Description of Research Project:

The theory of hyperplane arrangements has been a driving force in mathematics over many decades. It naturally lies at the crossroads of algebra, combinatorics and algebraic geometry. This proposal in turn lies at the very heart of these subject matters and algebraic Lie theory.

Much of the motivation for the study of arrangements comes from Coxeter arrangements. While the latter are well studied, their subarrangements are considerably less well understood. In this research proposal we want to investigate a particular class of arrangements which are associated with an ideal in the set of positive roots of the root system of a Weyl group, so called arrangements of ideal type. These were defined and investigated by Sommers and Tymoczko in 2006.

We propose two research strands stemming from two conjectures due to Sommers and Tymoczko.

The first of these conjectures concerns a multiplicative formula for the rank generating function of the poset of regions for an arrangement of ideal type which generalizes the well known factorization of the Poincare polynomial of the underlying Weyl group. Sommers and Tymoczko showed that this multiplicative formula holds for root systems of types A, B, C and small rank exceptional types.

The conjecture is still open in types D, E7 and E8. We propose a uniform approach to resolve this conjecture. By interpreting Sommers and Tymoczko's conjecture in the setting of rank-generating functions of the poset of regions for the underlying arrangements, we obtain a reduction to the case of ideals which do not contain any simple roots. Then we propose a further argument by induction on the cardinality of such ideals.

Our second research strand focuses on another conjecture by Sommers and Tymoczko. This concerns the freeness of the arrangements of ideal type. It was shown by Sommers and Tymoczko in case the Weyl group is classical or of exceptional type of small rank that each arrangement of ideal type is free. The general case was settled only very recently in a uniform manner for all types by Abe, Barakat,Cuntz, Hoge and Terao. This generalizes a seminal formula of Shapiro-Steinberg-Kostant relating the height distribution of the positive roots to the exponents of the Coxeter arrangement. Here we propose to investigate a number of questions concerning various stronger freeness properties for the arrangements of ideal type. It is known that the reflection arrangement of a Weyl group W itself is always inductively free. It is very likely that this is also the case for all the arrangements of ideal type. In general, we outline an inductive approach to this question by means of parabolic subgroups of the underlying Weyl group.

Apart from yielding new results for hyperplane arrangements, these results on arrangements of ideal type in turn provide new insight into the geometry of certain subvarieties of the flag variety associated with a complex reductive group G with Weyl group W, so called Hessenberg varieties.