Asymptotic Behavior of Spatially Inhomogeneous Balance Laws

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Abstract. We study the longtime behavior of spatially inhomogeneous scalar balance laws with periodic initial data and a convex flux.

Our main result states that for a large class of initial data the entropy solution will either converge uniformly to some steady state or to a discontinuous time-periodic solution. This extends results of Lyberopoulos, Sinestrari and Fan & Hale obtained in the spatially homogeneous case. The proof is based on the method of generalized characteristics together with ideas from dynamical systems theory.

A major difficulty consists of finding the periodic solutions which determine the asymptotic behavior. To this end we introduce a new tool, the Rankine-Hugoniot vector field, which describes the motion of a (hypothetical) shock with certain prescribed left and right states. We then show the existence of periodic solutions of the Rankine-Hugoniot vector field and prove that the actual shock curves converge to these periodic solutions.

Keywords: Poincaré-Bendixson; generalized characteristics; modulated traveling waves.

1. Introduction

In this paper we study the long-time behavior of scalar, spatially non-homogeneous hyperbolic balance laws

$$\begin{aligned} \frac{\partial}{\partial t}u + \frac{\partial}{\partial x}f(u) &= g(u,x) \\ u(x,0) &= u_0(x) \end{aligned} \tag{1.1}$$

with $x \in S^1 \sim \mathbb{R}/\mathbb{Z}$ and $t \geq 0$. The flux $f : \mathbb{R} \to \mathbb{R}$ is assumed to be a strictly convex function while $u_0 \in BV(S^1, \mathbb{R})$ belongs to the space of functions with bounded variation in the sense of Tonelli. Therefore a solution $u(x,t)$ maps the cylinder $S^1 \times \mathbb{R}^+$ into $\mathbb{R}$. In the case that f is a convex function a solution $u(x,t)$ is called an *admissible solution* if it satisfies for almost all $t > 0$ the entropy condition

$$u(x-,t) \geq u(x+,t) \tag{1.2}$$

where $u(x\pm,t)$ denotes $lim_{h\searrow 0} u(x \pm h,t)$. We can, and will, assume that admissible solutions, possibly after a modification on a set of zero measure, are continuous from the left and satisfy the entropy condition for all $t > 0$.

The Cauchy problem associated with a scalar hyperbolic conservation or balance law has a long history. It is known, that there are in general no classical solutions of this problem for all times, even if the initial data is smooth. In general there is no unique weak solution to equation (1.1), but the entropy condition singles out a unique one. In 1967, Volpert [16] could show that there is a unique solution that satisfies the entropy condition (1.2) if the initial data is continuous.

In 1970 Kružkov proved in [9], that there is a unique solution which satisfies the entropy condition, even if the initial data does not. At the points where the entropy condition is violated in the initial condition, rarefaction waves smoothen these points in arbitrarily short time. Moreover, a comparison principle holds: If we consider two solutions for which the initial data are ordered, then the solutions remain ordered for all positive times.

Lyberopoulos [12] studied equation (1.1) with $g(u,x) = u$ and f strictly convex. He was able to show that the solution to periodic initial data with period L and $\int_0^L u_0(x)\mathrm{d}x = 0$ was converging to a traveling wave for $t \to \infty$.

Fan and Hale [5] and Sinestrari [14] published a result for general source terms $g(u,x) = g(u)$ that possess only single zeros and satisfies the sign condition $g(u)u < 0$ for large $|u|$. They were able to show that the solution either converges uniformly to a spatially homogeneous steady state or that it converges to a discontinuous traveling wave with wave speed $f'(a_{2m})$, where a_{2m} is one of the zeros of g with $g'(a_{2m}) > 0$.

In this work we will consider more general source terms $g(u,x) \in C^1$ that depend explicitly on the space variable x. Technically, this causes several difficulties: The stationary solutions are no longer constant functions, comparison with spatially homogeneous solutions is not available any more and the characteristic system associated with (1.1) is not of skew-product structure as in the spatially homogeneous case.

Our main result states that the solutions either converge uniformly to a spatially inhomogeneous stationary solution or converge towards a shock solution that is periodic in time. A description via unstable fibers of periodic orbits of the associated characteristic equation shows that these time-periodic solutions are periodically modulated waves. The results in this article are based on the first author's thesis [4].

2. The method of generalized characteristics

Characteristics are a classical tool to study smooth solutions of first order partial differential equations. Dafermos [2] introduced the notion of generalized characteristics and showed how they can be used to get information about the structure of non-smooth solutions of scalar conservation laws with a convex flux. His method of generalized characteristics is crucial for our analysis and we recall in this chapter the notions and results we will use later on.

Definition 2.1. *A Lipschitz curve $x = \xi(t)$, defined on an interval $I = [a, b]$ is called a* generalized characteristic *associated with the solution u of (1.1) if it satisfies the differential inequality*

$$\dot{\xi} \in [f_u(u(\xi(t)+, t)), f_u(u(\xi(t)-, t))]$$

for almost all $t \in I$.

By the theory of Filippov [8] it can be shown, that there exists at least one forward and one backward characteristic through any point $(\bar{x}, \bar{t}) \in (\mathbb{R} \times \mathbb{R}^+)$.

It seems that Definition 2.1 does allow many different propagation speeds. However, as the following theorem shows, this is not the case:

Proposition 2.2. *Let $\xi : [a, b] \to \mathbb{R}$ be a characteristic. Then the following holds for almost all $t \in [a, b]$:*

$$\dot{\xi}(t) = \begin{cases} f_u(u(\xi(t)\pm, t)) & \text{if } u(\xi(t)-, t) = u(\xi(t)+, t) \\ \dfrac{f(u(\xi(t)+, t)) - f(u(\xi(t)-, t))}{u(\xi(t)+, t) - u(\xi(t)-, t)} & \text{if } u(\xi(t)-, t) > u(\xi(t)+, t) \end{cases} \tag{2.2}$$

The second equation in this theorem is called the *Rankine-Hugoniot condition* for the propagation speed of shocks.

Definition 2.3. *A characteristic on the interval $[a, b]$ is called* genuine, *if*

$$u(\xi(t)-, t) = u(\xi(t)+, t) \quad \text{for almost all } t \in [a, b].$$

The set of backward characteristics through $(\bar{x}, \bar{t})$ spans a funnel between the *minimal backward characteristic* $\xi^-(t; \bar{x}, \bar{t})$ and the *maximal backward characteristic* $\xi^+(t; \bar{x}, \bar{t})$ through $(\bar{x}, \bar{t})$.

Proposition 2.4. *Let $(\bar{x}, \bar{t}) \in \mathbb{R} \times \mathbb{R}^+$ be arbitrary. Then the minimal and the maximal backward characteristic $\xi^-(\cdot; \bar{x}, \bar{t})$ respectively $\xi^+(\cdot; \bar{x}, \bar{t})$ are genuine characteristics.*

The following theorem is central for our further analysis, as it relates the partial differential equation (1.1) to a system of two ordinary differential equations:

Proposition 2.5. *If $\xi(\cdot)$ is a genuine characteristic on the interval $[a, b]$ then there exists a function $v(\cdot)$ on $[a, b]$ such that $(\xi(\cdot), v(\cdot))$ is a continuously differentiable solution of the following* characteristic system *of ordinary differential equations:*

$$\begin{aligned} \dot{\xi}(t) &= f_u(v(t)) \\ \dot{v}(t) &= g(\xi(t), v(t)) \end{aligned} \tag{2.4}$$

If

$$v(a) = u(\xi(a), a)$$

then $v(t)$ is a solution of equation(1.1) on the characteristic $\xi(t)$ at time t. We therefore have:

$$u(\xi(t)-, t) = v(t) = u(\xi(t)+, t) \quad \forall t \in [a, b]\,. \tag{2.6}$$

From the theorem of Picard–Lindelöf we deduce:

Corollary 2.6. *Two genuine characteristics can only intersect at their endpoints, in particular backward characteristics do not intersect for $t > 0$.*

Corollary 2.7. *If the solution of the characteristic system (2.4) through any $(\bar{x}, \bar{v}, \bar{t}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$ on $[0, \bar{t}]$ is bounded, then every backward characteristic through any $(\bar{x}, \bar{t})$ is defined on the interval $[0, \bar{t}]$.*

Corollary 2.8. *If the solution through any $(\bar{x}, \bar{v}, \bar{t}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$ is bounded on $[\bar{t}, \infty)$, then there is no blow up in finite time on any forward characteristic, therefore it is defined for all times in $[\bar{t}, \infty)$.*

Proposition 2.9. *If the assumptions of the Corollaries 2.7 and 2.8 are satisfied, there is a unique forward characteristic through any point $(\bar{x}, \bar{t})$ with $\bar{t} > 0$.*

Forward characteristics can be used to construct the solution of a scalar hyperbolic balance law from the initial condition $u_0(x)$, even if the solution or the initial condition is not continuous.

The backward characteristics are useful to analyze the qualitative properties of a given solution. Mainly this aspect will be used in the following chapters.

3. Longtime behavior of solutions

In this chapter we will state the technical assumptions and formulate our main result, Theorem 3.6.

Like in [5] and [14] we need assumptions on the flux function f and on the source term g:

(A1) $f \in C^2(\mathbb{R}, \mathbb{R})$ and f is strictly convex, i.e. $f''(u) > c > 0$ for all u

(A2) $g \in C^1$ and there exists a constant $M > 0$ such that

$$u \cdot g(u, x) < 0 \text{ for all } |u| > M \text{ and all } x \in S^1 \tag{3.1}$$

(A3) The characteristic system (2.4) possesses no equilibria.

Condition **(A1)** is central for the method of generalized characteristics.

A direct consequence of **(A2)** is, that the solutions of (2.4) remain bounded for all positive times $t > 0$ because for v the following estimate holds:

$$|v(t)| \leq \max \left\{ \max_{x \in S^1} |u_0(x)|, |M| \right\} \quad \forall t \in [0, \infty).$$

From this we deduce with Proposition 2.9 the existence of a unique forward characteristic through any point (x, t) with $x \in S^1$ and $t > 0$.

As a direct consequence of the entropy condition (1.2) and the periodicity we have the following statement:

Lemma 3.1. *Let $a(\cdot)$ be a continuous function on $[\xi_1, \xi_2] \subseteq S^1$. If the solution of equation (1.1) at a fixed time $t > 0$ satisfies*

$$u(\xi_1, t) < a(\xi_1) \quad \textit{and} \quad u(\xi_2, t) > a(\xi_2) \tag{3.3}$$

then there is a $\tilde{\xi} \in (\xi_1, \xi_2)$ with $u(\tilde{\xi}, t) = a(\tilde{\xi})$.

Proof. Without loss of generality we can assume $\xi_1 < \xi_2$, then

$$\tilde{\xi} := \sup\{y \in [\xi_1, \xi_2];\ u(\xi, t) \leq a(\xi) \text{ for all } \xi_1 \leq \xi < y\}$$

has the desired property. To see this we note first that $u(\cdot, t)$ must be continuous at $x = \tilde{\xi}$. Otherwise the entropy condition (1.2) requires

$$u(\tilde{\xi}+, t) < u(\tilde{\xi}-, t)$$

and since BV functions possess one-sided limits $u(\xi, t) < a(\xi)$ would hold in some right neighborhood of $\tilde{\xi}$. This however contradicts the definition of $\tilde{\xi}$, so u is in fact continuous with respect to x at $x = \tilde{\xi}$.

The definition of $\tilde{\xi}$ now implies that u is strictly bigger than $a(\xi)$ in a right neighborhood of $\tilde{\xi}$. Together with the continuity at $\tilde{\xi}$ this shows that $u(\tilde{\xi}, t) = a(\xi)$.

□

To study the longtime behavior of solutions we need some further assumptions on the stationary solutions of (1.1). We will formulate them using the periodic solutions of equation (2.4). Due to assumption **(A3)** they have to wind around the cylinder $S^1 \times \mathbb{R}$ because contractible periodic orbits would have to contain some equilibrium in their interior. We require

(A4) All periodic orbits of the characteristic system (2.4) are hyperbolic in the o.d.e. sense, i.e. their nontrivial Floquet exponents do not lie on the imaginary axis.

Remark 3.2. By dissipativity **(A2)** and o.d.e hyperbolicity **(A4)** there are only finitely many periodic orbits of (2.4). Moreover, the sign of their Floquet exponents alternates, the first and the last being positive. In particular, the number of periodic orbits is odd.

Finally, all periodic orbits can be parameterized over ξ. This leads to the following notation:

Definition 3.3. *Let $\{(v_i(t), \xi_i(t)), 1 \leq i \leq k\}$ be the set of all periodic orbits of (2.4), $T_1, \dots, T_k$ denote their minimal periods and $\mu_1, \dots, \mu_k$ their nontrivial Floquet exponents.*

Before stating the main theorem of this paper we will prove an important theorem on the stationary solutions of the hyperbolic balance law.

Lemma 3.4. *The periodic solutions of the characteristic system can be identified with the continuously differentiable stationary solutions of the balance law, more precisely:*

There is a one-to-one correspondence between the periodic solutions $(v_i(t), \xi_i(t))$ of (2.4) and the stationary solutions $u(x,t) = a_i(x)$ of (1.1).

Proof. Assume first that $(v_i(t), \xi_i(t))$ is periodic: $v_i(T_i) = v_i(0)$ and $\xi_i(T_i) = \xi_i(0)$. Set

$$a_i(\xi_i(t)) := v_i(t).$$

Then a_i is a steady state solution of the balance law because differentiating both sides gives

$$\begin{aligned} & a_i'(\xi_i(t))\, \dot{\xi}_i(t) = \dot{v}_i(t) \\ \Rightarrow \qquad & a_i'(\xi_i(t))\, f'(v_i(t)) = g(\xi_i(t), v_i(t)) \\ \Rightarrow \qquad & a_i'(\xi_i(t))\, f'(a_i(\xi_i(t))) = g(\xi_i(t), a_i(\xi_i(t))). \end{aligned}$$

If, vice versa, $a_i(x)$ is a steady state of the balance law we let $(v(t), \xi(t))$ be the solution of the characteristic system (2.4) with initial condition $\xi(0) = 0$ and $v(0) = a_i(0)$.

We have to show that $a_i(\xi(t)) = v(t)$ holds for all $t > 0$. To this end we consider

$$\begin{aligned} \frac{d}{dt}\,(a_i(\xi(t)) - v(t)) &= a_i'(\xi(t))\, f'(v(t)) - g(\xi(t), v(t)) \\ &= \frac{g(\xi(t), a_i(\xi(t)))}{f'(a_i(\xi(t)))}\, f'(v(t)) - g(\xi(t), v(t)) \\ &= \frac{(g(\xi, a_i(\xi)) - g(\xi, v))\, f'(v) + g(\xi, v)\,(f'(v(t)) - f'(a_i(\xi(t))))}{f'(a_i(\xi(t)))} \end{aligned}$$

Since g and f' are Lipschitz and $f'(a_i(x))$ is uniformly bounded away from zero we can estimate

$$\frac{d}{dt}\,(a_i(\xi(t)) - v(t)) \leq L \cdot (a_i(\xi(t)) - v(t))$$

for some $L > 0$. A standard application of the Gronwall lemma now shows that $a_i(\xi(t)) - v(t) \equiv 0$. ◻

Remark 3.5. We note that there cannot exist discontinuous steady states if **(A3)** holds.

If f' changes its sign we have to include another assumption which restricts the possible initial data.

(A5) Let v_0 denote the zero of f'. We denote with a_r the smallest stationary solution which is larger than v_0. Then the initial data u_0 satisfies

$$u_0(x) \geq a_r(x) \quad \forall x \in S^1.$$

Now we can state the main theorem.

Theorem 3.6. *Consider the hyperbolic balance law*

$$\begin{aligned} \frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} f(u(x,t)) &= g(u(x,t),x) \\ u(x,0) &= u_0(x) \end{aligned} \tag{3.9}$$

with $u_0(x) \in BV(S^1, \mathbb{R})$. *Under the assumptions **(A1)–(A5)** the entropy solution* u *of equation (1.1) either satisfies*

$$\lim_{t\to\infty} ||u(\cdot,t) - a_{2m+1}(\cdot)||_\infty = 0$$

for some m, *or* $u(\cdot,t)$ *converges in* L^1 *to a* T_{2m}*-periodic solution. Except for the stationary case* $u_0(x) = a_{2m}(x)$ *this* T_{2m}*-periodic solution is discontinuous, i.e. a shock solution.*

Remark 3.7. If f' does not possess a zero we do not need assumption **(A5)**. In this case one should think of $a_r := -\infty$ in the statements and proofs.

Moreover, assumption **(A5)** may be replaced by

(A5') The initial data u_0 satisfies

$$u_0(x) \leq a_{r-1}(x) \quad \forall x \in S^1.$$

where a_{r-1} is the biggest stationary solutions which is smaller than v_0.

The proof can be reduced to the proof for **(A5)** after performing a transformation $u \mapsto v_0 - u$.

However, we strongly believe that the statement is true under the assumptions **(A1)–(A4)** alone.

4. Proof of Theorem 3.6

The proof will be accomplished by several lemmata. First we modify ideas of Fan & Hale [5] and Sinestrari [14] to be able to deal with the inhomogeneous source term. This will allow to prove that some (in fact, almost all) solutions converge uniformly to some steady state. In contrast to the case $g = g(u)$ these steady states may be spatially non-homogeneous. In addition, there are solutions which converge to a non-stationary and discontinuous asymptotic state. A difficulty arises when we want to identify these discontinuous asymptotic states. While in the spatially homogeneous case $g = g(u)$ the shocks propagate with constant speed, the location of the shock curves has to be determined differently in our case. To this end we introduce the Rankine-Hugoniot vector field which for any point (x,t) corresponds to the motion of an hypothetical shock located at this point with certain left and right states.

We begin by showing that the solution u after a sufficiently large time T can intersect at most one stationary solution.

Lemma 4.1. *There is a constant $T > 0$ only depending on f and the stationary solutions $a_1, a_2, \dots, a_k$ of (1.1), such that for all $t > T$ the set*

$$\left\{(u(\xi,t),\xi);\ \xi \in S^1\right\} \cap \bigcup_{i=1}^{k} \left\{(a_i(\xi),\xi);\ \xi \in S^1\right\}$$

is either empty or a subset of $\left\{(a_n(\xi),\xi);\ \xi \in S^1\right\}$ for some $n \in \{1, \cdots, k\}$.

To prove this we will first need two other lemmata. The first of them compares two solutions of the characteristic system (2.4) which run around the cylinder in the same direction but start in different points on the line $\xi = 0$. Here and in some other places it will be convenient to work in the extended phase space with $x, \xi \in \mathbb{R}$ instead of $x, \xi \in S^1$.

Lemma 4.2. *Let $(v_a(t), \xi_a(t))$ and $(v_b(t), \xi_b(t))$ be the two solutions of (2.4) with initial conditions $\xi_a(0) = \xi_b(0) = 0$ and $a_r(0) < v_a(0) < v_b(0) < \Omega$ where a_r is the smallest periodic solution bigger than v_0 and $\Omega > 0$ is some arbitrarily large constant.*

Consider these solutions in the extended phase space $(v,\xi) \in \mathbb{R} \times \mathbb{R}$ and let T_a and T_b be the times for which $\xi_a(T_a) = \xi_b(T_b) = 1$.
Then there exists a constant $\kappa > 0$ depending only on Ω but not on $v_{a,b}(0)$ such that

$$T_a - T_b > \kappa(v_b(0) - v_a(0)), \tag{4.2}$$

in particular, the solution with larger v is faster.
Moreover, $\xi_a(t) < \xi_b(t)$ for all $t \geq 0$.

Proof. For $a_r(0) < v < \Omega$ we have the estimate $0 < c_0 < f'(v) = \frac{\mathrm{d}\xi}{\mathrm{d}t}$ for some constant $c_0 > 0$. Therefore we can apply the inverse function theorem to $\xi(t)$ and obtain the inverse function $t(\xi)$ which satisfies the equation

$$\frac{\mathrm{d}t}{\mathrm{d}\xi} = \frac{1}{f'(\tilde{v}(\xi))}$$

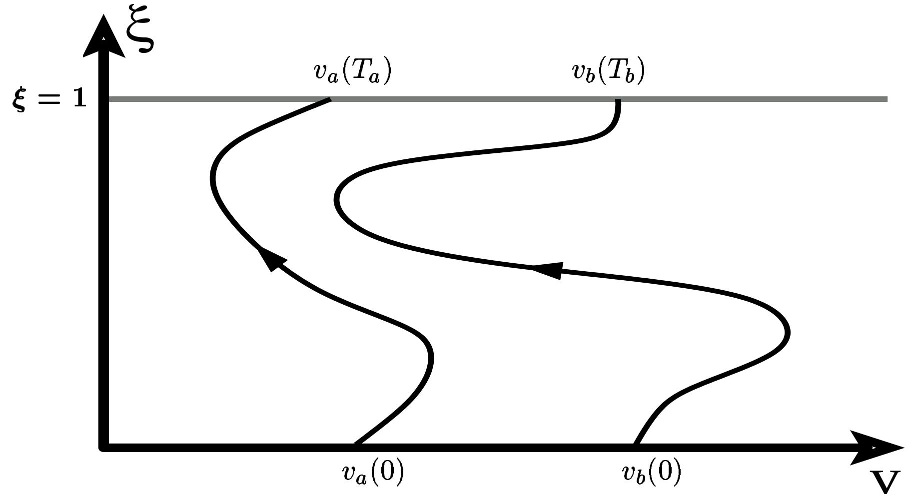

Fig. 1. Situation in Lemma 4.2. The relation $v_a(t) < v_b(t)$ for $t > 0$ does not follow immediately from $v_a(0) < v_b(0)$.

where $\tilde{v}(\xi) = v(t(\xi))$. Denote with $t_a(\xi)$ and $t_b(\xi)$ the inverse functions of ξ_a and ξ_b, then for any $\bar{t} \geq 0$

$$\bar{t} = \int_0^{\bar{t}} \mathrm{d}t = \int_0^{\xi_a(\bar{t})} \frac{\mathrm{d}\xi}{f'(\tilde{v}_a(\xi))} > \int_0^{\xi_a(\bar{t})} \frac{\mathrm{d}\xi}{f'(\tilde{v}_b(\xi))}$$

since $\tilde{v}_a(\xi) < \tilde{v}_b(\xi)$ for all $\xi \in \mathbb{R}$. The same calculation shows that

$$\bar{t} = \int_0^{\xi_b(\bar{t})} \frac{\mathrm{d}\xi}{f'(\tilde{v}_b(\xi))},$$

which implies $\xi_b(\bar{t}) > \xi_a(\bar{t})$ since $f'(\tilde{v}_b(\xi)) > 0$.

To prove the estimate (4.2) note first that the trajectories $\tilde{v}(\xi)$ of the characteristic system satisfy the equation

$$\frac{\mathrm{d}\tilde{v}}{\mathrm{d}\xi} = \frac{g(\xi, \tilde{v})}{f'(\tilde{v})}.$$

For $a_r(\xi) < \tilde{v}(\xi) < \Omega$ the right hand side is uniformly Lipschitz in $\tilde{v}$. A standard Gronwall argument then shows that there is some constant $c_1 > 0$ such that for all $\xi \in [0,1]$

$$\tilde{v}_b(\xi) - \tilde{v}_a(\xi) > c_1(\tilde{v}_b(0) - \tilde{v}_a(0)) = c_1(v_b(0) - v_a(0)).$$

This implies by convexity **(A1)** of the flux that

$$f'(\tilde{v}_b(\xi)) - f'(\tilde{v}_a(\xi)) > cc_1(v_b(0) - v_a(0)) \qquad \forall \xi \in [0,1].$$

In particular

$$\begin{aligned} T_a - T_b &= \int_0^1 \frac{\mathrm{d}\xi}{f'(\tilde{v}_a(\xi))} - \int_0^1 \frac{\mathrm{d}\xi}{f'(\tilde{v}_b(\xi))} \\ &= \int_0^1 \frac{f'(\tilde{v}_b(\xi)) - f'(\tilde{v}_a(\xi))}{f'(\tilde{v}_a(\xi))f'(\tilde{v}_b(\xi))} \,\mathrm{d}\xi \\ &\geq \frac{cc_1(v_b(0) - v_a(0))}{f'(\Omega)^2} =: \kappa(v_b(0) - v_a(0)). \end{aligned}$$

□

If (v_a, ξ_a) and (v_b, ξ_b) are periodic solutions of (2.4) this turns into a statement about their periods.

Corollary 4.3. *If $a_1 < a_2 < \ldots < a_{r-1} < v_0 < a_r < \ldots < a_k$ are the steady states of (1.1) then the periods of the associated periodic orbits of (2.4) satisfy*

$$T_1 < T_2 < \ldots < T_r \quad \text{and} \quad T_{r+1} > \ldots > T_{k-1} > T_k.$$

Lemma 4.4. *For any fixed $T > 0$ and any number $\delta > 0$ there exists a time $\bar{t} > 0$ with the following property: If $\xi_1(t)$ is a genuine characteristic on $[0, \bar{t}]$ corresponding to a T-periodic solution of (2.4) then there cannot be another genuine characteristic $\xi_2(t)$ defined on $[0, \bar{t}]$ such that*

$$sign(\dot{\xi}_1(t)) = sign(\dot{\xi}_2(t)) \qquad \forall t \in [0, \bar{t}].$$

and the time interval for one round trip of ξ_2 around the S^1 is always greater than $T + \delta$, i.e. if $t_i \in [0, \bar{t}]$ are the times with $\xi_2(t_i) = \xi_2(0)$ and $\Theta_i := t_i - t_{i-1}$ then

$$\Theta_i > T + \delta$$

cannot hold.

Similarly, there cannot be a genuine characteristic $\xi_2(t)$ defined on $[0, \bar{t}]$ with $sign(\dot{\xi}_1(t)) = sign(\dot{\xi}_2(t))$ and $\Theta_i < T - \delta$ for all i.

Proof. Suppose there are two such characteristics $\xi_1(\cdot)$ and $\xi_2(\cdot)$ for all times $\bar{t}$. We have to show, that they intersect in finite time. This contradicts Corollary 2.6.
Let $\Theta_i > T + \delta$ for arbitrary i.
We work in the extended phase space $\xi \in \mathbb{R}$ and assume without loss of generality that $\xi_1(0) < \xi_2(0)$.

We set $\Theta := T + \delta$ and choose $n \in \mathbb{N}$ large such that

$$(n+1)T < n\Theta.$$

From this we deduce

$$\xi_1((n+1)T) = \xi_1(0) + n + 1$$

but

$$\xi_2((n+1)T) < \xi_2(0) + n < \xi_1((n+1)T)$$

holds as well. From the intermediate value theorem we know that there is a time $t_s > 0$ when ξ_1 and ξ_2 intersect. This contradicts Corollary 2.6 and thereby proves the lemma. □

Proof of Lemma 4.1. We argue by contradiction and assume that the solution $u(\cdot, t)$ of (3.9) intersects two different periodic orbits of (2.4) for all $t > 0$. Since (2.4) has only finitely many periodic orbits there are $i, j \in \mathbb{N}$ such that

$$\{(u(\xi, t), \xi);\ \xi \in S^1\} \cap \bigcup_{i=1}^{k} \{(a_i(\xi), \xi);\ \xi \in S^1\}$$

is a subset of

$$\{(a_i(\xi), \xi);\ \xi \in S^1\} \cup \{(a_j(\xi), \xi);\ \xi \in S^1\}$$

but not a subset of only one of the two latter sets.
We will show how this leads to a contradiction.
By **(A5)** and comparison with the stationary solution a_r we can assume $a_r < a_i < a_j$. Here, as before, a_r is the smallest stationary solution bigger than v_0 where v_0 is the zero of f'. If f' does not possess a zero then we assume only $a_i < a_j$. From Corollary 4.3 we obtain some $\delta > 0$ such that $T_i - T_j > \delta$.

Choose $\bar{t}$ large such that it satisfies the assumptions of lemma 4.4. If the solution $u(\cdot, \bar{t})$ intersects both trajectories $a_i(\cdot)$ and $a_j(\cdot)$ in x_i and x_j respectively, then the corresponding backward characteristics $\xi_i(\cdot; x_i, \bar{t})$ and $\xi_j(\cdot; x_j, \bar{t})$ were both genuine characteristics on $[0, \bar{t})$ and T_i- respectively T_j-periodic.

From Lemma 4.4 we now deduce that this is impossible. Hence $u(\cdot, \bar{t})$ cannot intersect more than one periodic orbit of (2.4).

□

Lemma 4.5. *Let* $a_0(\xi) := -\infty$ *and* $a_{k+1}(\xi) := +\infty$. *If there are constants* $\delta > 0$, $t_0 \geq 0$ *and an integer* $0 \leq m \leq \frac{k-1}{2}$, *such that*

$$u(\xi, t_0) \in [a_{2m}(\xi) + \delta, a_{2m+2}(\xi) - \delta] \quad \textit{for all} \quad \xi \in S^1$$

then

$$u(\cdot, t) \longrightarrow a_{2m+1}(\cdot) \textit{ uniformly for } t \longrightarrow \infty\,.$$

Proof. Let $u(\xi, t_0) \subset [a_{2m}(\xi) + \delta, a_{2m+2}(\xi) - \delta]$ for all $\xi \in S^1$. The comparison principle applied to u and the stationary solutions a_{2m} and a_{2m+2} then shows that $u(\xi, t) \subset [a_{2m}(\xi), a_{2m+2}(\xi)]$ for all $t \geq t_0$ and all $\xi \in S^1$.

Without loss of generality we can assume $t_0 = 0$.
To prove the uniform convergence we argue indirectly and assume that $u(\cdot,t)$ does not converge uniformly to a_{2m+1}. Then we can find a sequence (x_k, t_k) such that $t_k \nearrow \infty$ and

$$u(x_k,t_k) \in (a_{2m}(x_k), a_{2m+1}(x_k) - \varepsilon) \cup (a_{2m+1}(x_k) + \varepsilon, a_{2m+2}(x_k))$$

for all $k \in \mathbb{N}$ and some $\varepsilon > 0$. Without loss of generality we may assume $u(x_k,t_k) \in (a_{2m}(x_k), a_{2m+1}(x_k) - \varepsilon)$.
Now we consider the minimal backward characteristics emanating from the points (x_k,t_k). Since the stationary solution a_{2m} corresponds to an unstable periodic orbit of (2.4), this orbit is asymptotically stable in backward time. This implies

$$u(\xi(0;x_k,t_k),0) \longrightarrow a_{2m}(\xi(0;x_k,t_k))$$

as $t_k \to \infty$. This contradicts the assumptions of the lemma. □

Lemma 4.6. *Either there exists a* $m \in \{0,\cdots,\frac{k-1}{2}\}$ *such that*

(I) $\lim_{t\to\infty} ||u(\cdot,t) - a_{2m+1}(\cdot)||_\infty = 0$

or there is a $m \in \{1,\cdots,\frac{k-1}{2}\}$ *such that the following holds:*

(II) $\forall t > 0\ \exists\, x \in S^1$ *such that* $u(x+,t) = a_{2m}(x)$ *or* $u(x-,t) = a_{2m}(x)$

Proof. If

$$\{u(\xi,\bar t),\xi);\ \xi \in S^1\} \cap \bigcup_{i=1}^{k} \{(a_i(\xi),\xi);\ \xi \in S^1\}$$

is empty or contained in $\{(a_{2m+1}(\xi),\xi);\ \xi \in S^1\}$ for sufficiently large $\bar t$, then we deduce that for every fixed $\xi \in S^1$

$$a_{2m}(\xi) < u(\xi,\bar t) < a_{2m+2}(\xi)\ .$$

Otherwise Lemma 3.1 shows that there would have to be a $\tilde\xi$ with $u(\tilde\xi,\bar t) = a_{2m}(\tilde\xi)$ or $u(\tilde\xi,\bar t) = a_{2m+2}(\tilde\xi)$.

Due to the compactness of the S^1 there is a $\delta > 0$ such that

$$u(\xi,\bar t) \subset [a_{2m}(\xi) + \delta, a_{2m+2}(\xi) - \delta]$$

With Lemma 4.5 we deduce **(I)**.

In the case that

$$\{(u(\xi,t),\xi);\ \xi \in S^1\} \cap \bigcup_{i=1}^{k} \{(a_i(\xi),\xi) | \xi \in S^1\}$$

is not empty and a subset of $\{(a_{2m}(\xi),\xi);\ \xi \in S^1\}$ for sufficiently large $\bar t$, we deduce that there must exist an extremal backward characteristic $\xi^+(t)$ or $\xi^-(t)$ such that

$$u(\xi^\pm(\bar t),\bar t) = a_{2m}(\xi(\bar t))$$

If we now solve the characteristic system (2.4) backward with the initial data $v_0 = a_{2m}(\xi(\bar{t}))$ and $\xi_0 = \xi(\bar{t})$ we obtain the characteristic claimed in **(II)** with

$$u(\xi(t), t) = a_{2m}(\xi(t)) \quad \forall t \in [0, \bar{t}). \qquad \square$$

The second case of Lemma 4.6 is the one we will now further analyze, because this is the case where the solution $u(\cdot, t)$ does not converge to a stationary solution.

For this reason we will assume for the rest of this chapter

(A6) For all sufficiently large t

$$\varnothing \neq \{(u(\xi, t), \xi);\ \xi \in S^1\} \cap \bigcup_{i=1}^{k} \{(a_i(\xi), \xi);\ \xi \in S^1\} \subseteq \{(a_{2m}(\xi), \xi);\ \xi \in S^1\}$$

holds for a $m \in \{1, \cdots, \frac{k-1}{2}\}$.

Remark 4.7.

(i) If **(A6)** is violated, we are automatically in case **(I)** of lemma 4.6.

(ii) **(A6)** implies the existence of at least one T_{2m}-periodic, genuine characteristic defined for all $t > 0$.

(iii) Combining **(A6)** and Lemma 3.1 shows that $a_{2m-1}(\xi) < u(\xi, t) < a_{2m+1}(\xi)$ for all $\xi \in S^1$ and all sufficiently large t.

Periodic genuine characteristics that exist for all times $t > 0$ will play a major role in the longtime behavior of those solutions which do not tend to a steady state. For this reason we need to characterize the set of genuine periodic characteristics which exist for all $t > 0$. To this end, we define the set $A(\bar{t}) \subset S^1$ of all initial points from which a genuine and periodic characteristic on $[0, \bar{t}]$ emanates. We will have to show that this set is not empty even for $\bar{t} \to \infty$.

Definition 4.8. *Fix $\bar{t} > 0$. Let $A(\bar{t}) \subseteq S^1$ be the set of intersections of extremal T_{2m}-periodic backward characteristics through points $(\bar{x}, \bar{t})$ with the x-axis:*

$$A(\bar{t}) := \{\xi^{\pm}(0; \bar{x}, \bar{t}) \in S^1;\ u(\bar{x}+, \bar{t}) = a_{2m}(\bar{x}) \text{ or } u(\bar{x}-, \bar{t}) = a_{2m}(\bar{x}), \bar{x} \in S^1\}$$

Lemma 4.9. *If **(A1)–(A6)** hold then $A(\bar{t})$ is non-empty and compact for arbitrary $\bar{t} > 0$.*

Proof. From **(A6)** and Lemma 4.6 we deduce that $A(\bar{t}) \neq \emptyset$.
$A(\bar{t}) \subseteq S^1$ is compact if and only if $A(\bar{t})$ is closed.
To show closedness we consider an arbitrary sequence $y_\ell \in A(\bar{t})$ and show that it has a convergent subsequence.

By definition of $A(\bar{t})$ there exist points $x_\ell \in S^1$ such that $y_\ell = \xi^{(\ell)}(0; x_\ell, \bar{t})$ where $\xi^{(\ell)}$ is a genuine characteristic on $[0, \bar{t}]$ with corresponding terminal value $v^{(\ell)} = a_{2m}(x_\ell)$.

Since the sequence x_ℓ is obviously bounded there exists a convergent subsequence

$$x_{\varphi(\ell)} \longrightarrow \tilde{x} \text{ where } \varphi(\ell) \text{ is monotone in } \ell.$$

Without restriction we may assume that $x_{\varphi(\ell)}$ converges to $\tilde{x}$ from the left. Then

$$u(\tilde{x},\bar{t}) = \lim_{\ell\to\infty} u(x_{\varphi(\ell)},\bar{t}) = \lim_{\ell\to\infty} a_{2m}(x_{\varphi(\ell)}) = a_{2m}(\tilde{x}).$$

This in turn implies that $\xi(\cdot;\tilde{x},\bar{t},a_{2m}(\tilde{x}))$ is an extremal backward characteristic

It remains to show that $\xi(0;x_{\varphi(n)},\bar{t}) \longrightarrow \xi^{\pm}(0;\tilde{x},\bar{t})$ if $n \longrightarrow \infty$.
This is a direct consequence of the continuity of a_{2m} and the continuous dependence of a solution of an ODE from the initial data. □

Remark 4.10. In [3] such globally defined genuine characteristics obtained as uniform limits of extremal backward characteristics are called *divides.*

Lemma 4.11. *If* $\bar{t} > \bar{s} > 0$ *then* $A(\bar{t}) \subseteq A(\bar{s})$.

Proof. Let $\tilde{x} \in A(\bar{t})$.
Then, by definition, there exists some extremal backward characteristic $\xi(\cdot;\bar{x},\bar{t})$ with $u(\bar{x}-,\bar{t}) = a_{2m}(\bar{x})$ or $u(\bar{x}+,\bar{t}) = a_{2m}(\bar{x})$. By Lemma 2.4 this is a genuine characteristic, in particular, for $\bar{s} < \bar{t}$ the solution $u(\cdot,\bar{s})$ is continuous in $\xi(\bar{s};\bar{x},\bar{t})$. Therefore

$$\tilde{x} = \xi(0;\bar{x},\bar{t}) = \xi(0;\xi(\bar{s};\bar{x},\bar{t}),\bar{s}) \quad \Rightarrow \quad \tilde{x} \in A(\bar{s}).$$ □

From this we deduce directly that

$$A(\infty) = \bigcap_{\bar{t}>0} A(\bar{t})$$

is a compact and non-empty subset of S^1 and can therefore be written as

$$A(\infty) = S^1 \setminus \bigcup_{n=1}^{\infty} (b_n, c_n)$$

with at most countably many disjoint open intervals (b_n, c_n).

For every $x \in A(\infty)$ there is a genuine T_{2m}–periodic characteristic ξ defined for all $t \geq 0$. In particular, there are characteristics β_n, γ_n such that

$$\left.\begin{array}{l}\beta_n(0) = b_n\\ \gamma_n(0) = c_n\end{array}\right\} \text{ with } \left\{\begin{array}{l}u(\beta_n(t),t) = a_{2m}(\beta_n(t))\\ u(\gamma_n(t),t) = a_{2m}(\gamma_n(t))\end{array}\right\} \quad \forall t > 0$$

For each n we consider the strip S_n on the cylinder $S^1 \times \mathbb{R}^+$ bounded by $\beta_n(\cdot)$ and $\gamma_n(\cdot)$:

$$S_n := \left\{(x,t) \in S^1 \times \mathbb{R}^+;\ x \in [\beta_n(t), \gamma_n(t)]\right\}$$

Outside the union of the strips S_n the solution is determined by the stationary solution a_{2m}:

$$u(x,t) = a_{2m}(x) \text{ if } (x,t) \notin \bigcup_{n=1}^{\infty} S_n$$

In particular, the special case of Theorem 3.6 where $u_0(x) \equiv a_{2m}$ for all $x \in S^1$ corresponds to $A(\infty) = S^1$.

The following lemma is a straightforward modification of Lemma 3.8 in [5].

Lemma 4.12. *For any $\varepsilon > 0$ there exists a time $T(\varepsilon) > 0$ such that the following holds: If $\xi(\cdot;\bar{x},\bar{t})$ is an extremal backward characteristic through a point $(\bar{x},\bar{t}) \in S_n$ with $\bar{t} > T(\varepsilon)$ then*

$$\xi(0;\bar{x},\bar{t}) \in [b_n, b_n+\varepsilon) \cup (c_n-\varepsilon, c_n].$$

Proof. If the claim of the lemma is false, we can find a sequence $(\xi_k, t_k) \in S_n$ with $t_k \nearrow \infty$ such that

$$\lim_{k\to\infty} \xi(0;x_k,t_k) = x_0 \text{ for some } x_0 \in (b_n, c_n).$$

The curve $\xi(t;x_k,t_k)$ is a genuine characteristic for all $t \in [0,t_k)$ and $u(\xi(t;x_k,t_k),t) = v(t)$ where (v,ξ) solves equation (2.4).

From Remark 4.7 and the convexity of f we deduce that the sequence of the characteristics $\xi(t;x_k,t_k)$ and their derivatives $\dot{\xi}(t;x_k,t_k)$ are uniformly bounded.

Restricting all these characteristics to an interval $[0,T]$ with $T > 0$ arbitrarily large the $\xi(\cdot;x_k,t_k)$ with $t_k > T$ are equicontinuous on $[0,T]$. By the Arzela-Ascoli theorem there exists a convergent subsequence converging to some $\eta(t)$ uniformly on the compact interval $[0,T]$.

Since T is arbitrary, we may use a diagonal procedure to construct a subsequence $\xi(t;x_{\varphi(k)},t_{\varphi(k)})$ which converges to a genuine characteristic $\eta(\cdot)$ uniformly on every compact interval. This process will give us a genuine characteristic η which is defined on $[0,\infty)$ with $\eta(0) = x_0$. We claim now that

$$u(\eta(0),0) = a_{2m}(x_0) \tag{4.40}$$

To prove this claim we show first that $u(x_0,0) > a_{2m}(x_0)$ is not possible. Since η is a genuine characteristic this would imply that $u(\eta(t),t) > a_{2m}(\eta(t))$ for all $t > 0$.
Since a_{2m} is unstable, $u(\eta(t),t)$ converges to $a_{2m+1}(\eta(t))$.
We now deduce from Lemma 4.2 that $\eta(\cdot)$ rotates always faster around the S^1 than the characteristics $\beta_n(\cdot)$ and $\gamma_n(\cdot)$.
If $t_1, t_2, t_3, \ldots$ is the sequence of times where $\eta(t_i) = 0$ then $t_{i+1} - t_i \to T_{2m+1}$ for $i \to \infty$.
Therefore we can deduce from Lemma 4.4, that η and γ_n would intersect in finite time.
This contradicts Corollary 2.6 because both η and γ_n are genuine characteristics on

$[0, \infty)$.
In the same way we can show that $u(x_0, 0) < a_{2m}(x_0)$ is also impossible. Together this proves (4.40).

Now from (4.40) we deduce $u(\eta(t), t) = a_{2m}(\eta(t))$ for all $t > 0$. Therefore $x_0 = \eta(0) \in A(\infty)$.
This contradicts the assumption $x_0 \in (b_n, c_n)$. $\square$

Basically, Lemma 4.12 tells us, that both β_n and γ_n are attractive in backward time. The following definition divides each of the strips S_n into two parts according to whether some extremal backward characteristic tends to the left or the right boundary of $[b_n, c_n]$.

Definition 4.13. *Fix $\varepsilon > 0$ small and let $T(\varepsilon)$ be sufficiently large as in Lemma 4.12. Then we define for $\bar{t} \geq T(\varepsilon)$*

$$\begin{aligned} S_n(\bar{t}) &:= [\beta_n(\bar{t}), \gamma_n(\bar{t})] \\ S_n^-(\bar{t}) &:= \{x \in S_n(\bar{t});\ \xi^-(0; x, \bar{t}) \in [b_n, b_n + \varepsilon]\} \\ S_n^+(\bar{t}) &:= \{x \in S_n(\bar{t});\ \xi^+(0; x, \bar{t}) \in [c_n - \varepsilon, c_n]\} \end{aligned}$$

From this definition we deduce directly with Lemma 4.12 that

$$S_n(\bar{t}) = S_n^-(\bar{t}) \cup S_n^+(\bar{t}).$$

Now we will have a closer look at the intersection of S_n^+ with S_n^-. We will be able to prove that this intersection is a generalized characteristic in the sense of Definition 2.1. This proof works like the proof in [5].

Lemma 4.14. *Fix some $\varepsilon > 0$ and let $T(\varepsilon)$ be as in Lemma 4.12. Then there exists some function $\chi : [T(\varepsilon), \infty) \to S^1$ such that for $\bar{T} \geq T(\varepsilon)$*

$$\begin{aligned} S_n^-(\bar{t}) &= [\beta_n(\bar{t}), \chi(\bar{t})] \\ S_n^+(\bar{t}) &= [\chi(\bar{t}), \gamma_n(\bar{t})] \end{aligned}$$

for all $\bar{t} > T(\varepsilon)$.

Proof. If $\bar{x} \in S_n^+(\bar{t})$ then $\xi^+(0; \bar{x}, \bar{t}) \in [c_n - \varepsilon, c_n]$.
Therefore $\xi^\pm(0; \tilde{x}, \bar{t}) \in [\xi^+(0; \bar{x}, \bar{t}), c_n]$ for all $\tilde{x} \in (\bar{x}, \gamma_n(\bar{t})]$, because otherwise backward characteristics would have to intersect.
From this we deduce

$$\tilde{x} \in S_n^+(\bar{t}) \quad \text{and} \quad \tilde{x} \notin S_n^-(\bar{t})\ .$$

Therefore, S_n^- and S_n^+ are connected and the right boundary of the interval $S_n^+(\bar{t})$ is equal to $\gamma_n(\bar{t})$.
We have proved our claim for S_n^+ if we have proved that it is closed. This is obvious for the right limit of the interval since γ_n is a genuine characteristic.

For the left boundary of the interval we consider a sequence $x_k \in S_n^+(\bar{t})$ converging to the boundary $\chi(\bar{t})$.

Since all backward characteristics are equicontinuous we can now apply the Arzela–Ascoli theorem to see that the sequence of maximal backward characteristics $\xi^+(\cdot; x_k, \bar{t})$ converges to $\xi^+(\cdot; \chi(\bar{t}), \bar{t})$, possibly after we have chosen a subsequence.
In particular $\lim_{k\to\infty} \xi^+(0; x_k, \bar{t}) = \xi^+(0; \chi(\bar{t}), \bar{t})$ holds. From this we obtain

$$\xi^+(0; \chi(\bar{t}), \bar{t}) \in [c_n - \varepsilon, c_n]$$

as $\xi^+(0; x_k, \bar{t}) \in [c_n - \varepsilon, c_n]$ for all k and $[c_n - \varepsilon, c_n]$ is closed. But from this $\chi(\bar{t}) \in S_n^+(\bar{t})$ follows.
The proof for $S_n^-(\bar{t})$ is analogous. □

Corollary 4.15. $S_n^-(\bar{t}) \cap S_n^+(\bar{t}) = \chi(\bar{t})$.

Lemma 4.16. *Let $T(\varepsilon)$ be again sufficiently large, then $\chi(t)$ is a Lipschitz continuous function, defined on $[T(\varepsilon), \infty)$.*

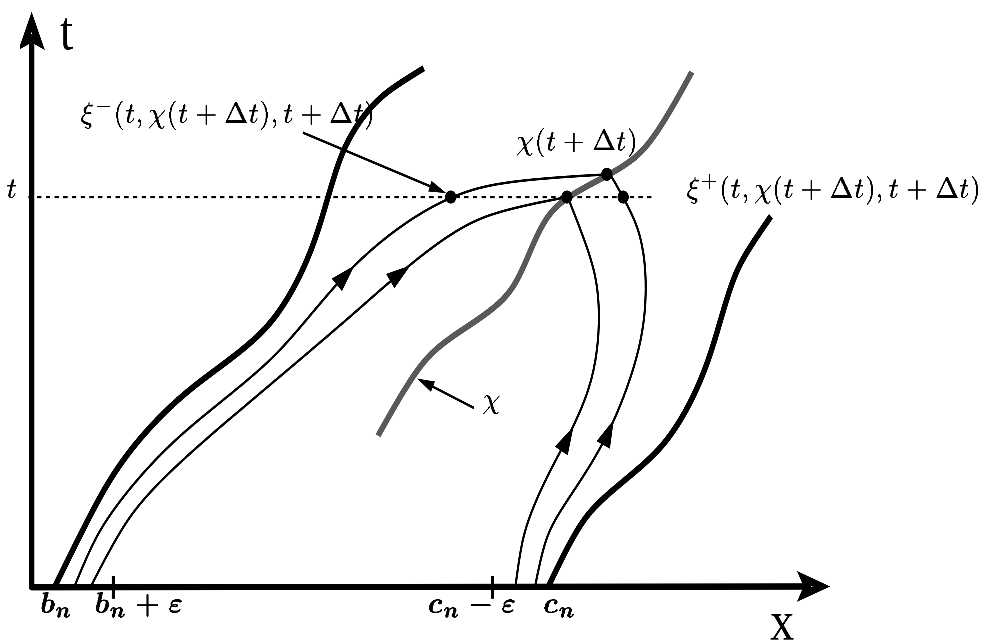

Fig. 2. Illustration to the proof of Lemma 4.16.

Proof. Suppose $\Delta t > 0$ and $t \geq T(\varepsilon)$.

Then

$$\begin{aligned} &\xi^+(0;\chi(t+\Delta t), t+\Delta t) \in [c_n-\varepsilon, c_n] \\ &\xi^+(0;\chi(t), t) \qquad\qquad\quad\; \in [c_n-\varepsilon, c_n] \end{aligned}$$

and

$$\begin{aligned} &\xi^-(0;\chi(t+\Delta t), t+\Delta t) \in [b_n, b_n+\varepsilon] \\ &\xi^-(0;\chi(t), t) \qquad\qquad\quad\; \in [b_n, b_n+\varepsilon\ .] \end{aligned}$$

From this follows

$$\begin{aligned} \chi(t) &\le \xi^+(t;\chi(t+\Delta t), t+\Delta t) \\ \chi(t) &\ge \xi^-(t;\chi(t+\Delta t), t+\Delta t)\ . \end{aligned}$$

because otherwise the maximal or minimal backward characteristics ξ^+ or ξ^- would have to intersect ($\rightarrow$ figure 2). Additionally, the following equation is true:

$$\chi(t+\Delta t) = \xi^{\pm}(t+\Delta t;\chi(t+\Delta t), t+\Delta t)$$

Therefore

$$\begin{aligned} &\tfrac{1}{\Delta t}[\chi(t+\Delta t)-\chi(t)] \le \\ &\qquad \tfrac{1}{\Delta t}\underbrace{\left[\xi^-(t+\Delta t;\chi(t+\Delta t), t+\Delta t) - \xi^-(t;\chi(t+\Delta t), t+\Delta t)\right]}_{\le C^+\Delta t} \\ &\tfrac{1}{\Delta t}[\chi(t+\Delta t)-\chi(t)] \ge \\ &\qquad \tfrac{1}{\Delta t}\underbrace{\left[\xi^+(t+\Delta t;\chi(t+\Delta t), t+\Delta t) - \xi^+(t;\chi(t+\Delta t), t+\Delta t)\right]}_{\ge -(C^-\Delta t)} \end{aligned} \tag{4.50}$$

where the $C^{\pm}$ are bounded by the maximal, respectively minimal slope of the backward characteristics. From this we deduce the Lipschitz condition

$$|\chi(t+\Delta t)-\chi(t)| \le C|\Delta t|\ .$$

□

Lemma 4.17. *The curve $\chi(t)$ is a characteristic.*

Proof. If Δt tends to zero in equation (4.50) we get:

$$\begin{aligned} &\lim_{\Delta t\to 0}\frac{\chi(t+\Delta t)-\chi(t)}{\Delta t} = \dot\chi(t) \in \underbrace{\left[\dot\xi^+(t,\chi(t),t), \dot\xi^-(t,\chi(t),t)\right]}_{=[f'(u(\chi(t)-,t), f'(u(\chi(t)+,t))]} \\ &\forall t\in[T(\varepsilon),\infty) \end{aligned}$$

$\Longrightarrow$ $\chi(t)$ satisfies the condition in Definition 2.1. □

Now we know, that $\chi(t)$ is a shock characteristic. In the spatially homogeneous case $g=g(u)$ (see [5,14]) the shock curve χ approaches a straight line. In our case we will see that it converges to a periodic curve. To prove this convergence we first have to identify this curve.

To this end we define the *Rankine-Hugoniot vector field* (RHV). The idea here is the following:

We extend the two curves β_n and γ_n by (2.4) for $-\infty < t \leq 0$. In any point $(\bar{x}, \bar{t}) \in S_n$ we then determine values $u^-(\bar{x}, \bar{t})$ and $u^+(\bar{x}, \bar{t})$ such that the extremal backward characteristics emanating from $(\bar{x}, \bar{t})$ with $u(\bar{x}, \bar{t}) = u^\pm(\bar{x}, \bar{t})$ will converge to $\beta_n(t)$ and $\gamma_n(t)$ respectively, as $t \to -\infty$:

$$\begin{aligned} \lim_{t \to -\infty} |\xi^+(t; \bar{x}, \bar{t}) - \gamma_n(t)| &= 0 \\ \lim_{t \to -\infty} |\xi^-(t; \bar{x}, \bar{t}) - \beta_n(t)| &= 0. \end{aligned} \tag{4.53}$$

Then $s(\bar{x}, \bar{t})$ is defined as the velocity of a (hypothetical) shock at $(\bar{x}, \bar{t})$ with left state u^- and right state u^+ via the Rankine-Hugoniot condition.

$$s(\bar{x}, \bar{t}) := \frac{f(u^+(\bar{x}, \bar{t})) - f(u^-(\bar{x}, \bar{t}))}{u^+(\bar{x}, \bar{t}) - u^-(\bar{x}, \bar{t})}. \tag{4.54}$$

Remark 4.18. In some of the proofs we consider the Rankine-Hugoniot vector field as an autonomous ordinary differential equation

$$\left. \begin{aligned} \dot{x} &= s(x, t) \\ \dot{t} &= 1 \end{aligned} \right\}$$

in the strip $S_n \subset S^1 \times \mathbb{R}$ (or $S_n \subset \mathbb{R} \times \mathbb{R}$ if we work in the extended phase space).

Later we will prove that the RHV possesses exactly one periodic solution which is unstable. In a last step we will then show, that $\chi(\cdot)$ converges towards this periodic orbit as $t \to \infty$.

4.1. *Properties of the Rankine-Hugoniot vector field*

We first have to verify that the Rankine-Hugoniot vector field is well-defined, i.e. that there are unique states $u^\pm(\bar{x}, \bar{t})$ with the desired properties.

From the o.d.e hyperbolicity **(A4)** of the periodic orbits the following estimates can be derived:

Proposition 4.19. *Consider the backward solution of (2.4) with terminal condition $(v(\bar{t}), \xi(\bar{t}))$ such that $a_{2m-1}(\xi(\bar{t})) < v(\bar{t}) < a_{2m+1}(\xi(\bar{t}))$.*

Denote with $(v_{2m}(t), \xi_{2m}(t))$ the periodic solution a_{2m} parameterized by t.

Then there exists a constant $C > 0$ and an asymptotic phase *$\theta \in S^1$ such that*

$$\begin{aligned} |v(\bar{t} + t) - v_{2m}(\bar{t} + t + \theta)| &\leq C e^{\mu_{2m} t} \\ |\xi(\bar{t} + t) - \xi_{2m}(\bar{t} + t + \theta)| &\leq C e^{\mu_{2m} t} \end{aligned}$$

for all $t < 0$.

For a proof see e.g. [6] or [1].

Remark 4.20. Recall that $\mu_{2m} > 0$ denotes the Floquet exponent of the unstable periodic orbit a_{2m}. The constant C approaches infinity if $v(\bar{t})$ approaches the points $a_{2m\pm1}(\bar{x})$ on the adjacent periodic orbits.

From **(A6)** and the compactness of S^1 one can deduce that all solutions we take into consideration are bounded away from the stable periodic solutions $a_{2m\pm1}$. We may therefore assume that the constant C in the previous proposition is fixed for all backward characteristics associated to the solution $u(\cdot, \bar{t})$ at a fixed time $\bar{t}$.

From the theory of dynamical systems it is known that the hyperbolic periodic orbit (v_{2m}, ξ_{2m}) possesses an unstable foliation where each fiber

$$\mathcal{F}(\theta) := \{(\xi(0), v(0));\ (\xi(t), v(t)) \text{ solves } (2.4) \text{ and } \lim_{t\to-\infty}(v_{2m}(t+\theta) - v(t)) = 0\}$$

corresponds to those points for which the asymptotic phase $\theta \in S^1$ coincides, see [7]. If θ corresponds to $\beta_n(\bar{t})$, i.e. $\beta_n(\bar{t}) = \xi_{2m}(t+\theta)$ then $\mathcal{F}(\theta)$ consists of the terminal values of those backward characteristics which converge to β_n as $t \to -\infty$.

In our relatively simple two-dimensional situation each fiber can be obtained in the following way: Each point p on the periodic orbit is a fixed point for the time-T_{2m}-map of the characteristic system (2.4). By o.d.e. hyperbolicity **(A4)** the linearization of this time-T_{2m}-map in p has the two eigenvalues 1 and $e^{\mu_{2m}T_{2m}}$. Hence, p possesses a one-dimensional unstable manifold which is as smooth as the time-T_{2m}-map, resp. the original vector field (2.4). This unstable manifold is precisely the unstable fiber corresponding to p. With respect to the base point p, one typically loses one order of regularity, see [7]. In our situation, **(A1)** and **(A2)** imply that the fibers $\mathcal{F}(\theta)$ are C^1-smooth and depend continuously on θ.

Lemma 4.21. *The Rankine-Hugoniot vector field defined in equations (4.54) and (4.53) is well-defined, Lipschitz in $\bar{x}$ and continuous with respect to $\bar{t}$. Moreover, it is T_{2m}–periodic in $\bar{t}$.*

Proof. We have to show that the states $u^+(\bar{x}, \bar{t})$ and $u^-(\bar{x}, \bar{t})$ are well-defined for each $(\bar{x}, \bar{t}) \in S_n$ and that they depend Lipschitz continuously on $\bar{x}$ and continuously on $\bar{t}$. These properties will then carry over to $s(\bar{x}, \bar{t})$ such that the Rankine-Hugoniot vector field possesses unique solutions for any initial condition.

Note first that in the extended phase space $(v, \xi) \in \mathbb{R} \times \mathbb{R}$ the unstable fibers $\mathcal{F}(\theta)$ are graphs over the ξ-axis. This is a consequence of the fact that solutions with smaller v are slower, see Lemma 4.2.

More precisely, if $\mathcal{F}(\theta)$ would intersect a line $\xi = \xi_0$ in two points $\underline{v}$ and $\overline{v}$ with $\overline{v} - \underline{v} =: \delta > 0$ then the backward characteristics with these two terminal values would differ by at least $\kappa\delta$ on any line $\xi = \xi_0 - k$ for any $k \in \mathbb{N}$. This however shows that they

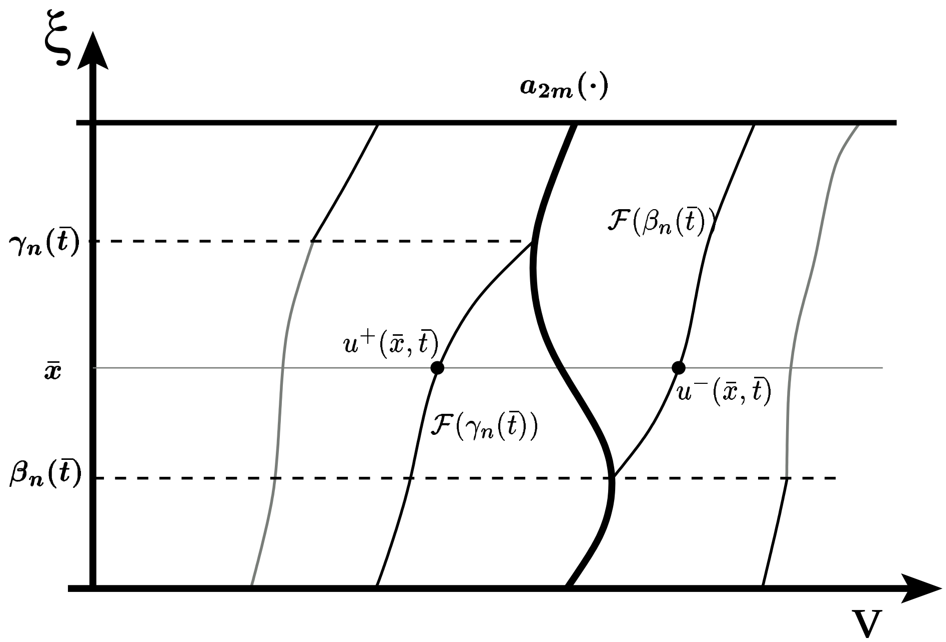

Fig. 3. Illustration of the unstable fibers.

cannot both correspond to solutions on the cylinder with the same asymptotic phase and hence cannot both lie in the same unstable fiber $\mathcal{F}(\theta)$.

Fix now $\bar{t} > 0$ and consider the fiber $\mathcal{F}(\theta)$ with θ corresponding to $\beta_n(\bar{t})$, see figure 4.1.

For any $\bar{x} > \beta_n(\bar{t})$ the line $\xi = \bar{x}$ intersects $\mathcal{F}$ in exactly one point $(\bar{x}, v^+)$. Now $u^+(\bar{x},\bar{t}) := v^+$ has the desired property, since points on the unstable fiber are precisely characterized by the fact that the backward characteristics with these terminal values converge to β_n as $t \to -\infty$.

Similarly, the value $u^-(\bar{x},\bar{t})$ is determined by the unstable fiber corresponding to $\gamma_n(\bar{t})$.

The Lipschitz continuity of $u^\pm$ with respect to $\bar{x}$ is just a consequence of the smoothness of the fibers, while the continuity with respect to $\bar{t}$ comes from the fact that the fibers we have to consider depend continuously on $\beta_n(\bar{t})$ and hence on $\bar{t}$.

The T_{2m}-periodicity of the Rankine-Hugoniot vector field follows from the T_{2m}-periodicity of the boundary characteristics β_n and γ_n. For $t = \bar{t} + T_{2m}$ we set $\tilde{u}^\pm(\bar{x}, \bar{t} + T_{2m}) := u^\pm(\bar{x},\bar{t})$. Then the backward characteristic to these terminal data converges for $t \to -\infty$ to the boundary characteristics β_n or γ_n as well. Because of the uniqueness of the Rankine-Hugoniot vector field we must have

$$u^\pm(\bar{x}, \bar{t} + T_{2m}) = \tilde{u}^\pm(\bar{x}, \bar{t} + T_{2m}) = u^\pm(\bar{x},\bar{t})$$

hence $u^{\pm}$ are T_{2m}–periodic. By construction this implies that the Rankine-Hugoniot vector field is itself T_{2m}–periodic. □

We are now able to study solutions of the Rankine-Hugoniot vector field. In particular, we prove that there is exactly one periodic solution, and that its period has to be T_{2m}. For this reason we analyze $s(\bar{x}, t)$ on vertical lines in the ξ–t–diagram. To this end, we work in the extended phase space $(x, t) \in \mathbb{R} \times \mathbb{R}$ again.

Lemma 4.22. *The Rankine-Hugoniot vector field points outside the strip S_n along the two boundary curves β_n and γ_n.*

Proof. It suffices to compare the slope $\left(\dot{\xi}_n(\bar{t})\right)^{-1}$ of the boundary curve β_n with the slope of the Rankine-Hugoniot vector field in $(\beta_n(\bar{t}), \bar{t})$.

It is clear that $u^-(\beta_n(\bar{t}), \bar{t}) = u(\beta_n(\bar{t}), \bar{t}) = a_{2m}(\beta_n(\bar{t}))$ because only with this terminal condition will the backward characteristic emanating in $(\beta_n(\bar{t}), \bar{t})$ converge to the curve β_n for $t \to -\infty$ without crossing β_n. We claim that $u^+(\beta_n(\bar{t}), \bar{t}) < a_{2m}(\beta_n(\bar{t}))$.

This can be best seen if we consider the characteristic curves in the extended phase space $(x, t) \in \mathbb{R} \times \mathbb{R}$. Here the curve β_n divides the plane in two sets one of which contains the curve γ_n. Backwards characteristics emanating from the point $(\beta_n(\bar{t}), \bar{t})$ with terminal value $v(\bar{t}) \geq a_{2m}(\beta_n(\bar{t}))$ will lie completely in the other region and can therefore not approach the curve γ_n.

By the mean value theorem the slope of the Rankine-Hugoniot vector field is

$$s(\beta_n(\bar{t}), \bar{t}) = \frac{f(u^+(\beta_n(\bar{t}), \bar{t})) - f(u^-(\beta_n(\bar{t}), \bar{t})))}{u^+(\beta_n(\bar{t}), \bar{t}) - u^-(\beta_n(\bar{t}), \bar{t})} = f'(\omega)$$

where ω lies between $u^+(\beta_n(\bar{t}), \bar{t})$ and $u^-(\beta_n(\bar{t}), \bar{t})$, in particular $\omega < a_{2m}(\beta_n(\bar{t}))$. By convexity of f this implies $f'(\omega) < f'(a_{2m}(\beta_n(\bar{t})))$ and hence $s(\beta_n(\bar{t}), \bar{t}) < \dot{\xi}_n(\bar{t})$. This shows that the RHV points to the exterior of S_n along β_n.

The proof for the other boundary γ_n can be carried out by the same arguments and is therefore omitted. □

For fixed $\bar{x} \in \mathbb{R}$ we now consider the vertical line

$$L_{\bar{x}} := S_n \cap \{x = \bar{x}\} \subset \mathbb{R} \times \mathbb{R}^+$$

in the extended phase space.

Lemma 4.23. *The Rankine-Hugoniot vector field $s(\bar{x}, t)$ is strictly decreasing in $\bar{t}$ on every vertical line $\{(\bar{x}, \bar{t});\ \bar{t} \in L_{\bar{x}}\}$.*

Proof. We will first prove that for fixed $\bar{x}$ the functions $u^{\pm}(\bar{x}, \bar{t})$ used in the definition of the Rankine-Hugoniot vector field are strictly decreasing in $\bar{t}$ on $L_{\bar{x}}$. Since both statements are proved in the same way we restrict ourselves to the function u^+.

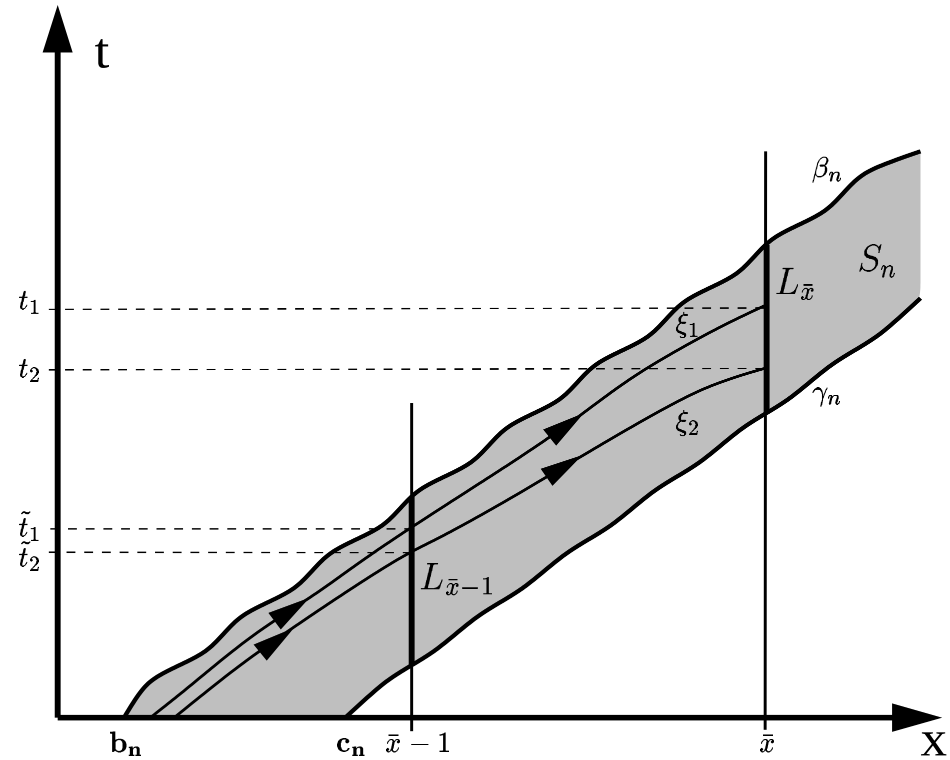

Fig. 4. Illustration for the proof of Lemma 4.23

We argue by contradiction and assume that there are $t_1 > t_2$ in $L_{\bar{x}}$ with $u_1^+ := u_1^+(\bar{x}, t_1) \geq u_2^+(\bar{x}, t_2) =: u_2^+$.

Let $\delta := t_1 - t_2 > 0$ and consider for $i = 1, 2$ the backward characteristics ξ_1 and ξ_2 emanating from $(\bar{x}, t_i)$ with terminal value u_i^+, see figure 4.

From Lemma 4.2 we know, that the trajectory starting in $(t_1, \bar{x})$ crosses the line $L_{\bar{x}-1}$ after a shorter time than the one that started in $(t_2, \bar{x})$. If $(\tilde{t}_1, \tilde{u}_1^\pm)$ and $(\tilde{t}_2, \tilde{u}_2^\pm)$ are the times and v-values where the backward trajectories cross the line $L_{\bar{x}-1}$ we have the inequalities $\tilde{t}_1 - \tilde{t}_2 > \delta$ and $\tilde{u}_1^+ > \tilde{u}_2^+$.
By induction the distance between the two backward characteristics ξ_{u_1} and ξ_{u_2} is greater than δ on every vertical line $L_{\bar{x}-j}$ with $j \in \mathbb{N}$. But this immediately shows that they cannot converge both to the boundary characteristic γ_n as demanded. Hence, our assumption $u_1^+ \geq u_2^+$ must be wrong.

The same argument can be used to prove the strict monotonicity of u^- with respect to $\bar{t}$ on $L_{\bar{x}}$.

Because f is convex, the monotonicity of $u^\pm$ carries over to $s(\bar{x}, \bar{t})$: If $t_1 > t_2$, then the inequality

$$\frac{f(u^+(\bar{x},t_1)) - f(u^-(\bar{x},t_1))}{u^+(\bar{x},t_1) - u^-(\bar{x},t_1)} - \frac{f(u^+(\bar{x},t_2)) - f(u^-(\bar{x},t_2))}{u^+(\bar{x},t_2) - u^-(\bar{x},t_2)} < 0$$

is also true. This shows that $s(\bar{x},t_1) < s(\bar{x},t_2)$. □

Lemma 4.24. *The Rankine-Hugoniot vector field $s(x,t)$ in the strip S_n possesses exactly one T_{2m}–periodic solution.*

Proof. We first show the existence of a periodic solution. In the strip S_n the Rankine-Hugoniot vector field corresponds to the ordinary differential equation

$$\left.\begin{aligned} \dot{x} &= s(x,t) \\ \dot{t} &= 1. \end{aligned}\right\} \tag{4.61}$$

Since there are no equilibrium points and $\dot{t} > 0$, following the solutions of the Rankine-Hugoniot vector field backward in time induces a continuous Poincaré map $\Pi_- : S(\bar{t}) \to S(\bar{t} - T_{2m})$.

As a consequence of Lemma 4.22 we have

$$\Pi_-\left([\beta_n(\bar{t}), \gamma_n(\bar{t})]\right) \supset [\beta_n(\bar{t} - T_{2m}), \gamma_n(\bar{t} - T_{2m})] = [\beta_n(\bar{t}), \gamma_n(\bar{t})]$$

The intermediate value theorem then implies that there exists at least one $\tilde{x}$ such that

$$\Pi_-(\tilde{x}) = \tilde{x}\ .$$

Due to the T_{2m}-periodicity of the Rankine-Hugoniot vector field the solution $\sigma(t)$ of $\dot{x} = s(x,t)$ with initial data $\sigma(\bar{t}) = \tilde{x}$ has to be a T_{2m}-periodic solution.

We now turn to uniqueness and show that σ is the only periodic orbit the Rankine-Hugoniot vector field can have within S_n.

If there were two such orbits $\sigma_1(t)$ and $\sigma_2(t)$ they cannot intersect each other as they solve the same ordinary differential equation (4.61) with (x,t) in the extended phase space $(x,t) \in \mathbb{R} \times \mathbb{R}$. We may therefore assume that σ_1 lies to the left and above σ_2. Hence, for any $t \in \mathbb{R}$ there exists some $\alpha(t) < t$ such that

$$\sigma_1(t) = \sigma_2(\alpha(t))$$

with

$$\alpha(t + T_{2m}) = \alpha(t) + T_{2m} \qquad \text{for all } t \tag{4.65}$$

since we assumed that both solutions σ_1 and σ_2 are T_{2m}–periodic on $S^1 \times \mathbb{R}$. Due to Lemma 4.23

$$\dot{\sigma}_1(t) = s(\sigma_1(t), t) < s(\sigma_2(\alpha(t)), \alpha(t)) = \dot{\sigma}_2(\alpha(t))$$

holds for all times $t \in \mathbb{R}$.

This implies

$$T_{2m} = \alpha(t + T_{2m}) - \alpha(t) = \int_0^{T_{2m}} \dot{\alpha}(t+\tau)\, d\tau > \int_0^{T_{2m}} 1\, d\tau = T_{2m}$$

contradicting (4.65). □

As a corollary we note that using Lemmas 4.21 and 4.24 infinitely many nontrivial T_{2m}-periodic solutions of the inhomogeneous balance law (1.1) can be constructed.

Corollary 4.25. *Choose finitely or countably many open intervals $(b_n, c_n) \subset S^1$. Let (β_n, v_n) and (γ_n, w_n) be the solutions of the characteristic system (2.4) with initial conditions $\beta_n(0) = b_n$, $v_n(0) = a_{2m}(b_n)$ and $\gamma_n(0) = c_n$, $w_n(0) = a_{2m}(c_n)$ respectively.*
As before we denote $S_n = \{(x,t) \in \mathbb{R} \times \mathbb{R};\ \beta_n(t) < x < \gamma_n(t)\}$. For each $(x,t) \in \cup_n S_n$ the states $u^-(x,t)$ and $u^+(x,t)$ as well as the Rankine-Hugoniot vector field $s(x,t)$ are well-defined. Let σ_n be the unique periodic solution of the Rankine-Hugoniot vector field in the strip S_n.
Then the function

$$u_{per}(x,t) = \begin{cases} u^-(x,t) & \text{if } \beta_n(t) < x \le \sigma_n(t) \\ u^+(x,t) & \text{if } \sigma_n(t) < x < \gamma_n(t) \\ a_{2m}(x) & \text{if } x \notin \cup_n S_n(t) \end{cases}$$

is a T_{2m}-periodic solution of the scalar balance law (1.1).

Let us quickly summarize at this point the rest of the proof: Any solution u of (1.1) which satisfies **(A6)** determines a set of finitely or countably many strips $S_n \subset S^1 \times \mathbb{R}_+$ where u does not coincide with a_{2m}. The previous corollary shows how a T_{2m}-periodic solution can be constructed from the strips S_n via the periodic solution σ_n and the Rankine-Hugoniot vector field. Moreover, we have seen that each of the strips S_n is divided by a curve χ into two parts S_n^+ and S_n^-. The next step consists in showing that after sufficiently large time the solution in the region S_n^+ is close to u^+ while it is close to u^- in S_n^-. The proof will then be completed by showing that in each Strip S_n the curves χ and σ_n approach each other as $t \to \infty$.

Lemma 4.26. *Let u be an entropy solution of the hyperbolic balance law (1.1). For every $\delta > 0$ there exists some time $\tau(\delta)$ such that the following holds for $t > \tau(\delta)$:*

$$x \in S_n^+(t) \Rightarrow |u(x,t) - u^+(x,t)| \le \delta$$
$$x \in S_n^-(t) \Rightarrow |u(x,t) - u^-(x,t)| \le \delta.$$

Proof. Recall that in Lemma 4.2 we have derived an inverse Lipschitz estimate for the difference between the times that two solutions of the characteristic equation need to go around the S^1.

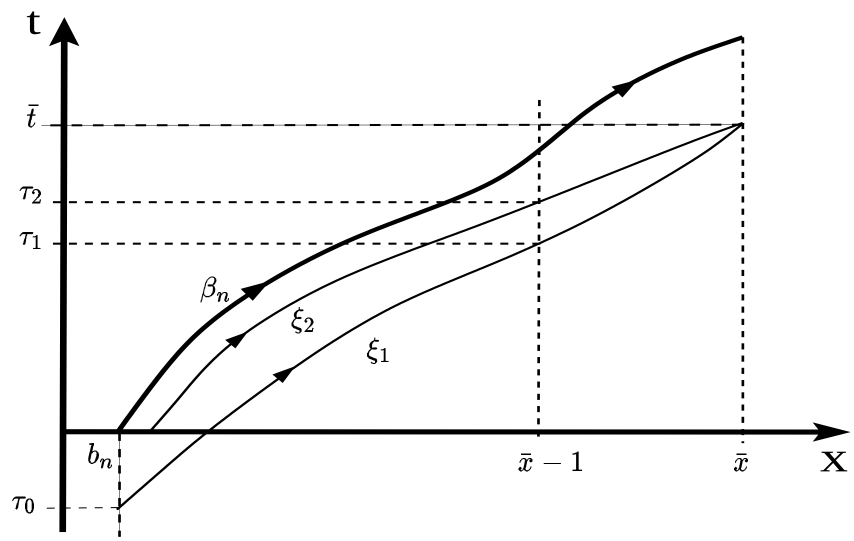

Fig. 5. The proof of Lemma 4.26

We fix some point $(\bar{x}, \bar{t}) \in S_n^-$ and consider the backward solutions of the characteristic system (2.4) with $\xi_1(\bar{t}) = \xi_2(\bar{t}) = \bar{x}$ and $v_1(\bar{t}) = u(\bar{x}, \bar{t})$, $v_2(\bar{t}) = u^-(\bar{x}, \bar{t})$ together with the corresponding backward characteristics in the x-t-plane, see the illustrating figure 5.

Along both of these backward characteristics an inequality $\dot{\xi} > m_1$ is satisfied for some positive constant m_1 since both v_1 and v_2 satisfy

$$v_i(t) > a_r(\xi(t)) \geq \min_{x \in S^1} a_r(x) > v_0.$$

Choose now $\varepsilon < m_1 \kappa \delta$ where $\kappa > 0$ is the constant from Lemma 4.2. We know from Lemma 4.12 that for $\bar{t} \geq T(\varepsilon)$ the minimal backward characteristic emanating from $(\bar{x}, \bar{t})$ with $\bar{x} \in S_n^-(\bar{t})$ will intersect the line $t = 0$ within the interval $[\beta_n(0), \beta_n(0) + \varepsilon]$.

On the other hand, the value $u^-(\bar{x}, \bar{t})$ is determined by the fact that the associated backward characteristic converges to β_n as $t \to -\infty$. In particular, by choosing $\bar{t}$ even larger, if necessary, we may assume that

$$\xi_2(0) - \beta_n(0) \leq \frac{\varepsilon}{2}.$$

It remains to show that $|u(\bar{x}, \bar{t}) - u^-(\bar{x}, \bar{t})| \leq \delta$ must hold in order that $|\xi_2(0) - \xi_1(0)| \leq \varepsilon$.

In the sequel we will argue in the extended phase space to be able to distinguish between $\bar{x}$, $\bar{x} - 1$, etc.

We argue by contradiction and assume that $|u(\bar{x}, \bar{t}) - u^-(\bar{x}, \bar{t})| > \delta$. Then the two backward characteristics ξ_1 and ξ_2 intersect the line $\xi = \bar{x} - 1$ at times τ_1 and τ_2 with $|\tau_1 - \tau_2| > \kappa\delta$ due to Lemma 4.2. The time difference between the two characteristics will only become bigger for $\xi < \bar{x} - 1$. In particular, if τ_0 is the time when the characteristic ξ_1 crosses the line $\xi = \xi_2(0)$ then $|\tau_0| > \kappa\delta$. This however proves that ξ_1 cannot intersect the line $t = 0$ in the interval $[\xi_2(0) - \varepsilon, \xi_2(0) + \varepsilon]$ and hence not in $[\beta_n(0), \beta_n(0) + \varepsilon]$. If such an intersection would take place the slope somewhere between the two points on ξ_1 with $t = 0$ and with $t = \tau_0$ would be bigger than m_1.

This proves that the backward characteristic ξ_1 misses the interval $[\beta_n(0), \beta_n(0) + \varepsilon]$ on the x-axis if $|u(\bar{x}, \bar{t}) - u^-(\bar{x}, \bar{t})| > \delta$ contrary to Lemma 4.12. Hence we must have $|u(\bar{x}, \bar{t}) - u^-(\bar{x}, \bar{t})| \leq \delta$ for $\bar{x} \in S_n^-(\bar{t})$. The proof of the estimate in $S_n^+(t)$ is completely analogous. □

We will now prove that χ and σ approach each other as $t \to \infty$. This is quite remarkable since σ is unstable as an integral curve of the Rankine-Hugoniot vector field. Before giving a formal proof we describe shortly the mechanism which forces χ and σ to be very close for large times: We know already that the characteristic χ exists for all $t > 0$ and the previous lemma implies that it behaves "almost" like a solution of the Rankine-Hugoniot vector field for sufficiently large times. If the distance between χ and σ was too big, χ would therefore leave the strip S_n precisely because of the instability of σ. Then, however, it would have to intersect one of the boundary characteristics β_n or γ_n and would not be defined on $\mathbb{R}_+$.

Lemma 4.27. *The shock curve approaches the curve σ as $t \to \infty$:*

$$\lim_{t \to \infty} |\chi(t) - \sigma(t)| = 0.$$

Proof. We will show that for each $\varepsilon > 0$ there exists some time $\mathcal{T}(\varepsilon)$ such that for $t > \mathcal{T}(\varepsilon)$ the curve $\chi(t)$ is contained in a strip

$$\Sigma_\varepsilon := \{(x, t) \in S_n;\ |t - \sigma^{-1}(x)| < \varepsilon\}$$

around σ. By Lemma 4.23 the Rankine-Hugoniot vector field is strictly monotone on every line $L_{\bar{x}}$. Due to periodicity of s this allows to find some $\nu > 0$ such that

$$s(x, \sigma^{-1}(x) + \varepsilon) + \nu < s(x, \sigma^{-1}(x)) < s(x, \sigma^{-1}(x) - \varepsilon) - \nu. \tag{4.73}$$

This implies that trajectories of the Rankine-Hugoniot vector field cannot enter Σ_ε from outside. From the definition of $s(x, t)$ and the continuity of f it follows that there exists some $\delta_0 = \delta_0(\varepsilon) > 0$ such that for $\omega^\pm \in \mathbb{R}$ the following holds:

$$|u^\pm(x, \sigma^{-1}(x) - \varepsilon) - \omega^\pm| < \delta_0 \Rightarrow \left| s(x, \sigma^{-1}(x) - \varepsilon) - \frac{f(\omega^+) - f(\omega^-)}{\omega^+ - \omega^-} \right| < \frac{\nu}{2} \tag{4.74}$$

and similarly

$$|u^\pm(x, \sigma^{-1}(x) + \varepsilon) - \omega^\pm| < \delta_0 \Rightarrow \left| s(x, \sigma^{-1}(x) + \varepsilon) - \frac{f(\omega^+) - f(\omega^-)}{\omega^+ - \omega^-} \right| < \frac{\nu}{2}. \tag{4.75}$$

We set $\mathcal{T}(\varepsilon) := \tau(\delta_0)$ where $\tau(\delta_0)$ is as in Lemma 4.26 and show that for $t > \mathcal{T}(\varepsilon)$ the curve χ has to lie within Σ_ε. To this end we prove that if $\chi(t)$ lies outside of Σ_ε for some

$t > \mathcal{T}(\varepsilon)$ then it has to remain outside for *all* $t > \mathcal{T}(\varepsilon)$. We will later show that this is impossible for a characteristic which is defined on $[0, \infty)$.

To see that $\chi(t)$ cannot enter Σ_ε for some $t \geq \mathcal{T}(\varepsilon)$ we assume $\chi(t) = \sigma(t - \varepsilon)$ and compare the slopes of $\chi(t)$ and the boundary $\sigma(t-\varepsilon)$ of Σ_ε. The other case $\chi(t) = \sigma(t+\varepsilon)$ can be treated in the same way.

The slope of χ is determined by the Rankine-Hugoniot condition. Our estimate (4.73) together with (4.74) then implies that

$$\dot{\chi}(t) > \dot{\sigma}(t - \varepsilon) + \frac{\nu}{2} \tag{4.76}$$

for $t > \mathcal{T}(\varepsilon) = \tau(\delta_0)$. The curve χ can therefore only leave the strip Σ_ε but not enter it.

In the next step we will show that it is not possible for χ to be outside Σ_ε for all sufficiently large t. Note that (4.76) implies that the t-difference between the curves χ and σ grows linearly with a fixed rate $\nu/2$ after any revolution around the cylinder.

Lemma 4.4 then tells us that in this case $\chi(t)$ would have to intersect one of the boundary characteristics β_n or γ_n in finite time. This however is a contradiction to the fact that $\chi(t)$ is a characteristic curve defined on $\mathbb{R}^+$. □

We can finally complete the proof of our main result.

Proof of Theorem 3.6. Let $u_0(x) \in BV(S^1, \mathbb{R})$. According to Lemma 4.1 we have for all $t > T$ with sufficiently large T

$$\left\{(u(\xi,t),\xi);\ \xi \in S^1\right\} \cap \bigcup_{i=1}^{k} \{(a_i(\xi),\xi);\ \xi \in S^1\}$$

is either empty or contained in $\{(a_n(\xi),\xi);\ \xi \in S^1\}$ for some $n \in \{1, \ldots, k\}$.

Lemma 4.6 now yields uniform convergence of the solution towards a profile a_{2m+1} with $m \in \left\{0, \cdots, \frac{k-1}{2}\right\}$ if

$$\left\{(u(\xi,\bar{t}),\xi);\ \xi \in S^1\right\} \cap \bigcup_{i=1}^{k} \{(a_i(\xi),\xi);\ \xi \in S^1\}$$

is either empty or contained in one of the sets $\{(a_{2m+1}(\xi),\xi);\ \xi \in S^1\}$.

In the other case **(A6)** holds and Lemma 4.17 shows the existence of at least one shock characteristic χ on some interval $[t_0, \infty)$. Lemma 4.27 in addition gives the asymptotic T_{2m}–periodicity of this shock characteristic $\chi(t)$.
The L^1-convergence of the solution for $t \longrightarrow \infty$ towards a discontinuous periodic solution follows from Lemma 4.26 and Lemma 4.27. The periodic shock solution is given by the functions $u^{\pm}$ as described in Corollary 4.25: To the left of the periodic orbit σ the solution u is given by u^- whereas to the right of σ it coincides with u^+.

□

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