COMPLETE REDUCIBILITY, GEOMETRIC INVARIANT THEORY AND BUILDINGS

Workshop at Bochum, February 15 - 19, 2016

Titles and Abstracts

Stephen Donkin (York)

Calculating characters of cohomology of line bundles on generalised flag varieties in positive characteristic

Let G be a semisimple algebraic group over an algebraically closed field K and B a Borel subgroup. The line bundles on the generalised flag variety G/B may be described via the one dimensional rational B-modules. If K has characteristic 0 then the characters of the cohomology modules of these line bundles is described by the beautiful Borel-Bott-Weil Theorem. (For each line bundle the cohomology is non-zero in at most one degree. There the character is given by Weyl's Character Formula). Very little is known in positive characteristic. I will describe a general recursive approach using the representation theory of G and of certain infinitesimal subgroups. In practice this is difficult to work with but does give a complete solution (in all characteristics) for G the special linear group of rank 3 and for the symplectic group of rank 4 in characteristic 2. The latter case is joint work with H. Geranios.

Oliver Goodbourn (Bochum)

Reductive pairs from representations of algebraic groups

Reductive pairs are a class of nice embeddings of reductive algebraic groups. They have been used to salvage some good behaviour observed in characteristic 0 in the positive characteristic case, for instance in work of Bate, Herpel, Martin and Röhrle on G-complete reducibility, and in providing uniform proofs of otherwise technical results. I will discuss work into determining when we get reductive pairs arising from representations of an algebraic group, including complete pictures for simple and Weyl modules for SL_2 in arbitrary characteristic.

Ralf Köhl (Gießen)

Generalized spin representations

The string theorists Damour, Kleinschmidt, Nicolai and independently De Buyl, Henneaux, Paulot have observed that the maximal compact subalgebra of the real Kac-Moody algebra of type E_{10} admits $\mathfrak{so}(32)$ as a quotient.

Together with Hainke and Levy we generalized this to arbitrary symmetrizable type and it turns out that in order to construct such a representation in the simply laced situation it suffices to construct a collection of matrices, one for each simple root, that square to -1 and commute, resp. anti-commute depending on whether the corresponding simple roots form a non-edge or an edge.

Jointly with Horn we used the theory of Clifford algebras in order to determine the quotients for the whole E_n series. And jointly with Ghatei, Horn and Weiß we integrated our findings to group level.

These generalized spin representations could be called 1/2-spin representations. Higher analogs $(3/2, 5/2, 7/2, \ldots)$ also exist for E_{10} by work of Kleinschmidt and Nicolai.

Paul Levy (Lancaster)

Dual singularities in exceptional type nilpotent cones

It is well-known that nilpotent orbits in $\mathfrak{sl}_n(\mathbb{C})$ correspond bijectively with the set of partitions of n, such that the closure (partial) ordering on orbits is sent to the dominance order on partitions. Taking dual partitions simply turns this poset upside down, so in type A there is an order-reversing involution on the poset of nilpotent orbits. More generally, if $\mathfrak{g}$ is any simple Lie algebra over $\mathbb{C}$ then Lusztig-Spaltenstein duality is an order-reversing bijection from the set of special nilpotent orbits in $\mathfrak{g}$ to the set of special nilpotent orbits in the Langlands dual Lie algebra $\mathfrak{g}^L$. It was observed by Kraft and Procesi that the duality in type A is manifested in the geometry of the nullcone. In particular, if two orbits $\mathcal{O}_1 < \mathcal{O}_2$ are adjacent in the partial order then so are their duals $\mathcal{O}_1^t > \mathcal{O}_2^t$, and the isolated singularity attached to the pair $(\mathcal{O}_1, \mathcal{O}_2)$ is dual to the singularity attached to $(\mathcal{O}_2^t, \mathcal{O}_1^t)$: a Kleinian singularity of type A_k is swapped with the minimal nilpotent orbit closure in $\mathfrak{sl}_{k+1}$ (and vice-versa). Subsequent work of Kraft-Procesi determined singularities associated to such pairs in the remaining classical Lie algebras, but did not specifically touch on duality for pairs of special orbits. In this talk, I will explain some recent joint research with Fu, Juteau and Sommers on singularities associated to pairs $\mathcal{O}_1 < \mathcal{O}_2$ of (special) orbits in exceptional Lie algebras. In particular, we (almost always) observe a generalized form of duality for such singularities in any simple Lie algebra.

Martin Liebeck (Imperial)

Regular orbits and generic stabilizers for linear actions of algebraic groups

If G is a simple algebraic group, and V an irreducible G-module such that $\dim V > \dim G$, we show that there is a non-empty open subset of V such that all the stabilizers of vectors in the subset are conjugate finite groups. Such a stabilizer is called a generic stabilizer, and it is usually trivial, although there are some interesting examples where it is not.

Alexander Lytchak (Köln)

Metric versions of the center conjecture in small dimension

In the investigations of convex subsets of symmetric spaces B.Kleiner and B.Leeb came across the following question generalizing the center conjecture. If a spherical building is equipped with its canonical metric structure, does any convex subset of the building has a canonical center, even if the subset is not a subcomplex? In the talk I will discuss an answer to this question in the case of dimension two, obtained in a joint work with A. Balser.

Ben Martin (Aberdeen)

Cocharacter-closed orbits and the Centre Conjecture

The study of closed orbits is central in geometric invariant theory. Let G be a reductive algebraic group over an algebraically closed field k and let V be an affine variety on which G acts. The Hilbert-Mumford Theorem gives a criterion in terms of cocharacters for the orbit $G \cdot v$ of a point $v \in V$ to be closed. This result was strengthened by Hesselink, Kempf and Rousseau via the concept of an "optimal destabilising cocharacter". The optimality theory has had many applications to algebraic groups and representation theory.

Now suppose k is not algebraically closed. Bate, Herpel, Martin and Röhrle introduced the notion of a *cocharacter-closed* orbit. There is no known counterpart to the theory of optimality in this setting, but we can prove that some of the consequences of optimality from the algebraically closed case carry over. I will discuss this work and explain how it is related to the Centre Conjecture for the spherical building of G.

George McNinch (Tufts)

Reductive subgroup schemes of parahoric group schemes

Let A be a complete DVR with residues k and fractions K, and let G be a connected and reductive algebraic group over K. The parahoric groups schemes Q attached to G are certain smooth affine A-group schemes having generic fiber $Q_K = G$; in general, such Q are not reductive over A. Suppose that G splits over an unramified extension of K. Under this assumption, we show that for any parahoric group scheme Q attached to G, there is a closed and A-split reductive subgroup scheme M of Q for which the special fiber M_k is a Levi factor of the identity component M_k^0, and for which the generic fiber M_K is a reductive subgroup of $G = Q_K$ which contains a maximal K-split torus of G.

Bernhard Mühlherr (Gießen)

Tame descent in buildings

Let Δ be a spherical building and $\Gamma \leq \mathrm{Aut}(\Delta)$. Then Γ is called completely reducible if its fixed point set in Δ is a building.

If Δ is the building associated with an isotropic, absolutely simple algebraic group over a field k, then any finite group $\Gamma \leq \mathrm{Aut}(\Delta)$ whose order is prime to the characteristic of k is completely reducible; another important class of completely reducible subgroups is provided by Galois actions.

There are spherical buildings of 'algebraic origin' which are not associated to an absolutely simple algebraic group. If we only consider irreducible spherical buildings of rank at least 2, then these buildings are characterized by the Moufang-condition. This is a consequence of a fundamental result of Tits and Weiss.

Suppose that Δ is an irreducible Moufang building of rank at least 2 and $\Gamma \leq \mathrm{Aut}(\Delta)$. In my talk I will present a criterion for the complete reducibility of Γ which is inspired by the two examples mentioned above. The criterion is an important step in a bigger joint project with Richard Weiss whose goal is to develop a coherent theory of affine Tits-indices for Bruhat-Tits buildings. I intend to give also an outline on these applications of our criterion.

Alexander Premet (Manchester)

Non-semisimple maximal subalgebras of exceptional Lie algebras

Let G be an exceptional simple algebraic group over an algebraically closed field of good characteristic and $\mathfrak{g} = \mathrm{Lie}(G)$. We show that any maximal non-semisimple Lie subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ has form $\mathfrak{m} = \mathrm{Lie}(P)$ for some maximal parabolic subgroup P of G. We show by example that the statement is no longer valid in bad characteristic. In my talk I shall explain the main steps of the proof.

Guy Rousseau (Nancy)

Buildings for Kac-Moody groups

I shall explain the buildings associated to split Kac-Moody groups, over a general field or a valuated field. The properties are less interesting than in the classical reductive case. So the application to some "Geometric Invariant Theory" seems difficult.

David Stewart (Newcastle)

Unipotent radicals of pseudo-reductive groups

Let G be an algebraic k-group over an arbitrary field k. Following Tits, G is said to be pseudo-reductive if the largest k-defined connected smooth normal unipotent subgroup $\mathscr{R}_{u,k}$ of G is trivial. This notion stands in contrast to the usual notion of reductivity; a pseudo-reductive group will in general have a non-trivial unipotent radical, $\mathscr{R}_u$, after base change to an algebraically closed field, hence will not be reductive. We report on some recent work with Bate, Martin and Röhrle to examine the group-theoretic properties of this unipotent radical. (They tend to be highly non-abelian.)

Donna Testerman (Lausanne)

Reductive overgroups of regular elements in simple algebraic groups

We discuss the following result obtained in joint work with A. Zalesski: Let H be a closed connected reductive subgroup of a simple algebraic group G such that H contains a regular unipotent element of G. Then H is G-irreducible. This has recently been shown to not extend to distinguished unipotent elements by my PhD student M. Korhonen. There are still some natural questions remaining to be answered in this direction. If time permits, we will discuss further work with Zalesski on reductive overgroups of regular semisimple elements in simple algebraic groups.

Adam Thomas (Bristol)

The Jacobson-Morozov Theorem and Complete Reduciblity of Lie Subalgebras

The well-known Jacobson-Morozov Theorem states that every nilpotent element of a complex semisimple Lie algebra $\mathfrak{g} = Lie(G)$ can be embedded in an $\mathfrak{sl}_2$-subalgebra. Moreover, a result of Kostant shows that this can be done uniquely, up to G-conjugacy. Much work has been done on extending these fundamental results to the modular case when G is a reductive algebraic group over an algebraically closed field of characteristic $p > 0$. I will discuss recent joint work with David Stewart, proving that the uniqueness statement of the theorem holds in the modular case precisely when p is larger than $h(G)$, the Coxeter number of G. In doing so, we consider complete reduciblilty of subalgebras of $\mathfrak{g}$ in the sense of Serre/McNinch. For example, we prove that every $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$ is completely reducible precisely when $p > h(G)$.