List of Topics

References: [L]=Levandosky, [E]=Evans, [F]=Folland, [FJ]=Fritz-John, [H]=Hunter, [HN]=Hunter-Nachtergaele, [R]=Rauch, [RR]=Renardy-Rogers, [S]=Schweizer(online version Feb 2012).

1. Background review, mostly without proofs: *Selected from [H, Ch 1] and [E, Appendices].*
   - Some notation.
   - L^p-spaces (and L^p_{loc}), Lebesgue convergence theorem, Banach and Hilbert spaces, Fubini, Hölder inequality, Minkowski inequality. (isometries, reflexivity, separability, complete orth.normal bases, dual space?).
   - The C_c, C_0, C^k, and $C^{k,\alpha}$ spaces. Density of C^∞ in L^p on $\mathbb{R}^n$.
   - Divergence theorem, Green's identities, integration by parts, averages, polar coordinates.
   - Convolution, mollifiers and smoothing. Density of $C^\infty(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$.
2. Introduction to PDE's.

   Types of PDE, Well-posedness, Modern strategy: distributional, weak, strong, classical solutions, *[S pg60].*
3. The Transport Equation (with constant coefficients).
   - Homogenous equation: $u_t + \sum_{i=1}^n b_i u_{x_i} = 0$ on $\mathbb{R}^n \times (0, \infty)$ with initial condition $u = g$ on $\mathbb{R}^n \times \{0\}$.
     - Special case $n = 1$ and $b = 1$ in *[S, pg 32-33].*
     - General case in *[E, pg 18].*
     - Characteristics of the equation. *[S, pg 34].*
   - Nonhomogenous equation: $u_t + \sum_{i=1}^n b_i u_{x_i} = f$ on $\mathbb{R}^n \times (0, \infty)$ with initial condition $u = g$ on $\mathbb{R}^n \times \{0\}$.
     - Special case $n = 1$ and $b = 1$ in *[S, pg 33-34].*
     - General case in *[E, pg 19].*
   - On a space-time cylinder: $(a, b) \times (0, \infty)$ and boundary conditions compatible with the characteristics *[S, pg 34].*

4. The Fourier Transform[*]. *Follow [L, Sec 2.3], plus references mentioned below for Riemann-Lebesgue Lemma.*

   - *Motivation: diagonalizing translation operator, and hence also differentiation. See me.*
   - Definition on $L^1(\mathbb{R}^n)$. Riemann-Lebesgue Lemma: $f \in L^1$ implies $\hat{f}$ is continuous and $\|\hat{f}\|_{C^0} \leq \|f\|_{L^1}$. *[HN, thm 11.34], see also [F, Lem 0.24].*
   - FT of Gaussians. Extension of FT to $L^2(\mathbb{R}^n)$ by density.
   - Properties of Fourier transform; Plancherel's theorem, derivative $\partial_i \to$ multiplication by x_i, convolution $\to$ multiplication, inversion formula.

5. The Heat Equation. *Follow [L, Sec 2.4]*

   - Solving the heat equation on $\mathbb{R}^n$: formally using Fourier transform, rigorous justification.
   - Fundamental solution of the Heat Equation.

6. Interlude on Distributions (Generalized functions). *Follow [L, Sec 2.5] plus references below.*

   - Definitions and examples *[L, Sec 2.5]*, convergence of distributions, *see also [H, Sec 3.3] if want more precision.* Regular distributions *[H, Example 3.11].*
   - Derivatives of distributions *[L, Sec 2.5].*
   - Weak derivatives of functions in L^1_{loc} and examples, *[H, pgs 47,48]. Also remarks on [H, pg 53] interpreting weak derivatives as distributional derivatives.* Basic properties of weak derivatives: uniqueness [E, pg 243], Leibniz rule [H, prop 3.16].
   - Distributional solution to transport equation when initial conditions are only in L^1_{loc}. *See [S, Beispiel 3.9].*

7. The Heat Equation continued. *Basically follow [L, Sec 4, pgs 30-44], leave out [L, Sec 2.7.3] and [L, Sec 2.7.4].* More details:

   - Fundamental solution of the Heat Equation revisited, *[L, Sec 2.5.4].*
   - Properties of solutions to the Heat equation, *[L, Sec 2.6].*
   - Inhomogenous Heat Equation on $\mathbb{R}^n$. Duhamel's Principle. *[L, Sec 2.7], omit sections 2.7.3, 2.7.4 unless have plenty of time.*
   - Maximum principle for bounded space domains, maximum principle for unbounded space domains *[L Sec 2.8]. Could leave uniqueness statement until end of section, then can compare with uniqueness result from energy.*
   - Energy estimates. *In [L, pg 44] energy is not stated as an inequality, but see [RR Lemma 1.20].*
   - Uniqueness: from max principle, and energy. Non-uniqueness without suitable bound. *For non-uniqueness see [FJ, pg211].*

[*]Careful: different texts use slighly different definitions for the Fourier transform. We will follow the definition appearing in most of the references we are using, such as [E], [L], [H].

8. Laplace Equation on $\mathbb{R}^n$. *Follow [L, Sec 3] plus reference below for Harnack's inequality. Can omit all of Sec 3.3 in [L].*

   - Fundamental solution of Laplace equation including proof in sense of distributions.
   - A solution to Poisson's equation, *(later we see is the unique bounded solution when* $n \geq 3$*).*
   - Harmonic functions: mean value properties, maximum principle, Harnack's inequality [H Sec 2.4], uniqueness on bounded domains, regularity, Liouville's theorem, uniqueness of bounded solutions on $\mathbb{R}^{n\geq 3}$ (representation formula).

9. Laplace Equation on a bounded domain $\Omega \subset \mathbb{R}^n$. *Follow [L, Sec 4] and comments below.*

   - Discussion of Dirichlet and Neumann Problems, including non-uniqueness of NP: *[F, pg 83-85].*
   - Green's functions (fundamental solution of Laplace equation on a bounded domain). Motivation, representation of a solution. Explicit computation of Green's function on $\mathbb{R}^n_+$. Poisson's formula for $\mathbb{R}^n_+$ and ball *(state without proof, details in [L, Sec 4]).*
   - Symmetry of Green's function: *If not enough time then only outline proof. See [F pg 86], and [L Sec 4.3, Lemma 13].*
   - Using Green's function to solve Poisson, i.e. assuming existence of Green's function prove existence of solution to Poisson/Dirichlet.

   State examples 6,7,8, 10 in [L, Sec 4] without details. Can also omit Theorem 12.

10. The Wave Equation.

    - On $\mathbb{R}^n \times [0, T]$; first $n = 1$, deducing D'Alembert's formula and proving it works *[S, Sec 2.3.3] and [E, Sec 2.4.1].*
    - $n = 3$: Kirchoff's formula via method of spherical means: *[E, pg 70-72].*
    - $n = 2$: Outline method of descent from $n = 3$ to $n = 2$. *[E, pg 73-74].*
    - Properties of solutions to the Wave Equation: *[E, pg 83-85] + [E, Rmks pg 78].* Include the following:
      - Huygens principle *(see also remarks in [F, pg 172]).*
      - Domain of dependence,
      - Finite speed of propagation,
      - Energy conservation,
      - Uniqueness from energy
      - No maximum principle; explain why (domain of dependence or finite speed of propagation).

      Omit discussion of: inhomogenous equation except to remark that a similar approach to the Heat equation motivated by Duhamel's principle works; wave equation in dimensions n>3 except to remark that procedure is similar to $n = 2, 3$ *namely first some* n *=odd by method of spherical means then descend to* n *=even.*

11. Potential Theory. *Follow [L, Sec 5]. Omit discussion of Exterior Problems. Focus on the Dirichlet problem rather than Neumann problem.*

    - Definitions of single and double layer potentials, preliminaries.
    - Gauss' Lemma.
    - Reduction of (interior) Dirichlet and Neumann Problems to integral equations.

12. Solving the integral equations for the Dirichlet and Neumann Problems using methods from Functional Analysis (with some details missing). *Mainly following [F, parts of Ch 3].*

    - Easier example where can solve an integral equation using contraction mapping theorem. Why this does not work in case of interest.
    - Compact operators: e.g. $C^{0,\alpha}(K) \hookrightarrow C^0(K)$, closed under operator convergence. Hilbert-Schmidt kernels. *Mostly found in [F, Ch 0 sec F].*
    - Weak convergence in Hilbert spaces, and weak compactness theorem. Compact operators take weakly convergent sequences to strongly convergent sequences. *[E, Appendix D].*
    - The Fredholm alternative (a special case): how uniqueness yields existence. *[E, Appendix D].*
    - Solving the Dirichlet and Neumann Problems. (This uses Young's inequality). *[F, parts of Ch 3].*
    - † Some further perspectives: ellipticity, L^2-theory, L^p-theory, Schauder theory, Perron's method, Pseudo-differential operators. *[H Sec 4.13].*

13. Solving the Dirichlet problem using variational/Hilbert space methods (with some details missing). There are three parts; can cover just the first two if not enough time.

    (A) Introduce Sobolev spaces, relate weak to classical derivatives via Sobolev's Lemma (Sobolev embedding theorem).

    (B) Prove existence and uniqueness of a weak solution $u \in H_0^1(\Omega)$ to

    $$\begin{aligned} -\Delta u = f \in L^2 \qquad &\text{on } \Omega, \\ u = 0 \qquad &\text{on } \partial\Omega. \end{aligned} \tag{1}$$

    (C) Show weak solutions satisfy the boundary conditions in a meaningful way (trace theorem), and that u is not just a weak solution but a strong solution (elliptic estimates), and is a classical solution if $f \in C^\infty(\overline{\Omega})$.

    More details:

(A) Sobolev spaces I:

  (i) Define $W^{k,p}(\mathbb{R}^n)$ and $H^k(\mathbb{R}^n)$, and on a bounded domain $\Omega \subset \mathbb{R}^n$ using distributional derivatives. Basic properties such as completeness. Smooth approximations $C_c^\infty(\mathbb{R}^n)$. *[S, Sec 3.3: pg 51-54]*. Definition of $H_0^k(\Omega)$, and completeness, *[S, Def 3.21]*.

  (ii) Lemma: equivalent characterization of $H^k(\mathbb{R}^n)$ using Fourier transform *[F, thm 6.1]*. Also can use to define $H^s(\mathbb{R}^n)$ for non-integer $s \geq 0$.

  (iii) The Sobolev Lemma: $H^k(\mathbb{R}^n)$ embeds in $C^r(\mathbb{R}^n)$ if $k > r + \frac{1}{2}n$, using Fourier transform. Corollary: $f \in H^k$ for all k implies $f \in C^\infty$. *[F, Lem 6.5, Cor 6.7]*.

(B) Dirichlet's principle and weak solutions of $-\Delta u = f$ on Ω with homogenous boundary conditions. *Nicely presented in [S, pg 97-102]*

  (i) Poincare inequality. *[S, Satz 6.1] but do only for* $p = 2$.

  (ii) Proof of existence of weak solution $u \in H_0^1(\Omega)$. *[S, Satz 6.3], but their boundary conditions are more general: we just set* $g = 0$. *Also, replace the space* X_g *by* $H_0^1(\Omega)$.

(C) Sobolev spaces II:

  (i) Trace theorem for Ω with smooth boundary. *Idea behind proof. See for example [S, Satz 3.20], or see me.*

  (ii) Interior elliptic estimates. *Proof is simple using Fourier transform, see step 1 in proof of [F, thm 6.28], or I can explain.*

  (iii) Elliptic estimates up to the boundary. *Statement without proof.*

  Combining (B) and (C) we can now show that our weak solution $u \in H_0^1(\Omega)$ to (1) actually satisfy: (See me)

  - $u \in H^2(\Omega')$ for all $\Omega' \subset\subset \Omega$ using Fourier transform.
  - $u = 0$ in the sense of trace.
  - $f \in C^\infty(\overline{\Omega}) \implies u \in C^\infty(\overline{\Omega})$.