(-1)}{2}. \tag{22}$$

(ii) if $u(z_j^0(t)-,t) < u(z_j^0(t)+,t)$, then the curve $z_j^0(t) = a_j + f'(0)t$ and hence the speed of the discontinuity is $f'(0)$.

Proof: By Lemma 5.3, we can choose a sequence $\{\varepsilon_n\}$ such that

$$u(x,t) := \lim_{n\to\infty} u_{\varepsilon_n}(x,t) \tag{23}$$

exists for almost all $(x,t)\in\mathbb{R}\times\mathbb{R}^+$ and $u(x,t) = +1$ or -1.
Moreover there exist curves $z_j(t)$, $j\in\mathcal{J}$ with $z_j(0) = a_j$, which are uniformly Lipschitz and which do not intersect each other except at the end points $t = T_j$. These curves separate regions where $u(x,t) = 1$ from regions where $u(x,t) = -1$. Hence, the set of points of discontinuity of $u(x,t)$ is exactly the union of the curves $z_j(t)$ with $j\in\mathcal{J}$.
There are two possibilities at $x = z_j(t)$:
Case I. $u(z_j(t)-,t) > u(z_j(t)+,t)$.
Note that in this case $u(z_j(t)-,t) = 1$ and $u(z_j(t)+,t) = -1$.
Consider some point $\bar{x} < z_j(\bar{t})$ and close enough to $z_j(\bar{t})$ at $t = \bar{t} < T_j$. Hence, $\lim_{n\to\infty} u_{\varepsilon_n}(\bar{x}-,\bar{t}) = 1$.
Let $\zeta_{\varepsilon_n}(t;\bar{x},\bar{t})$ be the minimal backward characteristic associated to u_{ε_n} defined by (15). At $t = 0$, we have either
(i) $\liminf_{n\to\infty} u_{\varepsilon_n}(\zeta_{\varepsilon_n}(0;\bar{x},\bar{t}),0) > 0$ or
(ii) $\liminf_{n\to\infty} u_{\varepsilon_n}(\zeta_{\varepsilon_n}(0;\bar{x},\bar{t}),0) = 0$.
For (i), we have that $u_{\varepsilon_n}(\zeta_{\varepsilon_n}(t;\bar{x},\bar{t}),t) \to 1$ as $n\to\infty$ uniformly for $t > \delta > 0$, where $\delta > 0$ is any constant. This leads to the expression

$$\lim_{n\to\infty} \zeta_{\varepsilon_n}(t;\bar{x},\bar{t})) = \bar{x} + f'(1)(t-\bar{t}). \tag{24}$$

Since u_0 is differentiable with isolated zeroes, one can assume for (ii) that

$$\lim_{n\to\infty} \zeta_{\varepsilon_n}(0;\bar{x},\bar{t})) = a \le \bar{x}$$

with $u_0(a) = 0$.
In both cases (i) and (ii),

$$a := \lim_{n\to\infty} \zeta_{\varepsilon_n}(0; \bar{x}, \bar{t})) \leq \bar{x} \tag{25}$$

Subcase I(1): $a < \bar{x}$ in (25).
In this case we set

$$x_1 := (\bar{x} + a)/2 \text{ and } t_1 := -(\bar{x} - x_1 - f'(1)\bar{t})/f'(1) > 0. \tag{26}$$

The equation for $\zeta_{\varepsilon_n}(t; \bar{x}, \bar{t})$, (15), and $|u_0(x)| < 1$ imply that

$$\frac{d\zeta_{\varepsilon_n}(t; \bar{x}, \bar{t})}{dt} < f'(1)$$

and hence $\zeta_{\varepsilon_n}(t_1; \bar{x}, \bar{t}) > x_1$.

$$Figure 5.1$$

As $\zeta_{\varepsilon_n}(0; \bar{x}, \bar{t}) \to a$ for $\varepsilon_n \to 0+$, we conclude from (17) and (18) that

$$u_1 := u_{\varepsilon_n}(\zeta_{\varepsilon_n}(t_1; \bar{x}, \bar{t}), t_1) \geq 1/2$$

for n sufficiently large.
Then system (15) implies that

$$\begin{aligned}
\frac{\bar{t} - t_1}{\varepsilon_n} &= \int_{u_1}^{\bar{u}} \frac{du}{u^\alpha (1+u)^\beta (1-u)^\beta} \\
&\leq 2^\alpha \int_{u_1}^{\bar{u}} \frac{du}{(1-u)^\beta} \\
&= \begin{cases} 2^\alpha \log\left(\frac{1-u_1}{1-\bar{u}}\right) & \text{if} \quad \beta = 1 \\ \frac{2^\alpha}{\beta - 1}\left(\frac{1}{(1-\bar{u})^{\beta-1}} - \frac{1}{(1-u_1)^{\beta-1}}\right) & \text{if} \quad \beta > 1. \end{cases}
\end{aligned}$$

Using the definition of t_1 from (26), we obtain the estimates

$$\frac{1}{(1-\bar{u})^{\beta-1}} \geq \mathcal{O}(1)\frac{\bar{t} - t_1}{\varepsilon_n} = \mathcal{O}(1)\frac{\bar{x} - x_1}{\varepsilon_n} \quad \text{for } \beta > 1$$

and

$$1 - \bar{u} = \mathcal{O}(1)\exp\left(-\frac{\bar{t} - t_1}{\varepsilon_n}\right) = \mathcal{O}(1)\exp\left(-\frac{\bar{x} - x_1}{\varepsilon_n}\right) \quad \text{for } \beta = 1.$$

This implies

$$|1 - \bar{u}| \leq \begin{cases} \mathcal{O}(1)\left(\frac{\varepsilon_n}{\bar{x} - a}\right)^{\frac{1}{\beta - 1}} & \text{for } \beta > 1 \\ \mathcal{O}(1)\exp\left(-\frac{\bar{x} - x_1}{\varepsilon_n}\right) & \text{for } \beta = 1 \end{cases} \tag{27}$$

We can prove the same estimate when $u_{\varepsilon_n}(\bar{x},\bar{t}) < 0$.
Independent of $\beta \geq 1$ we have

$$\frac{1}{\varepsilon_n} u_{\varepsilon_n}(\bar{x}-,\bar{t})^{\alpha}(1-(u_{\varepsilon_n}(\bar{x}-,\bar{t}))^2)^{\beta} \to 0 \tag{28}$$

uniformly for $\bar{x} < z_j(\bar{t})$ and close to $z_j(\bar{t})$.
Similarly one can prove that (28) also holds uniformly for $\bar{x} > z_j(t)$ and close to $z_j(t)$.
Applying this estimate to the the weak form of (3)

$$\int_0^{T_j}\int_{\mathbb{R}}\left(-u\phi_t - f(u)\phi_x - \frac{1}{\varepsilon_n}u_{\varepsilon_n}^{\alpha}(1-u_{\varepsilon_n}^2)^{\beta}\phi\right)dx\,dt = 0$$

for test functions with compact support confined near $x = z_j(t)$, $0 < t < T_j$, one sees that the shock $x = z_j(t)$, $0 < t < T_j$, is a weak solution of $u_t + f(u)_x = 0$.
Thus, the Rankine-Hugoniot condition

$$\frac{dz_j}{dt} = \frac{f(u(z_j(t)+,t)) - f(u(z_j(t)-,t))}{u(z_j(t)+,t) - u(z_j(t)-,t)} = \frac{f(1)-f(-1)}{2}$$

holds if $u(z_j(t)+,t) < u(z_j(t)-,t)$.
Subcase I(2): $a = \bar{x}$ in (25).
By (24), this case occurs only if $\liminf_{n\to\infty} u_0(\zeta_{\varepsilon_n}(0;\bar{x},\bar{t})) = 0$ and hence $a = a_j$ for some $j \in \mathcal{J}$.
We claim that this subcase cannot happen at all for small enough $\varepsilon_n > 0$.
Indeed, if $a = \bar{x}$ was true, we would have

$$\zeta_{\varepsilon_n}(t;\bar{x},\bar{t}) \equiv \bar{x} = a \tag{29}$$

from (25).
Given $\delta > 0$ small, it follows again from (17) and (18) that $0 \leq 1 - u_{\varepsilon_n}(x,\bar{t}/2) < \delta$ for $x \in [a_{j-1}+\delta, a_j-\delta]$ when $\varepsilon_n > 0$ is small enough.
To establish above claim, it suffices to prove that the forward characteristics $\xi_{\varepsilon_n}(t;\bar{x}-\delta,\bar{t}/2)$, $t > \bar{t}/2$ intersect $\zeta_{\varepsilon_n}(t;\bar{x},\bar{t})$ at some $t < \bar{t}$, which is impossible in view of Lemma 3.1.7.
To this end, we observe that before $x = \xi_{\varepsilon_n}(t;\bar{x}-\delta,\bar{t}/2)$ intersects $x = \zeta_{\varepsilon_n}(t;\bar{x},\bar{t})$, the estimate

$$u_{\varepsilon_n}(x,\bar{t}) > 0 \text{ for } \bar{x}-\delta < x < \bar{x} \tag{30}$$

holds due to $\bar{x} < z^{\varepsilon_n}(\bar{t})$. This implies

$$\frac{d\xi_{\varepsilon_n}(t;\bar{x}-\delta,\bar{t}/2)}{dt} > 0$$

and hence

$$\xi_{\varepsilon_n}(t;\bar{x}-\delta,\bar{t}/2) > \bar{x}-\delta.$$

For any point $x_1 \in [(a_{j-1}+a_j)/2, \xi_{\varepsilon_n}(t_1; \bar{x}-\delta, \bar{t}/2)]$, we have $\zeta(0; x_1, t_1) < \bar{x}-\delta$, since maximal backward characteristics cannot cross the forward characteristics $\xi_{\varepsilon_n}(t; \bar{x} - \delta, \bar{t}/2)$ from the left as t decreases. See Figure 5.2.

$$Figure 5.2$$

This is the case covered by Subcase I(1) or (i) before (23). Then our results for (i) and Subcase I(1) yields

$$0 \leq 1 - u_{\varepsilon_n}(\xi_{\varepsilon_n}(t; \bar{x} - \delta, \bar{t}/2)-, t) < \delta$$

when $\varepsilon_n > 0$ is sufficiently small. This infers that

$$\frac{d\xi_{\varepsilon_n}(t; \bar{x} - \delta, \bar{t}/2)}{dt} \geq f(1-\delta) - f(0)$$

before $x = \xi_{\varepsilon_n}(t; \bar{x} - \delta, \bar{t}/2)$ meeting $x = \zeta_{\varepsilon_n}(t; \bar{x}, \bar{t})$. Thus, the curve $x = \xi_{\varepsilon_n}(t; \bar{x} - \delta, \bar{t}/2)$ and $x = \zeta_{\varepsilon_n}(t)$ must intersect at some $t < \bar{t}$. This proves our claim.
Case II. $u(z_j(t)-, t) < u(z_j(t)+, t)$.
In this case, one has $u(z_j(t)-, t) = -1 = -u(z_j(t)+, t)$.
We claim that in this case, $z_j(\bar{t}) \equiv a_j$ for some $j \in \mathcal{J}$ and all $\bar{t} \in [0, T_j)$ under the assumption $f'(0) = 0$.
To this end, we consider two points x_1 and x_2 sufficiently close to $z_j(\bar{t})$ and $x_1 < z_j(\bar{t}) < x_2$. By definition of $z_j(t)$, $\text{sign}(u_{\varepsilon_n}(x_1-, \bar{t})) = 1 = -\text{sign}(u_{\varepsilon_n}(x_2-, \bar{t}))$ for large n.
From (15), the minimal backward characteristics through points $(x_1, \bar{t})$ and $(x_2, \bar{t})$ satisfy

$$\frac{d\zeta_{\varepsilon_n}(t; x_1, \bar{t})}{dt} < 0 < \frac{d\zeta_{\varepsilon_n}(t; x_2, \bar{t})}{dt}. \tag{31}$$

Since the sign of u_{ε_n} is constant along extremal backward characteristics, one has

$$x_1 < \zeta(t, x_1, \bar{t}) < z_j(t) < \zeta(t, x_2, \bar{t}) < x_2. \tag{32}$$

Now, let $x_1 \to z_j(t)-$ and $x_2 \to z_j(\bar{t})+$, estimates (31) and (32) imply that $z_j(t)$ is constant for $t \in [0, \bar{t}]$.
It follows immediately from the arbitrariness of $\bar{t} \in [0, T_j)$ and $z_j(0) = a_j$ that $z_j(t) \equiv a_j$ for all t in its domain of definition.
From the above analysis, we see that the the limit function $u(x, t)$ is completely determined by the curves $z_j(t)$, $j \in \mathcal{J}$.
Furthermore, these curves $z_j(t)$ are uniquely determined by the Rankine-Hugoniot condition (30) with $z_j(0) = a_j$ or is equal to a constant a_j for some $j \in \mathcal{J}$. In other words, no matter how the subsequence $\{\varepsilon_n\}$ are chosen, the limit functions $u(x, t) = \lim_{n\to\infty} u_{\varepsilon_n}(x-, t)$ are the same.
This proves the convergence of u_ε as $\varepsilon \to 0+$.

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