Besov Regularity of Solutions to the p-Poisson Equation

Philipps Universität Marburg

S. Dahlke, L. Diening, C. Hartmann, B. Scharf, M. Weimar

Technische Universität München

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Motivation

Goal: Theoretical foundation of adaptive (wavelet) methods for the p-Poisson equation on bounded Lipschitz domains $\Omega \subset \mathbb{R}^d$.

General setting [2]:

nonadaptive methods	$\curvearrowright$	linear approximation
adaptive methods	$\curvearrowright$	nonlinear approximation

$$u \in W^s(L_p(\Omega)) \quad \curvearrowright \quad \|u - u_N\|_{L_p(\Omega)} = O(N^{-s/d}) \quad \text{for linear approximation}$$

$$u \in B^\sigma_\tau(L_\tau(\Omega)),\ \frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p} \quad \curvearrowright \quad \|u - u_N\|_{L_p(\Omega)} = O(N^{-\sigma/d}) \quad \text{for nonlinear approximation}$$

Resulting Question: What is the Besov regularity of solutions to the p-Poisson equation?

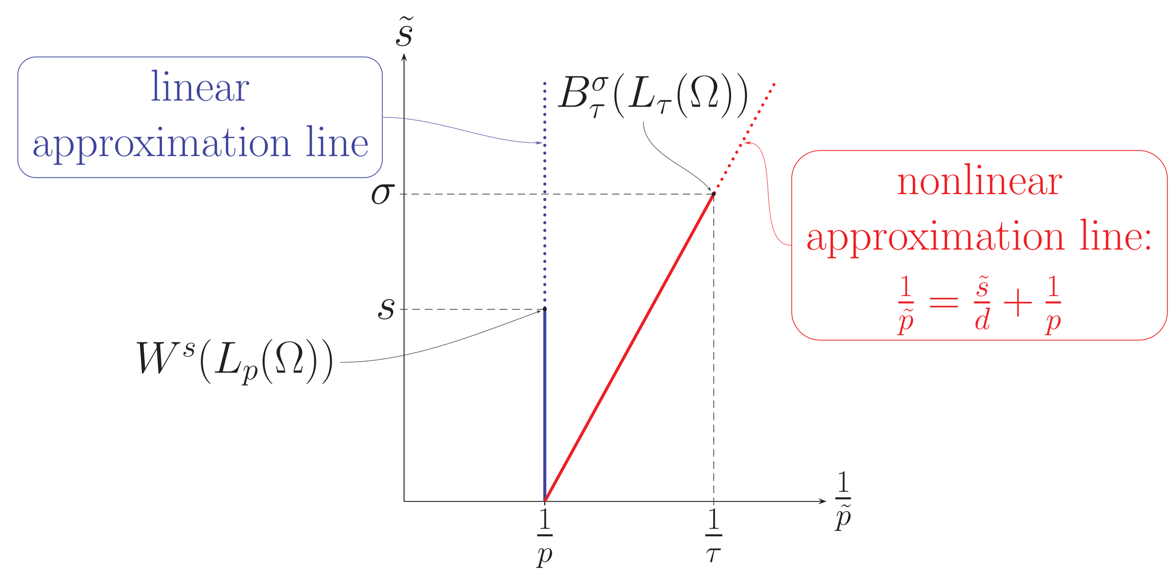

Figure 1: Linear vs. nonlinear appoximation illustrated in a so-called DeVore-Triebel-diagram.

The p-Poisson Equation

The p-Poisson equation:

$$\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = f \quad \text{in} \quad \Omega, \tag{1}$$

- $\Omega \subset \mathbb{R}^d$ bounded Lipschitz domain,
- $1 < p < \infty$,
- $f \in W^{-1}(L_{p'}(\Omega))$.

Variational formulation:

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \qquad \forall\, v \in W^1_0(L_p(\Omega)) \tag{2}$$

For given $g \in W^1(L_p(\Omega))$, equation (1) admits a unique weak solution $u \in W^1(L_p(\Omega))$ with $u - g \in W^1_0(L_p(\Omega))$.

A General Embedding

Let $C^{\ell,\alpha}(\Omega)$ be the Hölder space of all functions $g \in C^\ell(\Omega)$, for which

$$|g|_{C^{\ell,\alpha}(\Omega)} = \max_{|\nu|=\ell} \sup_{x \neq y \in \Omega} \frac{|\partial^\nu g(x) - \partial^\nu g(y)|}{|x-y|^\alpha} < \infty.$$

For the weight parameter $\gamma > 0$ we introduce the *locally weighted Hölder spaces*

$$C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega) = \{g : \Omega \to \mathbb{R} \mid g \in C^{\ell,\alpha}(K) \text{ for all compact } K \subset \Omega \text{ and } \sup_K \delta_K^\gamma |g|_{C^{\ell,\alpha}(K)} < \infty\},$$

where δ_K denotes the distance of K to the domain boundary.

General embedding theorem [1]:

Theorem 1. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain. Moreover, let $s > 0$ and $1 < p < \infty$, as well as $\ell \in \mathbb{N}_0$, $0 < \alpha \leq 1$, and $0 < \gamma < \ell + \alpha + 1/p$. If we define*

$$\sigma^* = \begin{cases} \ell + \alpha, & \text{if} \quad 0 < \gamma < \frac{\ell+\alpha}{d} + \frac{1}{p}, \\ \frac{d}{d-1}\left(\ell + \alpha + \frac{1}{p} - \gamma\right), & \text{if} \quad \frac{\ell+\alpha}{d} + \frac{1}{p} \leq \gamma < \ell + \alpha + \frac{1}{p}, \end{cases}$$

then for all

$$0 < \sigma < \min\left\{\sigma^*, \frac{d}{d-1}s\right\} \qquad \text{and} \qquad \frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p} \tag{3}$$

we have the continuous embedding

$$B^s_p(L_p(\Omega)) \cap C^{\ell,\alpha}_{\gamma,loc}(\Omega) \hookrightarrow B^\sigma_\tau(L_\tau(\Omega)).$$

Note that the second entry of the minimum in (3) is always greater than the Sobolev regularity parameter s by a factor of $d/(d-1)$. Thus, for appropriate parameters ℓ, α and γ, the function u actually gains some additional regularity in the considered scale of Besov spaces, compared to its Sobolev regularity.

Application to Solutions to the p-Poisson Equation

Sobolev Regularity

The following result is well-known [4].

Proposition 2. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain and let $1 < p < \infty$, as well as $f \in L_{p'}(\Omega)$. Then the unique solution $u \in W^1_0(L_p(\Omega))$ to the p-Poisson equation (1) with homogeneous Dirichlet boundary conditions satisfies*

$$u \in W^s(L_p(\Omega)) \qquad \text{for all} \qquad s < \overline{s} = \begin{cases} \frac{3}{2} & \text{if} \quad 1 < p \leq 2, \\ 1 + \frac{1}{p} & \text{if} \quad 2 < p < \infty. \end{cases}$$

Besov Regularity

In case of *polygonal* domains $\Omega \subset \mathbb{R}^2$ and *homogeneous* Dirichlet boundary conditions, an application of Theorem 1 to the solutions $u \in W^1_0(L_p(\Omega))$ of (1) yields the following result [1].

Theorem 3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and let $1 < p < \infty$, as well as $f \in L_q(\Omega)$ with $2 < q \leq \infty$ and $q \geq p'$. Then the unique solution $u \in W^1_0(L_p(\Omega))$ to the p-Poisson equation (1) with homogeneous Dirichlet boundary conditions satisfies*

$$u \in B^\sigma_\tau(L_\tau(\Omega)) \qquad \text{for all} \qquad 0 < \sigma < \overline{\sigma} \qquad \text{and} \qquad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p},$$

where

$$\overline{\sigma} = \begin{cases} 2 - \frac{2}{q} & \text{if} \quad 1 < p < 4/3 \text{ and } p' \leq q \leq \infty, \\ 2 - \frac{2}{q} & \text{if} \quad 4/3 \leq p \leq 2 \text{ and } 4 < q \leq \infty, \\ \frac{3}{2} & \text{if} \quad 4/3 \leq p < 2 \text{ and } p' \leq q \leq 4, \\ \frac{3}{2} & \text{if} \quad p = 2 \text{ and } 2 < q \leq 4, \\ 1 + \frac{1-2/q}{p-1} & \text{if} \quad 2 < p < \infty \text{ and } 2p < q \leq \infty, \\ 1 + \frac{1}{p} & \text{if} \quad 2 < p < \infty \text{ and } 2 < q \leq 2p. \end{cases}$$

Idea of proof: The proof is based on Theorem 1. Therefore, the Besov regularity $B^s_p(L_p(\Omega))$ of u is obtained from Proposition 2 and the embedding $W^s(L_p(\Omega)) \hookrightarrow B^{s-\varepsilon}_p(L_p(\Omega))$. The $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ regularity is obtained from local Hölder regularity results [3], together with an estimate for the parameter γ.

Sobolev Smoothness vs. Besov Regularity

Comparing Proposition 2 and Theorem 3, we see that in many relevant cases the Besov regularity of solutions to the p-Poisson equation is significantly higher than the Sobolev smoothness!

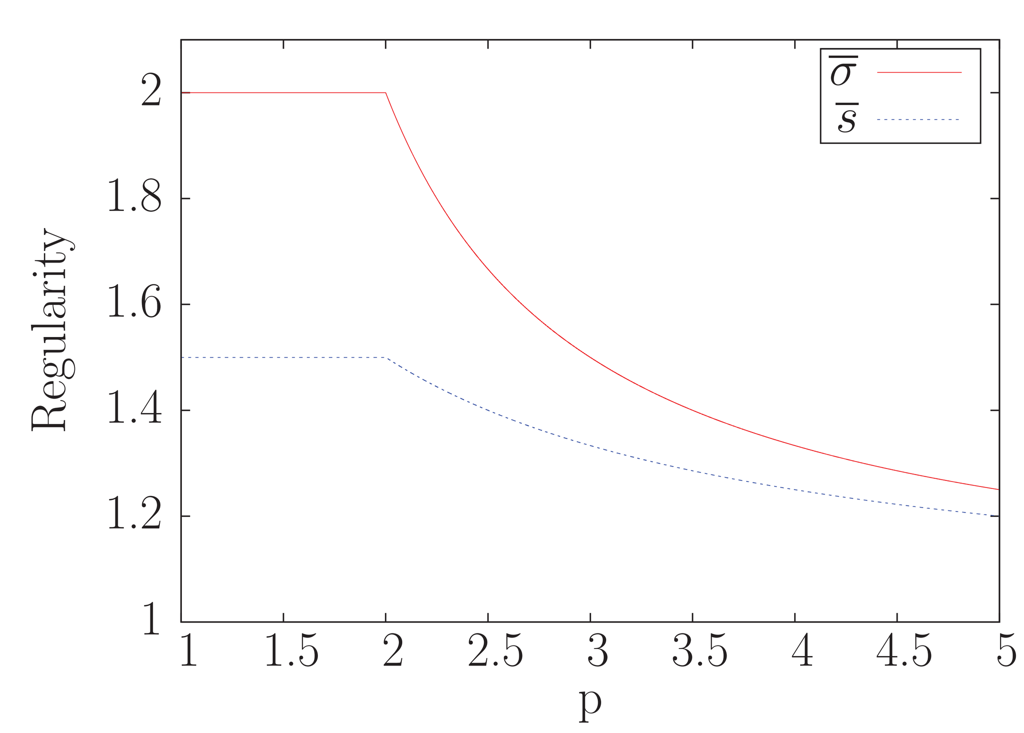

Figure 2:
Bounds $\overline{\sigma}$ and $\overline{s}$ for the regularity of solutions $u \in W^1_0(L_p(\Omega))$ to (1) with $f \in L_\infty(\Omega)$ on polygonal domains $\Omega \subset \mathbb{R}^2$, measured in $B^\sigma_\tau(L_\tau(\Omega))$, $1/\tau = \sigma/2 + 1/p$, and in $W^s(L_p(\Omega))$, respectively.

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