Equilibrium solutions of viscous scalar balance laws with a convex flux

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July 28, 1997

1 The Results

In this paper, we are going to study equilibrium solutions of viscous balance laws. These are parabolic partial differential equations of the form

$$u_t + (f(u))_x = \varepsilon u_{xx} + g(u), \quad x \in (0,1), \quad f, g \in C^3. \tag{1}$$

We impose Neumann boundary conditions

$$u_x(0) = u_x(1) = 0.$$

These boundary conditions turn out to make the problem analytically rather difficult but most of our methods apply to the study of other boundary conditions, too. Our goal is a complete description of all equilibrium (i.e. time-independent) solutions, including exact multiplicity results and statements about the stability of the solutions.

Without providing any details, we just mention that the information we collect during the proofs is sufficient to give a fairly good picture of the global attractor of (1) using the work of Fiedler and Rocha [2, 3].

Throughout the paper we make the following assumptions on f and g:

(H1) g is a dissipative function, i.e.

$$u \cdot g(u) < 0 \quad \forall\ |u| > R \tag{2}$$

for some (large) constant R.

(H2) f is a strictly convex function, i.e. $f''(u) > 0$ for all u.

(H3) the derivative of f does not vanish at zeroes of g.

(H4) the zeroes of g are simple. Then, by dissipativeness of g, there is a finite number of zeroes $u_1 < u_2 < \ldots < u_l$ where l is an odd number.

(H5) $f(u_i) \neq f(u_J)$ for all $1 \leq i < j \leq l$.

There are l trivial equilibrium solutions $u \equiv u_i$. Our first result tells that there are no other than those spatially homogenous solutions if f' has a definite sign on the compact interval that contains all zeroes of g.

Theorem 1.1 *Consider (1) and assume that $f'(u) \neq 0$ for all $u \in [u_1, u_l]$. Then there are no spatially non-homogenous equilibrium solutions of (1).*

In the other cases we can find equilibrium solutions with a boundary layer for ε sufficiently small.

Theorem 1.2 *Let u_i be one of the zeroes of g such that there exists $\tilde{u}_i$ with $f(\tilde{u}_i) = f(u_i)$ and*

$$g(\tilde{u}_i)(u_i - \tilde{u}_i) < 0. \tag{3}$$

Then there is a ε_0, such that for all $0 < \varepsilon \leq \varepsilon_0$ the parabolic equation (1) has an equilibrium solution u^ε with the properties:

(i) If $f'(u_i) > 0$ then $u^\varepsilon \rightarrow u_i$ uniformly on every interval $[0, 1-\kappa]$ with $\kappa > 0$ and $u^\varepsilon(1) \rightarrow \tilde{u}_i$.

(ii) If $f'(u_i) < 0$ then $u^\varepsilon \rightarrow u_i$ uniformly on every interval $[\kappa, 1]$ with $\kappa > 0$ and $u^\varepsilon(0) \rightarrow \tilde{u}_i$.

We are also interested in the stability of the equilibrium solutions with respect to the parabolic equation (1). To this end, consider the eigenvalue problem associated with the linearization of (1) at an equilibrium v:

$$\left.\begin{array}{rcl} w_{xx} - f'(v(x))w_x - f''(v(x))v_x(x)w + g'(v(x))w_x &=& \lambda w \\ w_x(0) = w_x(1) &=& 0 \end{array}\right\} \tag{4}$$

Definition 1.3 *An equilibrium v is called* **hyperbolic** *if 0 is not an eigenvalue of the linearization at v, i.e. if (4) has no nontrivial solution for $\lambda = 0$.*

Definition 1.4 *The* **Morse index** *$i(v)$ of a hyperbolic equilibrium v is the number of positive eigenvalues of the linearization at v.*

For the number of equilibria found and their Morse indices the following holds:

Theorem 1.5 *Let m be the number of zeroes u_i of g for which (3) holds. Then for ε sufficiently small there are exactly $l + m$ equilibria, all of them hyperbolic and*

- *$\frac{l+1}{2}$ equilibria have Morse index 0,*
- *$\frac{l-1+m}{2}$ equilibria have Morse index 1 and*
- *$\frac{m}{2}$ equilibria have Morse index 2.*

2 Equilibrium solutions

Since, by definition, equilibrium solutions do not depend on time t, we will write for these solutions simply $u(x)$ instead of $u(x,t)$. They solve the boundary value problem

$$\left.\begin{array}{rcl} \varepsilon u_{xx} - (f(u))_x + g(u) &=& 0 \\ u_x(0) = u_x(1) &=& 0. \end{array}\right\} \quad (5)$$

This singularly perturbed boundary value problem can be written as a first order system of Liénard type

$$\left.\begin{array}{rcl} \varepsilon u' &=& v + f(u) \\ v' &=& -g(u) \end{array}\right\} \quad (6)$$

with the boundary conditions

$$u'(0) = u'(1) = 0.$$

Here the prime stands for the derivative with respect to a new variable s which corresponds to the space variable x in the original balance law. However, since we treat this variable like some "time", we find it less confusing to write $(u(s), v(s))$ for solutions of (6). The boundary condition becomes then

$$v(s) + f(u(s)) = 0 \qquad \text{at } s = 0 \ \text{ and } \ s = 1.$$

Setting $\varepsilon = 0$ in equation (6) we get the "slow system" that describes partly the behaviour of the system as ε tends to 0:

$$\begin{array}{rcl} 0 &=& v + f(u) \\ v' &=& -g(u). \end{array}$$

The first equation defines a curve in (u,v)-space to which the flow is confined. This motivates the following

Definition 2.1 *The curve $\mathcal{C}$ given by the equation $v + f(u) = 0$ in the (u,v)-plane is called the* **singular curve**.

Multiplying the second equation in (6) by a factor ε and rescaling the variable s according to $s = \varepsilon\sigma$, we arrive at the fast system

$$\begin{array}{rcl} \dot{u} &=& v + f(u) \\ \dot{v} &=& -\varepsilon g(u). \end{array} \quad (7)$$

with the dot denoting the derivative with respect to the fast variable σ. This system will be used in the proof of theorem 1.2 for a normal form analysis.

Since we are interested mainly in solutions of a Neumann BVP, the following notion is useful:

Definition 2.2 *An* **admissible trajectory** *of system (6) is a trajectory that corresponds to a solution of the boundary value problem, i.e. it is a finite piece of a trajectory* $(u(s), v(s))$ *that satisfies* $v(0) + f(u(0)) = v(1) + f(u(1)) = 0$.

In [5] the following lemma was proved:

Lemma 2.3

(i) For any admissible trajectory $(u(s), v(s))$ *of* (6)

$$u_1 \le u(s) \le u_l \quad \forall s \in [0,1].$$

(ii) If $f'(u_i) < f'(u_{i+1}) < 0$, *then there is a positively invariant region between the curve* $\mathcal{C}$ *and the curve* $v + f(u) + (-1)^i k\varepsilon g(u) = 0$ *for some* k *chosen sufficiently large and all* ε *small. In this case the two equilibria* $(u_i, -f(u_i))$ *and* $(u_{i+1}, -f(u_{i+1}))$ *on the singular curve are connected by a heteroclinic orbit.*

(iii) If $0 < f'(u_i) < f'(u_{i+1})$, *there is a negatively invariant region between* $\mathcal{C}$ *and a curve* $v + f(u) + (-1)^{i+1} k\varepsilon g(u) = 0$ *and the two equilibria are also connected by a heteroclinic orbit.*

Remark: The heteroclinic orbits are part of the **slow manifold**, an invariant manifold that exists for small $\varepsilon > 0$ near the singular curve $\mathcal{C}$ except in a neighborhood of the extrema, cf. [1].

Proof of Theorem 1.1: Without restriction we assume that $f'(u) < 0$ for all $u \in [u_1, u_l]$, the case $f' > 0$ can be treated similarly (by reversing time and the use of negatively invariant regions instead of positively invariant ones). We proceed indirectly and suppose that there is a spatially nonhomogenous equilibrium solution of the viscous balance law. Then this solution corresponds to a non-constant admissible trajectory of (6). From lemma 2.3(i) we know that $u(0) \in (u_1, u_l)$ and because of the boundary condition $(u(0), v(0)) \in \mathcal{C}$. Again by lemma 2.3 there are positively invariant regions $R_1, R_2, \ldots, R_{l-1}$ enclosed between $\mathcal{C}$ and the heteroclinic orbit connecting the two equilibria $(u_i, -f(u_i))$ and $(u_{i+1}, -f(u_{i+1}))$. Hence, $(u(0), v(0))$ lies on the boundary of one of the positive invariant regions R_i and the trajectory enters R_i immediately . Since it has to stay inside that region, the right boundary condition $(u(1), v(1)) \in \mathcal{C}$ cannot be satisfied and the trajectory cannot be admissible. This contradiction completes the proof.

□

Now we turn to the case that f' changes its sign on $[u_1, u_l]$. Let therefore $\bar{u}$ be the minimum of f, i.e. $f'(\bar{u}) = 0$. By **(H3)** we can find $J \in \{1, 2, \ldots, l\}$ such that

$$u_J < \bar{u} < u_{J+1}$$

We start with a lemma which tells us that admissible trajectories do not cross the curve $\mathcal{C}$.

Lemma 2.4 *Let $(u(s), v(s))$ be an admissible solution. Then:*

(i) $v(s) + f(u(s)) < 0$ *for all* $0 < s < 1$.

(ii) $f'(u(0)) > 0$ *and* $f'(u(1)) < 0$.

Proof:

(i) Assume the contrary, hence $v(s_0)+f(u(s_0)) \geq 0$ for some $0 < s_0 < 1$. Two cases have to be distinguished depending on the sign of g on the interval (u_J, u_{J+1}). Consider first the case $g < 0$. We show that the trajectory cannot hit the singular curve $\mathcal{C}$ for $s > s_0$. Let

$$s_1 := \inf\{s > s_0; (u(s), v(s)) \in \mathcal{C}\}$$

Then it is clear that $u(s_1) > u_{J+1}$, since u grows until the trajectory hits $\mathcal{C}$. Hence, there is some $k > J$ such that

$$u_k < u(s_1) < u_{k+1}.$$

Consider now the curve $v + f(u) = \varepsilon k|g(u)|$ between u_k and u_{k+1}. As was shown in [5] by a simple calculation, for k large enough and all sufficiently small ε, the vectorfield of (6) along this curve points away from the curve $\mathcal{C}$. On the other hand, our trajectory has to cross this curve at some time between s_0 and s_1 in the opposite direction. By this contradiction we have thus shown that it is impossible for an admissible trajectory to cross $\mathcal{C}$ at time $s_0 \in (0, 1)$. The case of $g > 0$ is completely analogous. One considers backward trajectories and shows that it is impossible to find $s_1 < s_0$ such that $(u(s_1), v(s_1)) \in \mathcal{C}$.

(ii) Essentially the same arguments as in the proof of theorem 1.1 apply. If $f'(u(0))$ is positive and moreover $u(0) < u_J$, the trajectory immediately enters one of the positively invariant regions and can therefore not be admissible. Analogously $u(1) > u_{J+1}$ is impossible. The only cases we have to rule out are $u_J < u(0) \leq \bar{u}$ and $\bar{u} \leq u(1) < u_{J+1}$. Again, we can restrict ourselves to the first case and prove the second one by time reversal. Assume hence that $u_J < u(0) \leq \bar{u}$. Two things may happen: If g is positive on $[u_J, u_{J+1}]$ then for ε small the trajectories will approach the equilibrium $(u_J, -f(u_J))$ without intersecting $\mathcal{C}$. If however g is negative on $[u_J, u_{J+1}]$ then the trajectory will escape to infinity. In both cases the trajectory cannot be admissible.

□

Now it is rather clear how solutions of the BVP (6) look like: They have to start on $\mathcal{C}$ with $u(0) > \bar{u}$ and and end up with $u(1) < \bar{u}$. Since the trajectory has to stay below the curve $\mathcal{C}$, the profile $u(s)$ is monotonically decreasing. Moreover, outside a δ-neighborhood of $\mathcal{C}$ the velocity u' is of order $\mathcal{O}(\varepsilon^{-1})$ such that it takes only a time of order $\mathcal{O}(\varepsilon)$ until the orbit reenters a neighborhood of $\mathcal{C}$. To find a solution of the BVP we have to find some way such that solutions spend a time of order 1 either before they leave a neighborhood of $\mathcal{C}$ or between reentering a neighborhood of $\mathcal{C}$ and hitting $\mathcal{C}$ at $s = 1$. Therefore, in a next step, we investigate how long it takes a trajectory to leave a δ-neighborhood of $\mathcal{C}$. It will turn out that the trajectory has to start exponentially close to an equilibrium on $\mathcal{C}$ to take time of order 1 before the orbit leaves a δ-neighborhood of $\mathcal{C}$. Analogously, it could also finish exponentially close to an equilibrium. To prove this we have to go through a normal form analysis which is performed in the next section.

3 The Takens normal form

For $\varepsilon = 0$ the singular curve $\mathcal{C}$ and the curve corresponding to the (left or right) boundary condition coincide. To get some estimates on the time it takes a trajectory to leave a neighborhood of the singular curve we have to find out how these two curves separate for $\varepsilon \neq 0$. To that end we will put the fast vector field

$$\begin{aligned} \dot{u} &= v + f(u) \\ \dot{v} &= -\varepsilon g(u) \end{aligned}$$

in a nicer form. There are two different cases to be considered:
Case A: The normal form is computed near a point of the singular curve $\mathcal{C}$ where g is nonzero.
Case B: The normal form is computed near a point on $\mathcal{C}$ where g has a zero.

Without restriction we assume $f(0) = 0$ and compute the normal form near $u = v = 0$.

Lemma 3.1 *If $f, g \in C^3$ with $f'(0) \neq 0$, then there is a local C^2-change of variables*

$$(\tilde{u}, \tilde{v}, \tilde{\varepsilon}) = \mathcal{T}(u, v, \varepsilon)$$

such that $\tilde{\varepsilon} = \varepsilon$ and the transformed vector field is

$$\left.\begin{aligned} \dot{\tilde{u}} &= A(\tilde{v}, \varepsilon)\, \tilde{u} \\ \dot{\tilde{v}} &= R(\tilde{v}, \varepsilon). \end{aligned}\right\} \tag{8}$$

Here A and R are smooth functions of their arguments with $A(0,0) = f'(0)$ and $R(\tilde{v}, 0) = 0$. In case B also $R(0, \varepsilon) = 0$.

The proof is based on a normal form for vector fields near a nonhyperbolic equilibrium given by Takens [7].

Proposition 3.2 (Takens 1971) *Let 0 be a singular point of a C^∞-vector field X. If the eigenvalues of dX at 0 satisfy a nonresonance condition ('Sternberg $\alpha(dX(0),k)$-condition'), then there is a C^k-change of coordinates such that the vector field in the new coordinates is locally in the* standard form

$$X = \sum_{i=1}^{c} X_i(x_1,...,x_c)\frac{\partial}{\partial x_i} + \sum_{i,j=1}^{s} A_{ij}(x_1,...,x_c)y_j\frac{\partial}{\partial y_i} + \sum_{i,j=1}^{u} B_{ij}(x_1,...,x_c)z_j\frac{\partial}{\partial z_i}$$

where

(1) all c eigenvalues of $\frac{\partial X_i}{\partial x_j}$ (in $x_1 = ... = x_c = 0$) have real part zero,

(2) all s eigenvalues of $A_{ij}(0,...,0)$ have real part < 0 and

(3) all u eigenvalues of $B_{ij}(0,...,0)$ have real part > 0.

So, in the standard form the center manifold $W^c(0)$ is the linear space $\{y_1 = \ldots = y_s = z_1 = \ldots = z_u = 0\}$.

Proof of lemma 3.1: We will use the Takens standard form near singular points of the augmented fast system

$$\left.\begin{array}{rcl} \dot{u} &=& v + f(u) \\ \dot{v} &=& -\varepsilon g(u) \\ \dot{\varepsilon} &=& 0. \end{array}\right\} \quad (9)$$

This system was not assumed to be of class C^∞, but a look at Takens proof reveals that a C^k-version holds, too. If no resonances occur between eigenvalues, a C^{k+1}-vector field can be brought to the normal form by a C^k-change of coordinates. This is the reason why we assumed f and g to be of class C^3. Then the normal form is of class C^2 which is sufficient for us as all arguments below will not involve higher than second derivatives.

For the normal form analysis, we can assume that $u = v = \varepsilon = 0$ is an equilibrium point of (9). The Jacobian of the fast system at this point is

$$J = \begin{pmatrix} f'(0) & 1 & 0 \\ 0 & 0 & -g(0) \\ 0 & 0 & 0 \end{pmatrix}.$$

If $f'(0) \neq 0$, J has a double zero eigenvalue and one nonzero eigenvalue $f'(0)$. So there is only one eigenvalue with nonzero real part and all nonresonance conditions used in Takens proof will be automatically satisfied. By Takens' theorem a C^2-change of variables

$$(\tilde{u},\tilde{v},\tilde{\varepsilon}) = \mathcal{T}(u,v,\varepsilon)$$

transforms the equation to the form

$$\left.\begin{array}{rcl} \dot{\tilde{u}} &=& A(\tilde{v},\tilde{\varepsilon})\,\tilde{u} \\ \dot{\tilde{v}} &=& R(\tilde{v},\tilde{\varepsilon}) \\ \dot{\tilde{\varepsilon}} &=& S(\tilde{v},\tilde{\varepsilon}). \end{array}\right\} \quad (10)$$

Now we want to exploit two special features of our system:
1) There is a curve of equilibria that exists for $\varepsilon = 0$ (namely on $\mathcal{C}$) and
2) there is a smooth invariant foliation by planes $\{\varepsilon = const.\}$.

Moreover, in case B there is a stationary point $u = v = 0$ that exists for all ε. We perform a transformation involving only the coordinates $\tilde{v}$ and $\tilde{\varepsilon}$ on the center manifold such that the following two properties are satisfied:

(1) The curve of equilibria at $\varepsilon = 0$ is mapped onto the new $\tilde{v}$-axis. This yields $R(\tilde{v}, 0) = 0$.

(2) The leaves of the foliation $\{\varepsilon = const.\}$ are straightened each leaf is asigned its original ε such that we have $S \equiv 0$.

In case B we can do more. In each fiber $\{\varepsilon = const.\}$ there is one equilibrium point. Together these stationary points form a smooth curve. By a fiber-preserving diffeomorphism we can move this equilibrium to $\tilde{v} = 0$ such that $R(0, \varepsilon) \equiv 0$. All these transformations in $\tilde{v}$ and ε do not affect the structure of the $\tilde{u}$-equation, only the term $A(\tilde{v}, \tilde{\varepsilon})$ is changed but not the linearity in $\tilde{u}$. Moreover, and this explains the last claim, the transformations do not alter eigenvalues such that $A(0, 0) = f'(0)$.

□

3.1 Transition time analysis for case A

We want to compute the time a trajectory takes from the curve $\mathcal{C}$ until it leaves a δ-neighborhood of the singular curve.

Lemma 3.3 *The time σ_0 a trajectory needs between a point $(\tilde{u}_0, \tilde{v}_0)$ on the singular curve $\mathcal{C}$ and a section $\Delta := \{\tilde{u} = |\delta|\}$ is a function $\sigma_0(\tilde{v}_0, \varepsilon)$ with*

$$\sigma_0(\tilde{v}_0, \varepsilon) = \mathcal{O}(|\ln \varepsilon|)$$

as $\varepsilon \to 0$. In the original time s this corresponds to a transition time

$$s_0 = \varepsilon \sigma_0.$$

Proof: As a first step we find out how the boundary condition looks like in the new normal form coordinates. The points corresponding to the boundary condition $v + f(u) = 0$ form a two-dimensional manifold $\mathcal{B}$ in $(\tilde{u}, \tilde{v}, \varepsilon)$-space which contains the $\tilde{v}$-axis.

In the original coordinates, the tangent space $T_{(0,0,0)}\mathcal{B}$ to this manifold is spanned by the vectors $(-1, f'(u_0), 0)^T$ and $(0, 0, 1)^T$. The first one is the eigenvector of the Jacobian with eigenvalue 0 and therefore by the transformation $d\mathcal{T}$ mapped to $(0, 1, 0)^T$. The second vector $(0, 0, 1)^T$ however has a component in the orthogonal complement of the center manifold $W^c(0)$:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \Pi \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -\frac{g(0)}{(f'(0))^2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

where Π is the orthogonal projection onto the center manifold W^c. As a consequence, the distance between the two manifolds $\mathcal{B}$ and $\mathcal{C}$ is to first order proportional to ε and $\mathcal{B}$ can be written as the graph

$$\tilde{u} = \varepsilon \Psi(\tilde{v}, \varepsilon) \tag{11}$$

of a function Ψ with

$$\Psi(0,0) = -\frac{g(0)}{(f'(0))^2} \neq 0.$$

The geometric situation of this case is depicted in figure 1.

Using representation (11) of $\mathcal{B}$, the normal form from lemma 3.1 will yield an estimate for the time σ_0 from a point $(\tilde{u}_0, \tilde{v}_0)$ on $\mathcal{B}$ to the section Δ. It depends on the sign of $\Psi(0,0)$ whether $\{\tilde{u} = \delta\}$ or $\{\tilde{u} = -\delta\}$ is the right choice but since both cases are treated in exactly the same way we restrict ourselves to the first case that corresponds to $g(0) < 0$.

From lemma 3.1 we have $R(\tilde{v}, 0) \equiv 0$ and therefore

$$R(\tilde{v}, \varepsilon) = \varepsilon R_1(\tilde{v}, \varepsilon).$$

The $\tilde{v}$-equation from (8) then reads

$$\dot{\tilde{v}} = \varepsilon R_1(\tilde{v}, \varepsilon)$$

and the solution with initial value $\tilde{v}_0$ can be written as

$$\tilde{v}(\sigma) = \tilde{v}_0 + \tilde{v}_1(\varepsilon\sigma)$$

with

$$\tilde{v}_1(0) = 0.$$

Integrating the $\tilde{u}$-equation

$$\dot{\tilde{u}} = A(\tilde{v}, \varepsilon)\, \tilde{u}$$

from $\sigma = 0$ to the time σ_0 when the trajectory hits the section Δ yields the condition

$$\varepsilon \cdot \Psi(\tilde{v}_0, \varepsilon) \cdot \exp\left(\int_0^{\sigma_0} A(\tilde{v}(\sigma), \varepsilon)\, d\sigma \right) = \delta$$

on σ_0, which is equivalent to

$$\int_0^{\sigma_0} A(\tilde{v}(\sigma), \varepsilon)\, d\sigma = \ln\left(\frac{\delta}{\varepsilon \Psi(\tilde{v}_0, \varepsilon)} \right) =: L(\tilde{v}_0, \varepsilon, \delta). \tag{12}$$

In a next step we decompose $A(\tilde{v}(\sigma),\varepsilon)$ as

$$\begin{aligned} A(\tilde{v}(\sigma),\varepsilon) &= A(\tilde{v}_0,0) + A(\tilde{v}_0+\tilde{v}_1(\varepsilon\sigma),\varepsilon) - A(\tilde{v}_0,0) \\ &= A(\tilde{v}_0,0) + A_1(\tilde{v}_0,\varepsilon,\varepsilon\sigma) \end{aligned}$$

where A_1 is defined via the last equation and satisfies

$$A_1(\tilde{v}_0,0,0)=0.$$

Using this decomposition in (12) and with the slow time $s=\varepsilon\sigma$ as integration variable, one arrives at the equation

$$\sigma_0\cdot A(\tilde{v}_0,0)+\frac{1}{\varepsilon}\int_0^{\varepsilon\sigma_0} A_1(\tilde{v}_0,\varepsilon,s)\,ds = L(\tilde{v}_0,\varepsilon,\delta)$$

$$\Longleftrightarrow \quad \sigma_0\cdot\frac{A(\tilde{v}_0,0)}{L(\tilde{v}_0,\varepsilon,\delta)}+\frac{1}{\varepsilon L(\tilde{v}_0,\varepsilon,\delta)}\int_0^{\varepsilon\sigma_0} A_1(\tilde{v}_0,\varepsilon,s)\,ds = 1. \tag{13}$$

With the new variable χ that is defined as

$$1+\chi := \sigma_0\cdot\frac{A(\tilde{v}_0,0)}{L(\tilde{v}_0,\varepsilon,\delta)}$$

and compensates the asymptotic behaviour of σ_0 as ε tends to 0, it is possible to define a function $\mathcal{F}(\chi,\tilde{v}_0,\varepsilon)$ in a neighborhood of $\chi=\tilde{v}_0=\varepsilon=0$ such that (13) corresponds to $\mathcal{F}(\chi,\tilde{v}_0,\varepsilon)=0$. An application of the implicit function theorem will then yield a solution $\chi=\chi(\tilde{v}_0,\varepsilon)$ and from this solution χ it will be possible to calculate $\sigma_0(\tilde{v}_0,\varepsilon)$.

We define thus for $\varepsilon>0$

$$\mathcal{F}(\chi,\tilde{v}_0,\varepsilon) := \chi + \int_0^{(1+\chi)/A(\tilde{v}_0,0)} A_1(\tilde{v}_0,\varepsilon,\varepsilon L(\tilde{v}_0,\varepsilon,\delta)\tilde{s})\,d\tilde{s} \tag{14}$$

with the integration variable $\tilde{s}$ satisfying $\varepsilon L(\tilde{v}_0,\varepsilon,\delta)\tilde{s}=s$. The function $\mathcal{F}$ is obviously continuous for $\varepsilon>0$. Since

$$A_1(\tilde{v}_0,\varepsilon,\varepsilon L(\tilde{v}_0,\varepsilon,\delta)\tilde{s}) \to A_1(\tilde{v}_0,0,0)=0$$

as $\varepsilon\searrow 0$, we have

$$\lim_{\varepsilon\to 0}\mathcal{F}(\chi,\tilde{v}_0,\varepsilon)=\chi.$$

Hence we can continuously extend $\mathcal{F}$ as an odd function in ε by setting

$$\mathcal{F}(\chi,\tilde{v}_0,0):=\chi$$

and

$$\mathcal{F}(\chi,\tilde{v}_0,\varepsilon) := 2\chi-\mathcal{F}(\chi,\tilde{v}_0,-\varepsilon) \text{ for } \varepsilon<0.$$

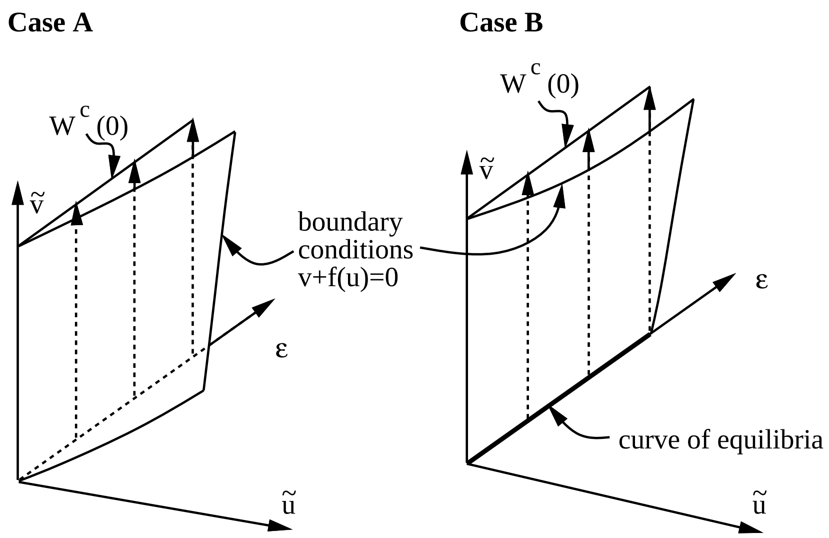

Figure 1: The geometry of the cases A and B

Note that in particular $\chi = \varepsilon = 0$ with any $\tilde{v}_0$ is a solution of the equation $\mathcal{F}(\chi, \tilde{v}_0, \varepsilon) = 0$. To apply the implicit function theorem near any such point, differentiability of $\mathcal{F}$ with respect to χ has to be proved. The derivative $\frac{\partial \mathcal{F}}{\partial \chi}$ is easily computed for $\chi, \varepsilon \neq 0$:

$$\frac{\partial \mathcal{F}}{\partial \chi}(\chi, \tilde{v}_0, \varepsilon) = 1 + \frac{1}{A(\tilde{v}_0, 0)} A_1 \left(\tilde{v}_0, \varepsilon, \varepsilon L(\tilde{v}_0, \varepsilon, \delta) \frac{1+\chi}{A(\tilde{v}_0, 0)} \right)$$

Again, $A_1(\tilde{v}_0, 0, 0) = 0$ shows that $\frac{\partial \mathcal{F}}{\partial \chi}$ tends to $\frac{\partial \mathcal{F}}{\partial \chi}(\chi, \tilde{v}_0, 0) = 1$ as $\varepsilon \to 0$, and the implicit function theorem applies near every point with $\chi = \varepsilon = 0$ and yields a solution $\chi = \chi(\tilde{v}_0, \varepsilon)$ of $\mathcal{F}(\chi, \tilde{v}_0, \varepsilon) = 0$. Thus, the time t_0 from the curve $\mathcal{C}$ to a section $\{\tilde{u} = \delta\}$ is

$$\sigma_0(\tilde{v}_0, \varepsilon) = \frac{1 + \chi(\tilde{v}_0, \varepsilon)}{A(\tilde{v}_0, 0)} \ln \left(\frac{\delta}{\varepsilon \Psi(\tilde{v}_0, \varepsilon)} \right) = \mathcal{O}(\ln |\varepsilon|)$$

and the lemma is proved.

□

3.2 Transition time analysis for case B

Unless in case A, in case B the manifold $\mathcal{B}$ and the center manifold are tangent to each other, so second order terms are needed to describe the separation of the two manifolds as $\varepsilon > 0$. We have to add second order terms of the Taylor expansion of the coordinate transformation $\mathcal{T}^{-1}$ and compare the coefficients of the vector field in old and new coordinates. To this end, the transformation $\mathcal{T}^{-1}$ is written in the form

$$\begin{aligned} u &= \phi(\tilde{u}, \tilde{v}) \\ v &= \psi(\tilde{u}, \tilde{v}) \end{aligned}$$

and the Taylor expansion up to quadratic terms are performed. From lemma 3.1 we know that in case B both $R(\tilde{v}, 0) = 0$ and $R(0, \varepsilon) = 0$ such that the Taylor expansion of R starts with quadratic terms. Plugging everything into equation (9) one gets after a lengthy but straightforward calculation by comparison of coefficients:

$$\begin{aligned} u &= \tilde{u} - \frac{1}{f'(0)}\tilde{v} + \frac{f''(0)}{2f'(0)}\tilde{u}^2 + \mathcal{O}\left(|\tilde{u}|^2 + (|\tilde{u}| + |\tilde{v}| + |\varepsilon|)(|\tilde{v}| + |\varepsilon|)\right) \\ v &= \tilde{v} - \frac{g'(0)}{f'(0)}\tilde{u}\varepsilon + \psi_{020}\tilde{v}^2 + \psi_{011}\tilde{v}\varepsilon + \psi_{002}\varepsilon^2 + \text{h. o. t.} \end{aligned}$$

With this coefficients of $\mathcal{T}$ the boundary condition $v + f(u) = 0$ in the new coordinates reads

$$\mathcal{F}_{bc}(\tilde{u}, \tilde{v}, \varepsilon) = 0$$

where

$$\mathcal{F}_{bc}(\tilde{u}, \tilde{v}, \varepsilon) := f'(0)\tilde{u} + \frac{f''(0)}{2}\tilde{u}^2 - \frac{f''(0)}{f'(0)}\tilde{u}\tilde{v} - \frac{g'(0)}{f'(0)}\varepsilon\tilde{u} - \frac{g'(0)}{f'(0)^2}\varepsilon\tilde{v} + \text{h. o. t.}$$

and since $f'(0) \neq 0$ we get by the implicit function theorem near $\tilde{u} = \tilde{v} = \varepsilon = 0$ a solution of $\mathcal{F}_{bc}(\tilde{u}, \tilde{v}, \varepsilon) = 0$ which is of the form $\tilde{u} = \tilde{u}_{bc}(\tilde{v}, \varepsilon)$. We have

$$\frac{\partial \tilde{u}_{bc}}{\partial \varepsilon}(0,0) = \frac{\partial \tilde{u}_{bc}}{\partial \tilde{v}}(0,0) = 0.$$

Important is the mixed derivative

$$\frac{\partial^2 \tilde{u}_{bc}}{\partial \varepsilon \partial \tilde{v}}(0,0) = -\left(\frac{\partial \mathcal{F}_{bc}}{\partial \tilde{u}}(0,0,0)\right)^{-1} \frac{\partial^2 \mathcal{F}_{bc}}{\partial \varepsilon \partial \tilde{v}}(0,0,0) = -\frac{g'(0)}{(f'(0))^3}.$$

Hence, the boundary condition $v + f(u) = 0$ is transformed into

$$\tilde{u} = \varepsilon \tilde{v} \Psi(\tilde{v}, \varepsilon)$$

with

$$\Psi(0,0) = -\frac{g'(0)}{(f'(0))^3}.$$

Now we can estimate the time a trajectory takes to leave a δ-neighborhood of $\mathcal{C}$:

Lemma 3.4 *Fix δ small and some $T > 0$. Then for any ε small enough there exists a point $(\tilde{u}_0, \tilde{v}_0)$ on $\mathcal{B}$ such that the time for a trajectory starting in $(\tilde{u}_0, \tilde{v}_0)$ to a section $\Delta := \{|\tilde{u}| = \delta\}$ is exactly T/ε.*

Given $\alpha > 0$, denote with σ_α the time this trajectory needs from $(\tilde{u}_0, \tilde{v}_0)$ to the section $\Delta_\alpha := \{|\tilde{u}| = \delta\varepsilon^\alpha\}$. Then $|\sigma_0 - \sigma_\alpha| = \mathcal{O}(\ln|\varepsilon|)$.

Proof: Since we have $R(\tilde{v}, 0) = 0$ as well as $R(0, \varepsilon) = 0$, the $\tilde{v}$-equation from (8) can be written as

$$\dot{\tilde{v}} = \varepsilon \tilde{v} R_1(\tilde{v}, \varepsilon)$$

with

$$R_1(\tilde{v}, \varepsilon) = \frac{g'(0)}{f'(0)} + o(|\tilde{v}| + |\varepsilon|).$$

We will assume that the domain $\{|\tilde{u}|, |\tilde{v}| \leq \delta, \varepsilon \leq \varepsilon_0\}$, where the normal form is valid, is taken so small that

$$\frac{1}{2}|f'(0)| \leq |A(\tilde{v}, \varepsilon)| \leq 2|f'(0)|, \tag{15}$$

$$\frac{1}{2}\left|\frac{g'(0)}{f'(0)}\right| \leq |R_1(\tilde{v}, \varepsilon)| \leq 2\left|\frac{g'(0)}{f'(0)}\right| \tag{16}$$

and

$$\frac{1}{2}\left|\frac{g'(0)}{f'(0)^3}\right| \leq |\Psi(\tilde{v}, \varepsilon)| \leq 2\left|\frac{g'(0)}{f'(0)^3}\right|. \tag{17}$$

From (16), we have the (crude) estimate

$$|\tilde{v}(\sigma)| \leq |\tilde{v}_0| \exp(2\varepsilon|g'(0)/f'(0)|\sigma) \tag{18}$$

as long as $|\tilde{v}(\sigma)| \leq \delta$.

We can now turn to the other equation

$$\dot{\tilde{u}} = A(\tilde{v}, \varepsilon)\, \tilde{u}.$$

We will use the equation

$$\int_0^{t_0} A(\tilde{v}(s), \varepsilon)\, ds = \ln\left|\frac{\delta}{\varepsilon \tilde{v}_0 \Psi(\tilde{v}_0, \varepsilon)}\right|$$

to estimate $\tilde{v}_0$ such that the time σ_0 a trajectory takes from $\mathcal{C}$ to the section Δ exactly T/ε. Using the above estimates (15), (16) and (17), we arrive at

$$\frac{\delta f'(0)^3}{2\varepsilon g'(0)} \exp\left(-\frac{2|f'(0)|T}{\varepsilon}\right) \leq |\tilde{v}_0| \leq \frac{2\delta f'(0)^3}{\varepsilon g'(0)} \exp\left(-\frac{|f'(0)|T}{2\varepsilon}\right) \tag{19}$$

provided that $\tilde{v}(\sigma)$ does not leave the domain of validity of our estimates. In other words, if a trajectory starts with $\tilde{v}_0$ larger than the upper bound in (19) then it will certainly leave a δ- neighborhood of $\mathcal{C}$ after a time which is strictly less than T/ε and if $\tilde{v}_0$ is smaller than the lower bound of (19) then it will certainly take longer than T/ε.

We have to check only, that for such an initial $\tilde{v}_0$ the condition $\tilde{v}(T/\varepsilon) \leq \delta$ is satisfied up to the time T/ε. With the estimates from (18) we have immediately that for the initial condition

$$\tilde{v}_0 = \frac{2\delta f'(0)^3}{\varepsilon g'(0)} \exp(-\frac{T}{2\varepsilon |f'(0)|})$$

we obtain

$$\tilde{v}(\frac{T}{\varepsilon}) \leq \frac{2\delta f'(0)^3}{\varepsilon g'(0)} \exp(-\frac{T}{2\varepsilon |f'(0)|} + \frac{2|g'(0)|T}{|f'(0)|}) \tag{20}$$

which clearly tends to 0 as $\varepsilon \to 0$. Thus, for ε sufficiently small, $\tilde{v}$ remains in the domain of the normal form long enough and we are finished. To prove the last claim it is sufficient to note that

$$\int_{\sigma_\alpha}^{\sigma_0} A(\tilde{v}(\sigma), \varepsilon)\, d\sigma = \alpha \ln |\varepsilon|.$$

With (15) the claim follows immediately

□

Proof of theorem 1.2: After the normal form calculations we return now to our original (slow) time coordinate s. We will restrict ourselves to part (i) since (ii) can be proved analogously. Fix $\delta > 0$ small. From lemma 3.4 we know that, for any sufficiently small ε, there exist u_-^ε and u_+^ε, both exponentially close to u_i such that the trajectory starting in $(u_-^\varepsilon, -f(u_-^\varepsilon))$ needs time $\frac{1}{2\varepsilon}$ to arrive at Δ and the trajectory starting in $(u_+^\varepsilon, -f(u_+^\varepsilon))$ needs time $2/\varepsilon$. Moreover, due to (20) the v-coordinate differs from $-f(u_i)$ only by $\mathcal{O}(\varepsilon)$. After leaving the δ-neighborhood $\mathcal{C}$, the u-velocity u' is of order $\mathcal{O}(\varepsilon^{-1})$ and hence it takes only a time of order $\mathcal{O}(\varepsilon)$ until the trajectory reenters a δ-neighborhood of the other branch of $\mathcal{C}$. The v-coordinate is still only $\mathcal{O}(\varepsilon)$ away from its initial value. This implies that the trajectory reenters a δ-neighborhood of $\mathcal{C}$ near $\tilde{u}_i$, in particular, at a point where g has the same sign as in $\tilde{u}_i$. Now two possibilities can occur. Either g is positive, then the stable manifold of the next equilibrium to the left of $\tilde{u}_i$ prevents the trajectory from reaching $\mathcal{C}$. It “blocks” the curve $\mathcal{C}$ and the trajectory will follow the stable manifold and converge to the equilibrium to the left of $\tilde{u}_i$. Hence, the trajectory does not reach the curve $\mathcal{C}$ and is therefore not admissible.

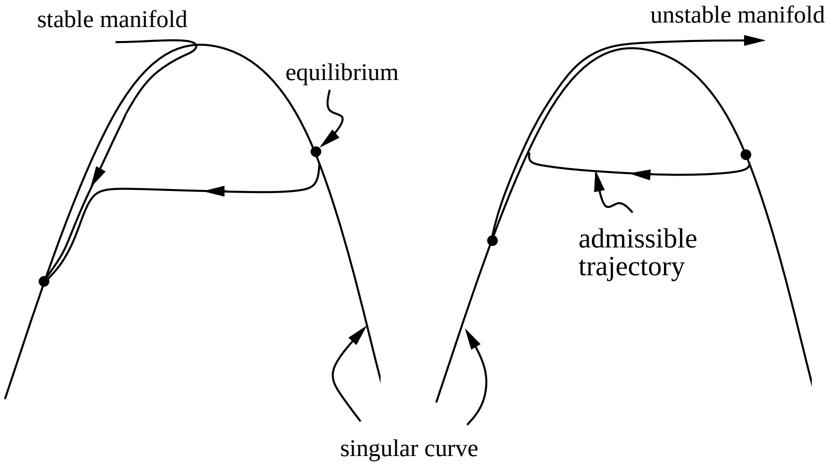

Figure 2: Trajectories for $g(\tilde{u}_i) > 0$ (left picture) and $g(\tilde{u}_i) < 0$ (right picture)

The situation is different when g is negative. Note that this is the case iff condition (3) is satisfied. There is no obstacle on the way to $\mathcal{C}$ and by lemma 3.3, the trajectory will need a time of order $\mathcal{O}(|\varepsilon \ln \varepsilon|)$ from entering a δ-neighborhood of $\mathcal{C}$ until it hits the curve $\mathcal{C}$. Altogether, the times $T_\pm$ from $(u^\varepsilon_\pm, -f(u^\varepsilon_\pm)$ to the first intersection with $\mathcal{C}$ are

$$T_- = \frac{1}{2} + \mathcal{O}(|\varepsilon \ln \varepsilon|) \text{ and } T_- = 2 + \mathcal{O}(|\varepsilon \ln \varepsilon|).$$

By continuity we can find between these trajectories a solution of the boundary value problem (5) and hence an equilibrium solution of our viscous balance law. From the trajectory in the Liénard plane we can easily deduce that the corresponding solution converges to u_i uniformly on any interval $[0, \kappa]$ and $u(1) \to \tilde{u}_i$.

□

4 Stability of the equilibria

In this section we are going to determine the Morse index of the equilibrium solutions found in theorem 1.2. For symmetry reasons we can again restrict ourselves to solutions that start near $(u_i, -f(u_i))$ and make a fast excursion at $s \approx 1$.

Lemma 4.1 *Consider a family u^ε of the equilibrium solutions we have found in theorem 1.2 starting near the same stationary point $(u_i, -f(u_i))$. Then there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ these equilibria u^ε are all hyperbolic with*

Morse index

$$\begin{aligned} i(u^\varepsilon) &= 1 \quad \textit{if } g'(u_i) < 0 \\ i(u^\varepsilon) &= 2 \quad \textit{if } g'(u_i) > 0. \end{aligned}$$

Proof of the lemma: We have to consider the eigenvalue problem

$$\left.\begin{aligned} \varepsilon w_{xx} - f'(u^\varepsilon) w_x - f''(u^\varepsilon) u_x^\varepsilon w + g'(u^\varepsilon) w &= \lambda w \\ w_x(0) = w_x(1) &= 0. \end{aligned}\right\} \tag{21}$$

at the family of equilibria u^ε of (6). The eigenvalue equation can be written as a first order system

$$\begin{aligned} \varepsilon w_x &= z + f'(u^\varepsilon) w \\ z_x &= -(g'(u^\varepsilon) - \lambda) w \\ w_x(0) &= w_x(1) = 0. \end{aligned}$$

Performing a Prüfer transformation

$$\varepsilon w = \varrho \cos\varphi \ , \quad z = -\varrho \sin\varphi$$

leads to equations for ϱ and φ. We will only need the φ-equation

$$\varphi_x = \sin^2\varphi + \frac{g'(u^\varepsilon) - \lambda}{\varepsilon} \cos^2\varphi - \frac{f'(u^\varepsilon)}{2\varepsilon} \sin 2\varphi \tag{22}$$

to determine the Morse index of u^ε. The Neumann boundary conditions for the original eigenvalue problem show up as initial resp. terminal condition

$$\tan\varphi(x) = \frac{f'(u^\varepsilon(x))}{\varepsilon} \text{ at } x = 0 \text{ and } x = 1.$$

So, if $\varphi_0(x; \varepsilon, \lambda)$ denotes the solution of (22) with initial value

$$\varphi_0(x = 0) = \arctan \frac{f'(u^\varepsilon(0))}{\varepsilon}$$

there is an easy criterium for λ to be an eigenvalue: The right boundary condition has to be satisfied and thus

$$\lambda \text{ is an eigenvalue} \iff \tan\varphi_0(1; \varepsilon, \lambda) = \frac{f'(u^\varepsilon(1))}{\varepsilon}.$$

We will use the following relation between the Morse index $i(u^\varepsilon)$ and $\varphi_0(1; \varepsilon, \lambda = 0)$:

Lemma 4.2 *Let* $\varphi_1 := \arctan\left(f'(u^\varepsilon(1))/\varepsilon\right)$. *Then the following holds:*

(i) If $\varphi_0(1; \varepsilon, 0) - \varphi_1 \neq k\pi$, $k = 0, 1, 2, \ldots$, *then the equilibrium* u^ε *is hyperbolic.*

(ii) If $\varphi_0(1;\varepsilon,0)-\varphi_1 \in ((k-1)\pi, k\pi)$, then $i(u^\varepsilon)=k$.

Proof : Part (i) is just a simple consequence of the characterization of eigenvalues. If there was a nontrivial solution of the eigenvalue problem with $\lambda = 0$ then the corresponding φ would satisfy $\tan\varphi = \tan\varphi_1$.

To prove part (ii) consider λ as a parameter and note that $\varphi_0(1;\varepsilon,\lambda)$ depends monotonically on λ and tends to $-\frac{\pi}{2}$ as $\lambda \to -\infty$ and to $+\infty$ as $\lambda \to +\infty$. For the eigenvalue λ_k $(k = 0,1,2,\ldots)$ we find

$$\varphi_0(1;\varepsilon,\lambda_k) = \varphi_1 + (k-1)\pi.$$

To determine the Morse index of u^ε is hence equivalent to counting how many of the numbers $\varphi_1 + k\pi$ lie between $-\frac{\pi}{2}$ and $\varphi_0(1;\varepsilon,0)$.

□

Hale and Sakamoto [4] have used the Prüfer transformation to compute Morse indices and eigenvalues for equilibria of the equation $u_t = \varepsilon u_{xx} + f(x,u)$. Although the details are quite different, we will see that with similar methods we can find the Morse index of u^ε.

The main idea is the following: Our goal is to compute $\varphi_0(1;\varepsilon,0)-\varphi_1$ accurately enough to decide to which of the intervals $((k-1)\pi, k\pi)$ it belongs. This will be achieved by finding invariant strips in the (x,φ)-plane. These strips allow to trace solutions for a time of order 1. Outside these invariant strips we will use a comparison with the solutions of (22) to different initial values. Note that u^ε_x solves the linearized equation (21) with Dirichlet boundary conditions and that Dirichlet boundary conditions translate into

$$\varphi(x) \equiv \frac{\pi}{2} \ (mod\ \pi) \text{ at } x = 0,1.$$

Hence, we already know two solutions of (22): The solution φ_- with initial value $\varphi_-(0;\varepsilon,0) = -\frac{\pi}{2}$ satisfies $\varphi_-(1;\varepsilon,0) = \frac{\pi}{2}$. Due to π-periodicity of equation (22) there is another solution φ_+ with initial value $\varphi_+(0;\varepsilon,0) = \frac{\pi}{2}$ and $\varphi_+(1;\varepsilon,0) = \frac{3\pi}{2}$.

The solution φ_0 with $\varphi_0(0;\varepsilon,0) = \arctan(f'(u^\varepsilon(0))/\varepsilon)$ is confined between φ_- and φ_+. As we will prove, φ_0 will after some time follow one of these solutions and end up very close to either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

The only property of u^ε that we need is, that it is almost constant over most of the interval $[0,1]$ and about the sign of f'. The latter fact shows up in the relation

$$f'(u^\varepsilon(1)) < 0 < f'(u^\varepsilon(0)).$$

By our estimates for the time the trajectory has to spend near the stationary point, we can choose $\bar{x} \in [0,1]$ (depending on ε) with

$$1 - \bar{x} = \mathcal{O}(|\varepsilon \ln \varepsilon|) \tag{23}$$

such that

$$u_x^\varepsilon \le C\varepsilon^2 \qquad \forall x \in [0, \bar{x}]. \tag{24}$$

and

$$f'(u^\varepsilon) > \gamma \qquad \forall x \in [0, \bar{x}]$$

for some constant $\gamma > 0$ not depending on $\varepsilon \in (0, \varepsilon_0]$.

We will first establish the existence of invariant strips. To this end define $\Phi_-(x)$ and $\Phi_+(x)$ as the two angles in $(-\frac{\pi}{2}, \frac{\pi}{2})$ where $\varphi_x = 0$, or in other words,

$$\varepsilon \sin^2 \Phi_\pm - f'(u^\varepsilon) \sin \Phi_\pm \cos \Phi_\pm + g'(u^\varepsilon) \cos^2 \Phi_\pm = 0 \tag{25}$$

leading to

$$\tan \Phi_\pm(x) = \frac{f'(u^\varepsilon(x)) \pm \sqrt{f'(u^\varepsilon(x))^2 - 4\varepsilon g'(u^\varepsilon(x))}}{2\varepsilon}.$$

As both $f'(u)$ and $g'(u)$ are bounded for $u \in [u_{min}, u_{max}]$ using (23) and the asymptotic behavior of arctan near $\frac{\pi}{2}$ we get for $x \in [0, \bar{x}]$ where $f'(u^\varepsilon(x)) > 0$:

$$\tan \Phi_-(x) = \frac{g'(u^\varepsilon(x))}{f'(u^\varepsilon(x))} + \mathcal{O}(\varepsilon),$$

$$\tan \Phi_+(x) = \frac{f'(u^\varepsilon(x))}{\varepsilon} + \mathcal{O}(1)$$

such that

$$\Phi_+(x) = \frac{\pi}{2} - f'(u^\varepsilon(x))^{-1}\varepsilon + \mathcal{O}(\varepsilon^2).$$

We show now the existence of a narrow negatively invariant strip around Φ_+.

Lemma 4.3 *Consider (22) for $\lambda = 0$, a given equilibrium u^ε and $x \in [0, \bar{x}]$. Then:*

There is a negatively invariant strip of width $\mathcal{O}(\varepsilon^3)$ around Φ_+.

Proof: We have to compare $\frac{d\Phi_+}{dx}$ with φ_x at $\varphi = \Phi_+ \pm k\varepsilon^3$ for some k. Let's start with

$$\frac{d\Phi_+}{dx} = \frac{2\varepsilon}{4\varepsilon^2 + \left(f' + \sqrt{f'^2 - 4\varepsilon g'}\right)^2} \cdot \left(f'' + \frac{f'f'' - 2\varepsilon g''}{\sqrt{f'^2 - 4\varepsilon g'}} \right) \cdot u_x^\varepsilon(x).$$

Here we have written f' as an abbreviation for $f'(u^\varepsilon(x))$, etc.

Expanding the square root one finds easily that the first term is of order $\mathcal{O}(\varepsilon)$ while the second is of order $\mathcal{O}(1)$. Together with (24) we get

$$\left| \frac{d\Phi_+}{dx} \right| \le C\varepsilon^2 \qquad \text{for } x \in [0, \bar{x}].$$

On the other hand, applying elementary addition formulas for sines and cosines,

$$\begin{aligned}
\frac{d\varphi}{dx}(\Phi_+ + k\varepsilon^3) &= \sin^2(\Phi_+ + k\varepsilon^3) + \frac{g'}{\varepsilon}\cos^2(\Phi_+ + k\varepsilon^3) - \frac{f'}{2\varepsilon}\sin(2\Phi_+ + 2k\varepsilon^3) \\
&= \cos 2k\varepsilon^3 \left(\sin^2\Phi_+ + \frac{g'}{\varepsilon}\cos^2\Phi_+ - \frac{f'}{2\varepsilon}\sin 2\Phi_+\right) \\
&\quad + \sin^2 k\varepsilon^3 \left(1 + \frac{g'}{\varepsilon}\right) \\
&\quad - \frac{1}{2}\sin 2k\varepsilon^3 \left(\frac{g'}{\varepsilon}\sin 2\Phi_+ + \frac{f'}{\varepsilon}\cos 2\Phi_+ - \sin 2\Phi_+\right).
\end{aligned}$$

The first term vanishes due to (25) and since

$$g'(u^\varepsilon)\sin 2\Phi_+ + f'(u^\varepsilon)\cos 2\Phi_+ = f'(u^\varepsilon) - 2\varepsilon\sin^2\Phi_+ \tan\Phi_+ = -f'(u^\varepsilon) + \mathcal{O}(\varepsilon) \tag{26}$$

the last term dominates and yields

$$\begin{aligned}
\frac{d\varphi}{dx}(\Phi_+ + k\varepsilon^3) &\geq \frac{1}{2}k\gamma\cdot\varepsilon^2 \\
\frac{d\varphi}{dx}(\Phi_+ - k\varepsilon^3) &\leq -\frac{1}{2}k\gamma\cdot\varepsilon^2
\end{aligned}$$

for ε small enough and $x \in [0, \bar{x}]$. Choosing k large enough gives negative invariance for the strip around Φ_+.

□

The next lemma provides us with some larger invariant strips with the additional property that $|\varphi_x|$ is large outside these strips.

Lemma 4.4 *Consider (22) for $\lambda = 0$, a family u^ε of equilibria and $x \in [0, \bar{x}]$. Then there exists some $\varepsilon_0 > 0$ and constants $\kappa, C_1, C_2, C_3 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$*

(i) the strip $[\Phi_+ - \kappa, \Phi_+ + \kappa]$ is negatively invariant and for solutions φ_1, φ_2 inside this strip

$$|\varphi_1(x_2) - \varphi_2(x_2)| \geq e^{\frac{C_1}{\varepsilon}(x_2 - x_1)}|\varphi_1(x_1) - \varphi_2(x_1)| \text{ for } 0 \leq x_1 \leq x_2 \leq \bar{x}.$$

Analogously, the strip $[\Phi_- - \kappa, \Phi_- + \kappa]$ is positively invariant and for solutions φ_1, φ_2 inside this strip

$$|\varphi_1(x_2) - \varphi_2(x_2)| \leq e^{-\frac{C_2}{\varepsilon}(x_2 - x_1)}|\varphi_1(x_1) - \varphi_2(x_1)| \text{ for } 0 \leq x_1 \leq x_2 \leq \bar{x}.$$

(ii) Outside these strips

$$|\varphi_x| \geq \frac{C_3}{\varepsilon}.$$

Proof: (i) The existence of the invariant strips is proved in a similar fashion as in the preceding lemma, so we omit the proof here.

To prove the contraction and expansion properties we denote with $H(\varphi, \varepsilon)$ the right hand side of (22) with $\lambda = 0$, i.e.

$$H(\varphi, \varepsilon) = \sin^2 \varphi + \frac{g'(u^\varepsilon)}{\varepsilon} \cos^2 \varphi - \frac{f'(u^\varepsilon)}{2\varepsilon} \sin 2\varphi.$$

By (26)

$$\frac{\partial H}{\partial \varphi}(\Phi_+, x) = \frac{f'(u^\varepsilon(x))}{\varepsilon} + \mathcal{O}(1)$$

such that

$$\frac{\partial H}{\partial \varphi}(\varphi, x) > \frac{C_1}{\varepsilon}$$

for ε small enough and $\varphi \in [\Phi_+ - \kappa, \Phi_+ + \kappa]$.

Similarly,

$$\frac{\partial H}{\partial \varphi}(\Phi_-, x) = -\frac{f'(u^\varepsilon(x))}{\varepsilon} + \mathcal{O}(1)$$

and

$$\frac{\partial H}{\partial \varphi}(\varphi, x) < -\frac{C_2}{\varepsilon}$$

for ε small enough and $\varphi \in [\Phi_- - \kappa, \Phi_- + \kappa]$.

(ii) Define

$$C_3 := \frac{1}{2} \inf |g'(u^\varepsilon(x)) \cos^2 \varphi - f'(u^\varepsilon(x)) \sin \varphi \cos \varphi| > 0$$

where the infimum is taken over the compact region that is obtained from the rectangle $[0, \bar{x}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ by removing strips

$$\{(x, \varphi);\ 0 \le x \le \bar{x},\ \Phi_\pm - \kappa < \varphi < \Phi_\pm + \kappa\}.$$

Then for $\varepsilon < \varepsilon_0 := C_3$, obviously

$$\varepsilon |H(\varphi, \varepsilon)| \ge C_3$$

and the proof is complete.

□

The important observation is now that the initial value $f'(u^\varepsilon(0))/\varepsilon$ of the solution φ_0 we are mainly interested in lies above or below this negatively invariant $\mathcal{O}(\varepsilon^3)$-strip around Φ_+ depending on the sign of $g'(u^\varepsilon(0))$ which is the same as the sign of $g'(u_E)$. This follows simply from the expansion

$$\begin{aligned} \tan \Phi_+(0) &= \frac{f'(u^\varepsilon(0)) - \sqrt{f'(u^\varepsilon(0))^2 - 4\varepsilon g'(u^\varepsilon(0))}}{2\varepsilon} \\ &= \frac{f'(u^\varepsilon(0))}{\varepsilon} - \frac{g'(u^\varepsilon(0))}{f'(u^\varepsilon(0))} + \mathcal{O}(\varepsilon). \end{aligned}$$

Using the Laurent series of arctan at $\pi/2$ one gets

$$\Phi_+(0) = \frac{\pi}{2} - \frac{1}{f'(u^\varepsilon(0))}\varepsilon - \frac{g'(u^\varepsilon(0))}{f'(u^\varepsilon(0))}\varepsilon^2 + \mathcal{O}(\varepsilon^3). \tag{27}$$

In particular, for ε small enough and $g'(u_E) < 0$ the initial value of φ_0 lies below the negatively invariant strip. Thus φ_0 has to stay below Φ_- at least up to $x = \bar{x}$. Similarly, for $g'(u_E) < 0$ the initial value of φ_0 lies above the negatively invariant strip and φ_0 stays above this strip.

Having established the existence of invariant strips, we want to take a solution of (22) that stays inside the negatively invariant strip as a reference. Let φ_N be a solution on $[0, \bar{x}]$ inside the negatively invariant strip. Clearly φ_N can, for instance, be obtained by solving (22) backward starting with $\varphi_N(\bar{x}) = \Phi_+(\bar{x})$.

Using lemma 4.4, we are now able to describe the behavior of $\varphi_0(\cdot;\varepsilon,0)$.

1) $g'(u_i) < 0$ **:** In this case we compare first φ_0 and φ_N. Since φ_N lies in the $\mathcal{O}(\varepsilon^3)$-strip around Φ_+ and from (27) we have that

$$\varphi_N(0) - \varphi_0(0;\varepsilon,0) \geq C_4\varepsilon^2.$$

Thus we can find some $\tilde{x}_1$ of order $\mathcal{O}(|\varepsilon \ln \varepsilon|)$ such that φ_0 leaves the negatively invariant strip around Φ_+ at $x = \tilde{x}_1$. Outside the strip, φ_x is large, so there is $\tilde{x}_2$ with

$$\tilde{x}_2 - \tilde{x}_1 = \mathcal{O}(\varepsilon)$$

such that φ_0 enters the positively invariant strip around Φ_- at $x = \tilde{x}_2$. The situation is depicted in figure 3.

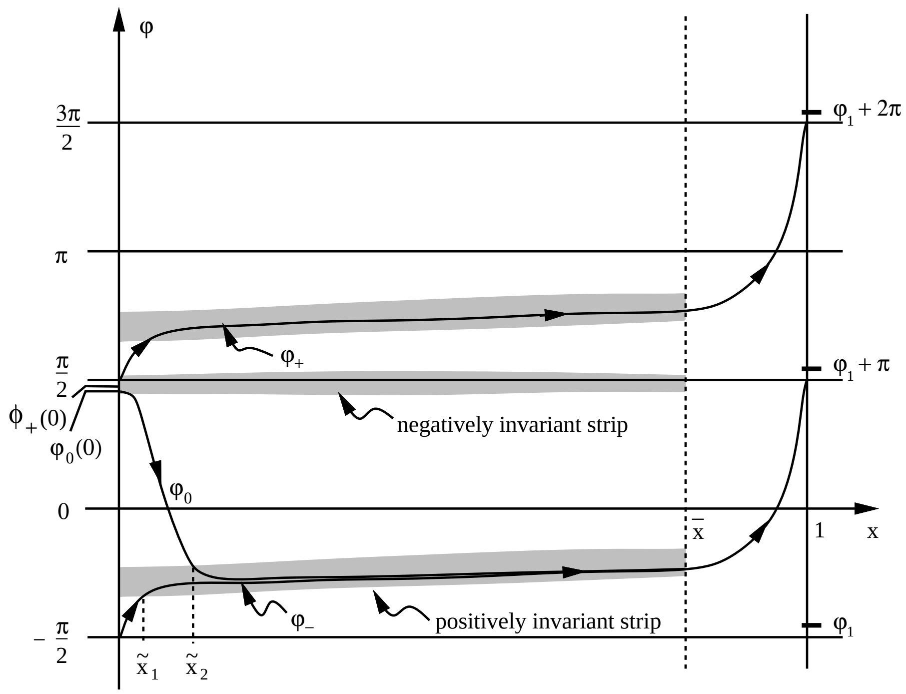

Figure 3: The case $g'(u_E) < 0$

Analogous arguments show that at $x = \tilde{x}_2$ also φ_- must have entered the positively invariant strip from below. Hence

$$|\varphi_0(\tilde{x}_2; \varepsilon, 0) - \varphi_-(\tilde{x}_2; \varepsilon, 0)| \leq 2\kappa$$

and due to the contraction property inside the positively invariant strip

$$\varphi_0(\bar{x}; \varepsilon, 0) - \varphi_-(\bar{x}; \varepsilon, 0) = \mathcal{O}(\exp(-C_2/\varepsilon)).$$

Even though φ_0 and φ_- may separate at an exponential rate of order C/ε on the interval $[\bar{x}, 1]$ the difference between the two solutions stays small since the length of this interval is only of order $\mathcal{O}(|\varepsilon \ln \varepsilon|)$:

$$\varphi_0(1; \varepsilon, 0) - \varphi_-(1; \varepsilon, 0) = \mathcal{O}(\varepsilon^{-C} \exp(-C_2/\varepsilon)).$$

Thus, the difference tends to 0 very rapidly as $\varepsilon \searrow 0$ and since we know that

$$\varphi_-(1; \varepsilon, 0) = \frac{\pi}{2}$$

we can conclude that for ε sufficiently small

$$\varphi_0(1;\varepsilon,0) \leq \varphi_1 + \pi = \frac{\pi}{2} - \frac{1}{f'(u^\varepsilon(1))}\varepsilon + \mathcal{O}(\varepsilon)$$

since $f'(u^\varepsilon(1)) < 0$.

Hence,

$$\varphi_0(1;\varepsilon,0) - \varphi_1 \in (0,\pi)$$

and by lemma 4.2 the solution u^ε is for all small ε a hyperbolic equilibrium solution with Morse index $i(u^\varepsilon) = 1$.

2) $g'(u_i) > 0$ **:** Again we compare first φ_0 and φ_N. The difference consists of the fact that φ_0 lies above the $\mathcal{O}(\varepsilon^3)$-strip that is negatively invariant and leaves the strip $[\Phi_+ - \kappa, \Phi_+ + \kappa]$ at the top. Therefore, φ_0 enters the positively invariant strip $[\Phi_- - \kappa + \pi, \Phi_- + \kappa + \pi]$ at some $\tilde{x}_2$ which is of order $\mathcal{O}(|\varepsilon \ln \varepsilon|)$. The solution φ_+ enters this strip as well and as above one shows that

$$\varphi_0(1;\varepsilon,0) - \varphi_+(1;\varepsilon,0) = \mathcal{O}(\varepsilon^{-C}\exp(-C_2/\varepsilon)).$$

Since

$$\varphi_+(1;\varepsilon,0) = \frac{3\pi}{2}$$

this yields

$$\varphi_0(1;\varepsilon,0) - \varphi_1 \in (\pi, 2\pi)$$

and by lemma 4.2 the equilibrium u^ε of the viscous balance law is hyperbolic with Morse index $i(u^\varepsilon) = 2$.

□

Lemma 4.5 *For any u_i that satisfies (3) and ε sufficiently small there is exactly one equilibrium solution u^ε of the type found in theorem 1.2.*

Proof: Suppose there were two equilibrium solutions u^ε and $\tilde{u}^\varepsilon$ both starting near the same stationary point. Consider now the interval $[u^\varepsilon(0), \tilde{u}^\varepsilon(0)]$. It contains only finitely many points that correspond to solutions of the boundary value problem. Otherwise there would be a non-hyperbolic equilibrium solution in contrast to the preceding lemma. So, without restriction, we may suppose that there is no other solution $\hat{u}^\varepsilon$ with $\hat{u}^\varepsilon(0) \in [u^\varepsilon(0), \tilde{u}^\varepsilon(0)]$. Consider now the curve

$$\mathcal{S} = \{((u(1), v(1)) \mid (u,v) \text{ solves}(6)\text{with } u(0) \in [u^\varepsilon(0), \tilde{u}^\varepsilon(0)] \text{ and } v(0) = 0\}$$

which is a part of the shooting curve used by Fiedler and Rocha [2]. Its endpoints lie on the u-axis $\{v = 0\}$ and there are no other intersections with the

u-axis as they would correspond to other equilibrium solutions. We concentrate on the angle between $\mathcal{S}$ and the u-axis at the two endpoints. Rocha [6] has shown that this angle is exactly the value $\varphi_0(1;\varepsilon,0)$ we have calculated in the proof of lemma 4.1. So denote with φ_0 this angle calculated for the equilibrium u^ε and with $\tilde{\varphi}_0$ the corresponding angle for $\tilde{u}^\varepsilon$. Since $\mathcal{S}$ lies on one side of the u-axis not both of the angles φ_0 and $\tilde{\varphi}_0$ can lie in the same interval $(k-1)\pi, k\pi)$ and hence not both equilibria are of the same Morse index. This contradiction shows that there can be only one equilibrium u^ε.

□

5 Proof of theorem 1.3

We have to show that there are no other equilibrium solutions as the spatially homogenous ones and those found in theorem 1.2. By the previous lemma 4.5 each of those is unique for ε sufficiently small. The first part is rather easy. From lemma 2.4 we know that any admissible solution must be a monotone trajectory from the arc of $\mathcal{C}$ where $f' > 0$ to the left arc of $\mathcal{C}$ where $f' < 0$. Since this has to take time 1 the trajectory either has to stay near an equilibrium at the beginning or at the end. In lemma 3.4 we have seen that this means starting or ending up exponentially close to an equilibrium point. Note that not both can happen sine no two equilibrium points have the same v-coordinate by **(H5)**. Hence the possible solutions are exactly the ones we have found in theorem 1.2.

In lemma 3.2 of [5] it was shown that the spatially homogenous equilibria $u \equiv u_i$ are hyperbolic provided that **(H4)** holds and that the Morse index is 0 for $g'(u_i) < 0$ and the Morse index is 1 for $g'(u_i) > 0$. Hence we have $(l+1)/2$ spatially homogenous equilibria with Morse index 0 and $(l-1)/2$ with Morse index 1.

We also know the number of non-homogenous equilibria. By the results of the previous section this is simply the number m of g-zeroes for which (3) holds. It remains only to determine how many of those m equilibria have Morse index 2. We will prove that the number of non-homogenous equilibria with Morse index 2 is equal to the number of non-homogenous equilibria with Morse index 1 by grouping them into pairs where one has index 1 while the other has index 2. To this end, consider a permutation $\{v_1, v_2, \ldots, v_l\}$ of the homogenous equilibria $\{u_1, u_2, \ldots, u_l\}$ such that

$$f(v_1) < f(v_2) < \ldots < f(v_l).$$

It is a simple observation that $g'(v_{2i-1})$ and $g'(v_{2i})$ have different sign. To see this, note that it is obviously true for $i = 1$. Then induction applies: Either $f'(v_{2i-1})$ and $f'(v_{2i})$ are of the same sign and the claim is true or they are of different sign and the claim follows by an induction step. Moreover, condition

(3) either holds for both v_{2i-1} and v_{2i} or for neither of them. Again, if $f'(v_{2i-1})$ and $f'(v_{2i})$ are of the same sign this claim is trivial, and in the other case it is a simple matter of checking several possibilities for the sign of g on the intervals $[v_{2i}, \tilde{v}_{2i-1}]$ and $[v_{2i-1}, \tilde{v}_{2i}]$. If (3) holds for both v_{2i-1} and v_{2i} then theorem 1.2 gives us two non-homogenous equilibria corresponding to v_{2i-1} and v_{2i}. By lemma 4.1 one of them has Morse index 1 while the other has Morse index 2.

This argument is valid for $v_1, v_2, \ldots, v_{l-1}$. To finish the proof we note that it is a consequence of the dissipativeness condition **(H1)** that for the last homogenous equilibrium v_l condition (3) never holds.

□

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