# PERIODIC APPROXIMATIONS OF IRRATIONAL PSEUDO-ROTATIONS USING PSEUDOHOLOMORPHIC CURVES

BARNEY BRAMHAM

Abstract. We prove that every $C^\infty$-smooth, area preserving diffeomorphism of the closed 2-disk having not more than one periodic point is the uniform limit of periodic $C^\infty$-smooth diffeomorphisms. In particular every smooth irrational pseudo-rotation can be $C^0$-approximated by integrable systems. This partially answers a long standing question of A. Katok regarding zero entropy Hamiltonian systems in low dimensions. Our approach uses pseudoholomorphic curve techniques from symplectic geometry.

## Contents



## 1. Introduction

1.1. **The main result.** In this paper we prove a statement which is a significant first step towards answering the following general question of Katok.

> *In low dimensions is every conservative dynamical system with zero topological entropy a limit of integrable systems?*

This is stated as problem 1 in [29], but relates also to the Anosov-Katok constructions [1] in 1970. Low dimensions means maps on surfaces or flows on 3-dimensional manifolds. The result in this article is concerned with Katok's question for area preserving disk maps.

Arguably one of the main obstacles to an affirmative answer to this question is the presence of ergodic components of positive measure. Indeed, ergodic maps (with respect to Lebesgue measure) exhibit strongly different dynamical behavior to integrable ones. In particular, almost every point is the initial condition for a dense orbit.

With regard to area preserving disk maps, the only known ergodic examples with zero entropy are so called irrational pseudo-rotations. These are the area preserving disk maps with precisely one periodic point, see definition 3. Using pseudoholomorphic curve methods

we show that all irrational pseudo-rotations are in some sense limits of integrable systems as Katok's question suggests.

More precisely, for each $t \in \mathbb{R}$ let $R_{2\pi t} : D \to D$ denote the rigid rotation through angle $2\pi t$ on the disk $D = \{(x, y) \,|\, x^2 + y^2 \leq 1\}$. Let $\mathrm{Diff}^\infty_+(D)$ and $\mathrm{Diff}^\infty(D, \omega_0)$ be the spaces of $C^\infty$-smooth diffeomorphisms of the disk which preserve orientation and the area form $\omega_0 = dx \wedge dy$ respectively.

**Theorem 1** (Main result)**.** *Suppose $\varphi \in \mathrm{Diff}^\infty(D, \omega_0)$ fixes the origin and has no other periodic points. Then it is the $C^0$-limit of a sequence of maps of the form $\varphi_k = g_k^{-1} R_{2\pi p_k/q_k} g_k$, for a sequence of conjugating maps $g_k \in \mathrm{Diff}^\infty_+(D)$ which fix the origin, and a sequence of rationals $p_k/q_k$ converging to an irrational number.*

Concerning the convergence of the approximation maps in theorem 1 it is natural to ask whether $C^0$-convergence is so weak as to allow "almost anything" to be obtained in the limit A more natural topology to consider Katok's question, as discussed in [29], is at least a $C^{1,\varepsilon}$-topology, $\varepsilon > 0$, in which the topological entropy is lower semi-continuous. The author would therefore like to thank Patrice Le Calvez for pointing out the following and the idea of its proof using work of Franks. A slightly more general statement is proven in appendix A.1.

**Proposition 2.** *Suppose $\varphi \in \mathrm{Diff}^\infty(D, \omega_0)$ is the $C^0$-limit of a sequence of maps of the form $\varphi_k = g_k^{-1} R_{2\pi p_k/q_k} g_k$, for a sequence of conjugating maps $g_k \in \mathrm{Diff}^\infty_+(D)$ fixing the origin, and a sequence of rationals $p_k/q_k$ converging to an irrational number. Then $\varphi$ necessarily has precisely one periodic point. In particular it is an irrational pseudo-rotation.*

Thus $C^0$-convergence is still strong enough to guarantee that the limit object is an irrational pseudo-rotation. In particular that it has zero entropy. (Due, for example, to Katok's theorem for $C^{1,\varepsilon}$ surface diffeomorphisms [28] that bounds entropy from above by the exponential growth rate of periodic points.)

The existence of ergodic area preserving disk maps with zero entropy was established back in 1970 by Anosov and Katok [1]. Previous to their constructions it was even an open question in the non-conservative setting; Shnirelman 1930 [39] found a (non-area preserving) homeomorphism of the disk with a dense orbit, a more detailed discussion of which can be found in [11].

It is interesting to compare the Anosov-Katok construction to the statement of theorem 1. They construct an ergodic map $\varphi : D \to D$ as the $C^\infty$-limit of a sequence of maps $\varphi_k : D \to D$ which are inductively constructed with the following form. For each $k \in \mathbb{N}$ there exists $(p_k, q_k) \in \mathbb{Z} \times \mathbb{N}$ relatively prime, and $g_k \in \mathrm{Diff}^\infty(D, \omega_0)$, also fixing the origin, so that

$$\varphi_k = g_k^{-1} \circ R_{2\pi p_k/q_k} \circ g_k. \tag{1}$$

The maps $g_k$ are arranged so that the orbits of $\varphi_k$ increasingly spread out over the disk as $k \to \infty$. Consequently, the sequence $\{g_k\}$ blows up in every $C^r$ topology. But by interatively choosing $q_{k+1} - q_k$ sufficiently large depending on the size of $\|g_k\|_{C^k}$, the $C^k$ norm of $\varphi_k$ can be controlled. A limiting subsequence converges to a map $\varphi$ with the desired "pathological" behavior such as a dense orbit, or ergodicity, or even weak mixing. More details of this method, other results and questions, are in Fayad-Katok [11]. See also Fayad-Saprykina [12].

In some sense then, theorem 1 reverses the limiting process just described above. However our conclusions are in two respects weaker than a word for word converse to the Anosov-Katok construction. Firstly, the convergence $\varphi_k \to \varphi$ in [1] is in the $C^\infty$-sense. Secondly, each $\varphi_k$

in [1] preserves the standard area form. We do not show this for the approximation maps in theorem 1. This raises natural questions for further investigation.

A remark on integrability: The notion of integrability for a map on a surface that appears to be referred to in [29] is that the map should admit a "first integral", that is, a smooth real valued function on the surface that is not constant on any open set but is constant on the orbits of the given map. It is obviously in this sense that each of our approximation maps in theorem 1 is integrable. A natural question is whether approximation maps can be found that are integrable in the Liouville-Arnold sense. This would follow if they were area preserving.

1.2. **Outline of the proof.** Let $\varphi : D \to D$ be an irrational pseudo-rotation. That is, $\varphi$ is a smooth area and orientation preserving diffeomorphism that fixes the origin and has no other periodic points. As a consequence the circle map on the boundary $\varphi : \partial D \to \partial D$ has irrational rotation number $[\alpha] \in \mathbb{R}/\mathbb{Z}$ say.

We pick a closed loop of Hamiltonians $H_t : D \to D$, over $t \in \mathbb{R}/\mathbb{Z}$, which generate a symplectic isotopy whose time-one map is $\varphi$. Denote the 1-periodic path of Hamiltonian vector fields on the disk by $X_{H_t}$. For each $n \in \mathbb{N}$ equip the solid torus

$$Z_n := \mathbb{R}/n\mathbb{Z} \times D$$

with coordinates $(\tau, z)$. Then the vector field $R_n(\tau, z) = \partial_\tau + X_{H_\tau}(z)$ defines a flow on $Z_n$ with time-one map $\varphi$ and first return map $\varphi^n$. For each $n \in \mathbb{N}$ there is a unique simple periodic orbit in $Z_n$ (corresponding to the unique fixed point of $\varphi^n$ at the origin). We can choose $t \mapsto H_t$ so that this periodic orbit in $Z_n$ passes through the center of each disk slice, indeed is parameterized by

$$\begin{aligned} \gamma_n : \mathbb{R}/n\mathbb{Z} &\to Z_n \\ t &\mapsto (t, 0). \end{aligned}$$

Consider the 4-manifold $W_n := \mathbb{R} \times Z_n$ with the unique almost complex structure satisfying

$$\begin{cases} J_n \partial_{\mathbb{R}} = R_n \\ J_n|_{TD} = i \end{cases} \tag{2}$$

where $\partial_{\mathbb{R}}$ is the vector field dual to the $\mathbb{R}$-coordinate on $W_n$. Then $(W_n, J_n)$ is a so called cylindrical, symmetric, almost complex manifold. That is, it is compatible in a precise way with the necessary symplectic structures for the compactness framework from symplectic field theory [4] to apply to $J_n$-holomorphic curves. In section 7 we adapt techniques developed by Hofer, Wysocki, and Zehnder [22, 23, 24, 26, 27] to construct, for each $n \in \mathbb{N}$, a pair of finite energy foliations $\mathcal{F}_n^+$ and $\mathcal{F}_n^-$ of the almost complex manifold $(W_n, J_n)$. In this introduction we will only need to refer to the sequence $(\mathcal{F}_n^+)$ which we therefore abbreviate by $(\mathcal{F}_n)$. We describe these foliations in a moment. However, having constructed the foliations, the proof of theorem 1 consists of three steps:

(1) For each $n \in \mathbb{N}$ we use the leaves in $\mathcal{F}_n$ to define a disk map $\varphi_n = \varphi_{\mathcal{F}_n} : D \to D$. Since this is defined by following a path along the leaves we will refer to it as the *holonomy map of* $\mathcal{F}_n$. Each map $\varphi_n$ inherits the smoothness of the foliation.
(2) A symmetry in the leaves of $\mathcal{F}_n$ implies that $\varphi_n$ is integrable in the sense that $(\varphi_n)^n = \mathrm{id}_D$. Indeed, by classical results this implies that $\varphi_n$ is at least topologically conjugate to a rotation $R_{2\pi q/n} : D \to D$, for some $q \in \{0, 1, \ldots, n-1\}$.
(3) There exists a subsequence $n_j$ such that $\varphi_{n_j}$ converges to the given irrational pseudo-rotation $\varphi$.

We now outline these steps in more detail.

**Step 1:** For each $n \in \mathbb{N}$, $\mathcal{F}_n$ is a 2-dimensional foliation of the 4-manifold $\mathbb{R} \times Z_n = \mathbb{R} \times \mathbb{R}/n\mathbb{Z} \times D$ by surfaces called the leaves of $\mathcal{F}_n$. Each leaf $F \in \mathcal{F}_n$ is the image of a properly embedded, non-compact $J_n$-holomorphic map satisfying a finite energy condition. For precise definitions see section 3. Let us here describe the geometric properties of the leaves in $\mathcal{F}_n$. There is a distinguished leaf $C_n \in \mathcal{F}_n$ that is homeomorphic to $\mathbb{R} \times S^1$, which we call the *cylinder* in $\mathcal{F}_n$, see Figure 1. This has the following simple description:

$$C_n = \big\{(a, \gamma_n(t)) \in \mathbb{R} \times Z_n \,\big|\, a \in \mathbb{R},\ t \in S^1\big\}.$$

Thus $C_n$ intersects each slice $\{c\} \times Z_n$ in the trace of the unique simple periodic orbit $\gamma_n$. The other leaves are homeomorphic to $[0, \infty) \times S^1$ and will be referred to as *half cylinders*. Each half cylinder $F$ can be parameterized by a smooth embedding $\tilde{u} : [0, \infty) \times \mathbb{R}/n\mathbb{Z} \to W_n = \mathbb{R} \times Z_n = \mathbb{R} \times \mathbb{R}/n\mathbb{Z} \times D$ of the form $(s, t) \mapsto (s - s_0, t, z(s, t))$ for some constant $s_0$ depending on $F$. The loops $s \mapsto (\cdot, z(s, \cdot))$ begin in the boundary of $Z_n$ at $s = 0$ and converge in $C^\infty(\mathbb{R}/n\mathbb{Z}, Z_n)$ to the unique simple periodic orbit $\gamma_n$ as $s \to +\infty$.

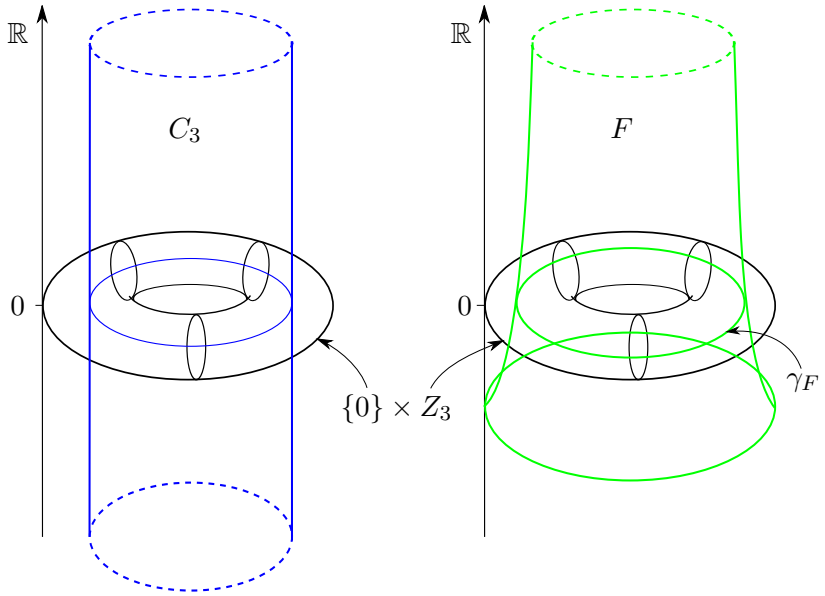


FIGURE 1. Illustrating $n = 3$: both figures separately represent leaves in $\mathbb{R} \times Z_3$ coming from the foliation $\mathcal{F}_3$. On the left the leaf is the cylinder leaf $C_3$, on the right a typical half cylinder leaf $F$. Both leaves intersect the hypersurface $\{0\} \times Z_3$ transversely in embedded closed loops.

Infact the half cylinders come in a 2-parameter family. One parameter comes from translations in the $\mathbb{R}$-direction: if $F \in \mathcal{F}_n$ is a half cylinder then for each $c \in \mathbb{R}$ the translated set

$$F_c := \big\{(a + c, m) \in \mathbb{R} \times Z_n \,\big|\, (a, m) \in F\big\}$$

will also be a half cylinder in $\mathcal{F}_n$.

For a half cylinder $F \in \mathcal{F}_n$ consider the intersection

$$\gamma_F := F \cap \big(\{0\} \times Z_n\big).$$

Identifying the hypersurface $\{0\} \times Z_n$ with $Z_n$ we can view

$$\gamma_F \subset Z_n.$$

It turns out that $\gamma_F$ is either empty or is an embedded circle that closes up after going "once" around the solid torus. For any point $p \in Z_n$ there exists a leaf $F \in \mathcal{F}_n$ such that $p \in \gamma_F$. Moreover if $G$ is another leaf for which $p \in \gamma_G$, then $G$ and $F$ are related by an $\mathbb{R}$-translation in $\mathbb{R} \times Z_n$, and so $\gamma_G = \gamma_F$. Thus the set

$$\Delta(\mathcal{F}_n) := \left\{ \gamma_F \subset Z_n \,\middle|\, F \in \mathcal{F}_n \right\}$$

is some kind of filling of the mapping torus $Z_n$ by pairwise disjoint embedded circles. Note that the trace of the periodic orbit $\gamma_n$ also lies in $\Delta(\mathcal{F}_n)$, since

$$\gamma_{C_n} = \text{image}(\gamma_n).$$

It turns out that each circle $\gamma \in \Delta(\mathcal{F}_n)$ intersects each disk slice $\{\tau\} \times D \subset Z_n$ transversely and uniquely. This is a consequence of the positivity of intersections phenomenon between pairs of distinct pseudoholomorphic curves; the disks $\{a, \tau\} \times D \subset \mathbb{R} \times Z_n$ are also images of embedded $J_n$-holomorphic maps. We can therefore define a new disk map

$$\varphi_n = \varphi_{\mathcal{F}_n} : D \to D$$

which maps the point $\xi \in D$ to $\xi' \in D$ if there exists a loop $\gamma \in \Delta(\mathcal{F}_n)$ connecting the points $(0, \xi)$ and $(1, \xi)$, see figure 2. We refer to $\varphi_n$ as the *holonomy map associated to* $\mathcal{F}_n$.

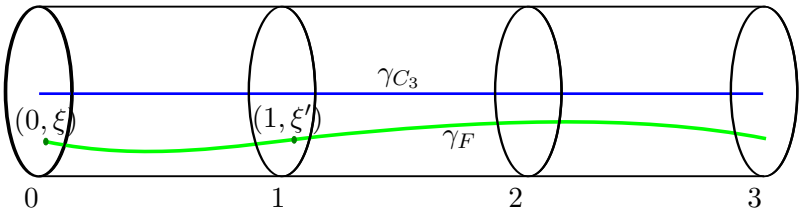


FIGURE 2. Illustrating the definition of the holonomy map $\varphi_3 : D \to D$. The tube represents the mapping torus $Z_3$ opened out. Identify this space with the hypersurface $\{0\} \times Z_3$ inside $\mathbb{R} \times Z_3$. Then the curves $\gamma_{C_3}$ and $\gamma_F$ are the closed loops from figure 1 where the leaves $C_3$ and $F$ respectively intersect $\{0\} \times Z_3$. We then set $\varphi_3(\xi) = \xi'$.

From the properties above the holonomy map is a bijection between the disks. Since $\text{image}(\gamma_n) \in \Delta(\mathcal{F}_n)$ we know that $\varphi_{\mathcal{F}_n}$ fixes the origin. This completes our definition and geometric description of the holonomy maps.

**Step 2:** The integrability of $\varphi_n$ comes from the following symmetry within the foliation $\mathcal{F}_n$. Our 3-manifold $Z_n$ has a natural $\mathbb{Z}_n$-action on it given by the deck transformations. The corresponding $\mathbb{Z}_n$-action on $W_n$ generated by the map

$$\begin{aligned} \mathcal{T} : \mathbb{R} \times Z_n &\to \mathbb{R} \times Z_n \\ (a, \tau, z) &\mapsto (a, \tau + 1, z) \end{aligned} \tag{3}$$

also preserves the almost complex structure $J_n$ defined in (2). Thus if $F$ is a leaf in $\mathcal{F}_n$ then $\mathcal{T}(F)$ is the image of another $J_n$-holomorphic map, and we can ask whether $\mathcal{T}(F)$ is also a leaf in $\mathcal{F}_n$. This turns out to be the case, the key reason being the positivity of intersections

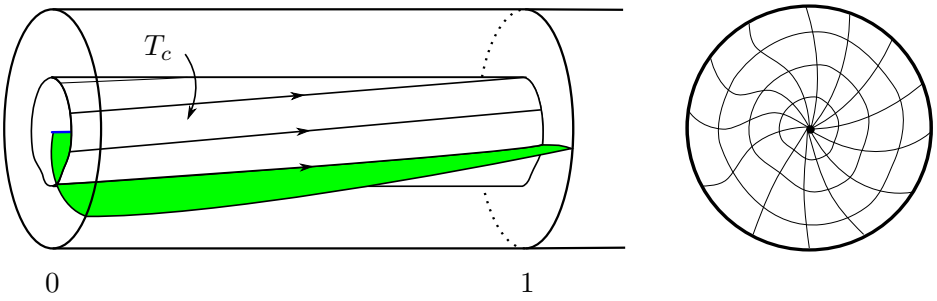


FIGURE 3. Illustrating a 2-torus in $Z_n$, which corresponds to an invariant circle for the $n$-th holonomy map $\varphi_{\mathcal{F}_n}$. The shaded strip is a portion of the projection into $Z_n$ of a leaf $F \in \mathcal{F}_n$. This strip would close up in time $n$. For each $c \in (-\infty, 0]$ we obtain a 2-torus $T_c \in Z_n$ of which one is illustrated. The set $T_c$ is defined as the union of all closed loops $\gamma_G$ as $G \in \mathcal{F}_n$ runs over all leaves with boundary condition $\partial G \in \{c\} \times \partial Z_n$ for the given $c \in (-\infty, 0]$. (For $c > 0$ this defines an empty set.) As $c \to -\infty$ the tori $T_c$ collapse onto the periodic orbit $\gamma_n$ at the center. Note that although $T_c$ is comprised of $n$-periodic loops, $T_c$ itself is 1-periodic due to the deck transformation invariance of $\mathcal{F}_n$. The concentric circles in the figure on the right illustrate how these 2-tori intersect the disk $\{0\} \times D$ in circles invariant under $\varphi_{\mathcal{F}_n}$. Each radial arc would get mapped to another radial arc under $\varphi_{\mathcal{F}_n}$.

phenomenon for pseudoholomorphic curves, plus certain convenient features of the dynamics. We conclude then that

$$\mathcal{T}(\mathcal{F}_n) := \left\{ \mathcal{T}(F) \,\middle|\, F \in \mathcal{F}_n \right\} = \mathcal{F}_n$$

for each $n \in \mathbb{N}$. We say that $\mathcal{F}_n$ has *deck transformation invariance*. The deck transformation invariance of $\mathcal{F}_n$ implies that the set of embedded loops $\Delta(\mathcal{F}_n) \subset Z_n$ is invariant under

$$\begin{aligned} \hat{\mathcal{T}} : Z_n &\to Z_n \\ (\tau, z) &\mapsto (\tau + 1, z). \end{aligned} \tag{4}$$

That is, if $\gamma \in \Delta(\mathcal{F}_n)$ then $\mathcal{T}(\gamma) \in \Delta(\mathcal{F}_n)$. As a consequence the holonomy map is $n$-periodic. That is,

$$\left(\varphi_{\mathcal{F}_n}\right)^n = \mathrm{id}_D \,.$$

To see this consider the disk maps

$$\psi_0,\ \psi_1,\ \ldots,\ \psi_{n-1}$$

where for each $k \in \mathbb{Z}_n$, $\psi_k$ is defined to take $\xi \in D$ to $\xi' \in D$ if the loop $\gamma \in \Delta(\mathcal{F}_n)$ that contains $(k, \xi)$ also passes through $(k+1, \xi')$. In particular $\psi_0 = \varphi_{\mathcal{F}_n}$. Now fix any point $\xi \in D$. Consider the unique loop $\gamma \in \Delta(\mathcal{F}_n)$ that contains $(0, \xi) \in Z_n$. Following $\gamma$ around the solid torus $Z_n$ it must intersect the disk slice $\{1\} \times D$ at the point $(1, \psi_0(\xi))$, the slice $\{2\} \times D$ at $(2, \psi_1 \circ \psi_0(\xi))$, and so on. Continuing, going around $Z_n$ once, $\gamma$ will intersect $\{0\} \times D$ again at the point $(0, \psi_{n-1} \circ \ldots \circ \psi_0(\xi))$. On the other hand $\gamma$ closes up after going once around $Z_n$ and so this point is also $(0, \xi)$. Thus,

$$\psi_{n-1} \circ \ldots \circ \psi_0 = \mathrm{id}_D \,.$$

However it follows from the deck transformation invariance of $\Delta(\mathcal{F}_n)$ that each of these maps is the same, and so

$$\psi_k = \varphi_{\mathcal{F}_n}$$

for all $0 \leq k \leq n-1$. Therefore $\left(\varphi_{\mathcal{F}_n}\right)^n = \mathrm{id}_D$.

**Step 3:** We claim that there exists a subsequence of the holonomy maps $\varphi_{\mathcal{F}_{n_k}} : D \to D$ that converges to the given pseudo-rotation $\varphi$ as $k \to \infty$.

To explain this heuristically, start with any mapping torus $(Z_n, R_n)$ and consider the following way in which a trajectory $\gamma : \mathbb{R} \to Z_n$, that is a solution to $\dot{\gamma}(t) = R_n(\gamma(t))$, trivially gives rise to an immersed surface $F_\gamma$ in $W_n = \mathbb{R} \times Z_n$ by taking the product space

$$F_\gamma := \mathbb{R} \times \gamma(\mathbb{R}) \subset W_n. \tag{5}$$

Since $J_n$ couples the $\mathbb{R}$-direction with the $R_n$-direction this surface is an (immersed but not in general embedded) $J_n$-holomorphic curve parameterized by $\tilde{u} : \mathbb{C} \to W_n$, $\tilde{u}(s,t) = (s, \gamma(t))$. Moreover, for two trajectories $\gamma_1$ and $\gamma_2$ the surfaces $F_{\gamma_1}$ and $F_{\gamma_2}$ are either equal or disjoint, and indeed the union

$$\mathcal{F}^{\mathrm{vert}} := \left\{ F_\gamma \,\middle|\, \gamma \right\}$$

over all trajectories is a foliation of $\mathbb{R} \times Z_n$ by $J_n$-holomorphic curves. We will call this the (unique) vertical foliation of $(W_n, J_n)$, because each leaf is invariant under translations in the $\mathbb{R}$-direction in $W_n = \mathbb{R} \times Z_n$ and it is standard to draw the $\mathbb{R}$-axis in this context pointing vertically. The holonomy map associated to this foliation is clearly just the time-one map of the flow of $R_n$ which is the irrational pseudo-rotation $\varphi$. So we may formally write

$$\varphi_{\mathcal{F}^{\mathrm{vert}}} = \varphi.$$

Amongst all $J_n$-holomorphic curves in $(W_n, J_n)$ the ones which take the form of a product with a trajectory, as in (5), are also distinguished by the vanishing of a certain integral over the surface. More precisely, let $\omega$ be the exact differential 2-form $dx \wedge dy + d\tau \wedge dH$ on $Z_n = \mathbb{R}/n\mathbb{Z} \times D$, where $(x, y)$ are standard coordinates on $D$, and $\tau$ is the $\mathbb{R}/n\mathbb{Z}$-coordinate, and $H : Z_n \to \mathbb{R}$ is the Hamiltonian. Then for any vector $v$ in the tangent space to $W_n$ one can check that $\omega(v, Jv) \geq 0$ with equality if and only if $v \in \mathrm{span}\{\partial_{\mathbb{R}}, R_n\}$. Thus if $S \subset W_n$ is an immersed surface with $J_n$-invariant tangent bundle then the integral

$$\int_S \omega \geq 0$$

is non-negative and vanishes if and only if $S$ is tangent to the 2-plane distribution with fibers $\mathrm{span}\{\partial_{\mathbb{R}}, R_n\}$. That is, if and only if $S$ is contained in a "vertical" surface of the form $\mathbb{R} \times \gamma(\mathbb{R})$ for an $R_n$-trajectory $\gamma$, as in (5).

We can therefore view the value of the $\omega$-integral (also called the $\omega$-energy) of each leaf in the foliation $\mathcal{F}_n$ as a measure of how close to the vertical foliation it is, or alternatively how close the loop $\gamma_F$ is to an $R_n$-trajectory. Using Stokes' theorem the $\omega$-integral for the leaves can infact be calculated. For the cylinder leaf the integral vanishes, while for each half cylinder leaf $F \in \mathcal{F}_n$ one gets the more interesting quantity

$$\int_F \omega = \pi\{n\alpha\}$$

where $\alpha$ is the rotation number of the pseudo-rotation on the boundary of the disk, and $\{x\} \in [0, 1)$ denotes the fractional part of $x \in \mathbb{R}$.

Since $\alpha$ is irrational there exists a subsequence $n_k \in \mathbb{N}$ for which $\{n_k\alpha\} \to 0$. Thus for the corresponding subsequence of foliations $(\mathcal{F}_{n_k})$ the above integrals over the leaves decays uniformly to zero as $k \to \infty$. Compactness properties of pseudoholomorphic curves imply that these foliations converge in some sense to the vertical foliation $\mathcal{F}^{\text{vert}}$, because the latter is the unique foliation for which the $\omega$-integral vanishes on all leaves. From this one can conclude that the subsequence of holonomy maps $(\varphi_{\mathcal{F}_{n_k}})$ converges to the holonomy map of $\mathcal{F}^{\text{vert}}$. That is,

$$\lim_{k\to\infty} \varphi_{\mathcal{F}_{n_k}} = \varphi_{\mathcal{F}^{\text{vert}}}.$$

However, as we observed above, the right hand side is simply the pseudo-rotation $\varphi$.

1.3. **Results in the literature of related interest.** Using different techniques, Le Calvez proved in 2004, see Theorem 1.9 in [32], that every minimal $C^1$-diffeomorphism of the 2-torus that is homotopic to the identity can be $C^0$-approximated by periodic $C^1$-diffeomorphisms. Recall that a diffeomorphism is minimal if every point is the initial condition for a dense orbit. Thus, in this result also strongly non-integrable maps are approximated by, in some sense, integrable ones.

An interesting result about irrational pseudo-rotations in the class of homeomorphisms of the open and closed annulus homotopic to the identity, was obtained by Béguin-Crovisier-LeRoux-Patou [2] and Béguin-Crovisier-LeRoux [3]. Stated for maps on the closed disk this is as follows. Let $\varphi$ be an orientation preserving, measure preserving, homeomorphism of the disk with a single periodic point and boundary rotation number $\alpha \in \mathbb{R}/\mathbb{Z}$. Then the rigid rotation $R_{2\pi\alpha}$ is the $C^0$-limit of maps (not necessarily area preserving) conjugate to $\varphi$. The authors of these papers note that one does not know from their approach that $\varphi$ is in the closure of the set of maps conjugate to $R_{2\pi\alpha}$.

*Note added in Proof.* Patrice Le Calvez recently announced in [34] a proof of a version of theorem 1 using generating functions. The result he obtains is for $C^1$-pseudo-rotations which satisfy a condition on the boundary.

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## 2. Irrational pseudo-rotations

**Definition 3.** A (smooth) *irrational, pseudo-rotation* is a $C^\infty$-diffeomorphism $\varphi : D \to D$ of the closed 2-disk $D$ with the following properties: (1) $\varphi$ preserves the volume form $dx \wedge dy$. (2) $\varphi(0) = 0$. (3) $\varphi$ has no periodic points in $D\backslash\{0\}$.

There are equivalent definitions which admit generalizations to rational pseudo-rotations which we will not need. See for example [33] and [2].

If $\varphi : D \to D$ is an irrational pseudo-rotation, then the restriction of $\varphi$ to the boundary is an orientation preserving circle diffeomorphism without periodic points. It therefore has irrational rotation number on the boundary.

More precisely, let $\pi : \mathbb{R} \to \partial D$ be the projection map $x \mapsto e^{2\pi i x}$. Then for any lift $f : \mathbb{R} \to \mathbb{R}$ of $\varphi|_{\partial D}$ via $\pi$, the limit

$$\tau(f) := \lim_{n\to\infty} \frac{f^n(x) - x}{n} \in \mathbb{R} \tag{6}$$

exists and is independent of $x$, see for example [30], and is called the translation number of $f$. Furthermore, the element $[\tau(f)] \in \mathbb{R}/\mathbb{Z}$ in the quotient space, is even independent of the choice of lift $f$, and is called the rotation number of $\varphi|_{\partial D}$.

**Definition 4.** Let $\varphi : D \to D$ be an irrational pseudo-rotation. Then we define the *rotation number* of $\varphi$ to be the value on the circle

$$\mathrm{Rot}(\varphi) := [\tau(f)] \in \mathbb{R}/\mathbb{Z}$$

for any lift $f : \mathbb{R} \to \mathbb{R}$ of the restriction $\varphi : \partial D \to \partial D$.

A preferred homotopy $\{\varphi_t\}_{t\in[0,1]}$ from $\varphi_0 = \mathrm{id}_{\partial D}$ to $\varphi_1 = \varphi|_{\partial D}$ gives us a preferred lift of $\varphi|_{\partial D}$. Namely the terminal map of the unique lift to a homotopy in the universal covering space which begins at $\mathrm{id}_{\mathbb{R}}$. In particular, any Hamiltonian generating $\varphi$ as its time-one map, restricts to a homotopy on the boundary of the disk from $\mathrm{id}_{\partial D}$ to $\varphi|_{\partial D}$ and thus determines a canonical lift of the latter. Using this we define:

**Definition 5.** Let $\varphi : D \to D$ be an irrational pseudo-rotation. Let $H_t \in C^\infty(D, \mathbb{R})$ be a path of Hamiltonians on $(D, \omega_0 = dx \wedge dy)$ generating $\varphi$ as its time-one map. Then we define the *rotation number of $\varphi$ with respect to $H$* to be the real number

$$\mathrm{Rot}(\varphi; H) := \tau(f) \in \mathbb{R}$$

where $f : \mathbb{R} \to \mathbb{R}$ is the canonical lift of $\varphi : \partial D \to \partial D$ determined by $H$.

If $\varphi$ is an irrational pseudo-rotation, then the unique periodic point is non-degenerate in the sense that for all $n \in \mathbb{N}$, the linearization $D\varphi^{(n)}(0)$ does not have eigenvalue 1. The proof is a well known application of the Poincaré-Birkhoff fixed point theorem. See appendix A.2.

## 3. Finite energy foliations

For any $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R})$ with $H_t := H(t, \cdot)$ constant on the boundary of $D$ for each $t \in \mathbb{R}/\mathbb{Z}$, the smooth time-dependent vector field $X_H(t, \cdot) := X_{H_t}$ on $D$ defined by

$$\omega_0(X_{H_t}(z), \cdot) = -dH_t(z)$$

for all $z = (x, y) \in D$ is tangent to $\partial D$ and therefore generates a 1-parameter family of diffeomorphisms $\phi^t : D \to D$ over $t \in \mathbb{R}$. Using that the disk is simply connected it is well known that one may find an $H$ for any element $\varphi \in \mathrm{Diff}^\infty(D, \omega_0)$ so that $\varphi = \phi^1$. Then $H$ is said to generate $\varphi$.

From now on let $\varphi : D \to D$ be a fixed irrational pseudo-rotation. Unless stated otherwise $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R})$ is a 1-periodic time-dependent Hamiltonian generating $\varphi$.

*Remark* 1. By precomposing $H$ with a suitable closed loop in $\mathrm{Diff}^\infty(D,\omega_0)$ based at the identity, we may assume that the unique 1-periodic orbit of $X_{H_t}$ corresponding to the fixed point $0 \in D$ of $\varphi$ is the constant trajectory $t \mapsto 0 \in D$ for all $t \in \mathbb{R}$. This is not necessary, but makes the proof of theorem 8 slightly easier.

Define a smooth vector field $R_H$ on the solid torus $Z := \mathbb{R}/\mathbb{Z} \times D$ by

$$R_H(\tau, z) := \partial_\tau + X_H(\tau, z) \tag{7}$$

for all $\tau \in \mathbb{R}/\mathbb{Z}, z \in D$. The first return map on $\{0\} \times D$ is canonically identified with the pseudo-rotation $\varphi$.

For each $n \in \mathbb{N}$ let $Z_n$ be the 3-manifold-with-boundary $\mathbb{R}/n\mathbb{Z} \times D$, and $R_n$ the vector field on $Z_n$ that projects down to $R_H$ under the natural projection $\pi_n : Z_n \to Z$. Clearly the first return map of the flow generated by $R_n$ is the $n$-th iterate $\varphi^n : D \to D$. We will refer to the pair $(Z_n, R_n)$ as *the mapping torus of length-n associated to* $H$. It will also be useful to denote by $Z_\infty := \mathbb{R} \times D$ the universal covering of each $Z_n$.

All the dynamical information on $(Z_n, R_n)$ can be captured by an almost complex structure on the 4-manifold $\mathbb{R} \times Z_n$ as follows. For each $n \in \mathbb{N} \cup \{\infty\}$ define $J_n$ on $\mathbb{R} \times Z_n$ by

$$\begin{cases} J_n(a,\tau,z)\partial_{\mathbb{R}} = R_n \\ J_n(a,\tau,z)|_{T_z D} = i \end{cases} \tag{8}$$

for all $(a, \tau, z) \in \mathbb{R} \times Z_n$. Here, $\partial_{\mathbb{R}}$ is the vector field dual to the $\mathbb{R}$-coordinate on $\mathbb{R} \times Z_n$, and $i$ denotes the constant almost complex structure on the disk coming from the standard integrable complex structure on $\mathbb{C}$ [*]. In other words $i\partial_x = \partial_y$ and $i\partial_y = -\partial_x$. Observe that $J_n$ is independent of the $\mathbb{R}$-coordinate on $\mathbb{R} \times Z_n$, referred to as $\mathbb{R}$-*invariance*. This idea of coupling a suitable conservative vector field in an odd-dimensional manifold with the $\mathbb{R}$-direction in the product 4-manifold by an almost complex structure is due to Hofer [18].

There is a 1-parameter family of 2-tori

$$L_c := \{c\} \times \partial Z_n$$

for $c \in \mathbb{R}$, that fill the boundary $\mathbb{R} \times \partial Z_n$ of the 4-manifold. Each $L_c$ is totally real with respect to the almost complex structure $J_n$, that is

$$TL_c \oplus J_n T(L_c)$$

is the full 4-dimensional tangent space at each point of $L_c$. These will form the boundary conditions for our pseudoholomorphic curves with boundary.

Let us describe the $J_n$-holomorphic half infinite cylinders with totally real boundary conditions that we are interested in. Let $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^- = (-\infty, 0]$. For $n \in \mathbb{N}$ we consider maps $\tilde{u} = (a, \tau, z) \in C^\infty(\mathbb{R}^\pm \times \mathbb{R}/n\mathbb{Z}, \mathbb{R} \times Z_n = \mathbb{R} \times \mathbb{R}/n\mathbb{Z} \times D)$ for which there exists $c \in \mathbb{R}$ such that

$$\begin{cases} \partial_s \tilde{u}(s,t) + J_n(\tilde{u}(s,t))\partial_t \tilde{u}(s,t) = 0 & \text{for all } (s,t) \in \mathbb{R}^\pm \times \mathbb{R}/n\mathbb{Z} \\ \tilde{u}(0,t) \in L_c & \text{for all } t \in \mathbb{R}/n\mathbb{Z} \\ \tau(0,\cdot) : \mathbb{R}/n\mathbb{Z} \to \mathbb{R}/n\mathbb{Z} & \text{has degree } +1, \end{cases} \tag{9}$$

having so called finite total energy, which we define in a moment.

[*]For convenience we use this "non-generic" almost complex structure, although it is not necessary. The pseudoholomorphic curves we encounter are either orbit cylinders or embedded with genus zero, one boundary component, and Fredholm index 2. Such curves are automatically regular.

This setting is a special case of that described in [4]. In particular $(\mathbb{R}\times Z_n, J_n)$ is a cylindrical symmetric almost complex manifold, and the almost complex structure $J_n$ is compatible with the stable Hamiltonian structure $(\omega_n, \lambda_n)$ on $Z_n$ given by

$$\begin{cases} \omega_n = dx \wedge dy + d\tau \wedge dH \\ \lambda_n = d\tau \end{cases} \tag{10}$$

in coordinates $(\tau, (x, y))$ on $\mathbb{R}/n\mathbb{Z}\times D$. Recall that this means that $\lambda_n \wedge \omega_n > 0$ and $\ker(\omega_n) \subset \ker(d\lambda_n)$, see for example [4] or [8]. The compactness theory in [4] leads us to consider the following two quantities for a solution to (9) which we will refer to as the $\omega$-energy, the $\lambda$-energy, and the sum of them as the total energy. In our context the $\lambda$-energy of a solution $\tilde{u} = (a, \tau, z) \in \mathbb{R} \times \mathbb{R}/n\mathbb{Z} \times D$ to (9) is the quantity

$$E_\lambda(\tilde{u}) := \sup_{\psi\in\mathcal{C}} \int_{\mathbb{R}^+\times\mathbb{R}/n\mathbb{Z}} \tilde{u}^*\Big(\psi(a)da \wedge d\tau\Big) \in [0, +\infty] \tag{11}$$

where $\mathcal{C}$ is the set of smooth functions $\psi : \mathbb{R} \to [0, \infty)$ for which $\int_{\mathbb{R}} \psi(s)ds = 1$. The second energy, that which in the more general context of [4] is called the $\omega$-energy, is

$$E_\omega(\tilde{u}) := \int_{\mathbb{R}^+\times\mathbb{R}/n\mathbb{Z}} \tilde{u}^*\omega_n \in [0, +\infty]. \tag{12}$$

In section 6 we will prove the following.

**Lemma 6.** *Let $n \in \mathbb{N}$. Suppose $\tilde{u} \in C^\infty(\mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}, \mathbb{R} \times Z_n)$ is a solution to (9) with $E_\lambda(\tilde{u}) < \infty$. Then there exists $(s_0, t_0) \in \mathbb{R} \times \mathbb{R}/n\mathbb{Z}$ so that*

$$\tilde{u}(s + s_0, t + t_0) = (s, t, z(s, t)) \tag{13}$$

*for all $(s, t) \in [-s_0, \infty) \times \mathbb{R}/n\mathbb{Z}$, where $z \in C^\infty([-s_0, \infty) \times \mathbb{R}/n\mathbb{Z}, D)$ satisfies the Floer equation*

$$\partial_s z(s, t) + i\Big(\partial_t z(s, t) - X_H(t, z(s, t))\Big) = 0 \tag{14}$$

*for all $(s, t) \in [-s_0, \infty) \times \mathbb{R}/n\mathbb{Z}$.*

This is a converse to "Gromov's trick" [16]. It follows from this lemma that if a solution $\tilde{u} = (a, \tau, z)$ to (9) has finite $\lambda$-energy then the two energies have rather nice expressions:

$$E_\lambda(\tilde{u}) = n, \tag{15}$$

and

$$E_\omega(\tilde{u}) = \frac{1}{2}\int_{s=-s_0}^{+\infty}\int_{t=0}^{n} \big|\partial_s z(s, t)\big|^2 + \big|\partial_t z(s, t) - X_H(t, z(s, t))\big|^2 ds dt. \tag{16}$$

In particular the $\omega$-energy of $\tilde{u}$ becomes equal to the Floer energy of $z$. Let us verify (15) and (16) now. First,

$$\begin{aligned} E_\lambda(\tilde{u}) &= \sup_{\psi\in\mathcal{C}} \int_{\mathbb{R}^+\times\mathbb{R}/n\mathbb{Z}} \psi(s - s_0)ds \wedge dt \\ &= \sup_{\psi\in\mathcal{C}} \int_{s=-s_0}^{\infty} \psi(s)ds \cdot \int_{\mathbb{R}/n\mathbb{Z}} dt \\ &\leq n, \end{aligned}$$

and clearly there exists $\psi \in \mathcal{C}$ such that $\int_{s=-s_0}^{\infty} \psi(s)ds = 1$, so $E_\lambda(\tilde{u}) = n$. Similarly (16) follows from lemma 6 as follows. Abbreviating $\omega_0 = dx \wedge dy$ for the area form on the disk,

$$E_\omega(\tilde{u}) := \int_{\mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}} \tilde{u}^*\omega = \int_{[-s_0,\infty)\times\mathbb{R}/n\mathbb{Z}} z^*\omega_0 - z^*dH_t \wedge dt.$$

The integrand simplifies to

$$\begin{aligned}
&= \Big(\omega_0(z_s, z_t) - dH_t(z)[z_s]\Big)ds \wedge dt \\
&= \Big(\omega_0(z_s, iz_s) + \omega_0(z_s, X_H(t,z)) - dH_t(z)[z_s]\Big)ds \wedge dt \\
&= \Big(|z_s|^2 + dH_t(z)[z_s] - dH_t(z)[z_s]\Big)ds \wedge dt \\
&= |z_s|^2 ds \wedge dt.
\end{aligned}$$

The expression (16) then follows immediately using (13) again.

**Definition 7.** For a solution $\tilde{u} = (a, \tau, z) \in C^\infty(\mathbb{R}^\pm \times \mathbb{R}/n\mathbb{Z}, \mathbb{R} \times Z_n)$ to (9) we refer to the degree of the circle map

$$z(0, \cdot) : \mathbb{R}/n\mathbb{Z} \to \partial D$$

as the *boundary index* of $\tilde{u}$.

For $x \in \mathbb{R}$ we write

$$\begin{aligned}
\lceil x \rceil &:= \min\big\{k \in \mathbb{Z} \,\big|\, k > x\big\} \\
\lfloor x \rfloor &:= \max\big\{k \in \mathbb{Z} \,\big|\, k \leq x\big\}.
\end{aligned}$$

In section 7 we will prove the following existence result concerning solutions to (9). Each $\mathcal{F}_n^+$ is the $n$-th finite energy foliation referred to in section 1.2.

**Theorem 8.** *Let $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R})$ be a Hamiltonian generating an irrational pseudo-rotation $\varphi$. Let $(Z_1, R_1), (Z_2, R_2), \ldots$ be the corresponding sequence of mapping tori. For each $n \in \mathbb{N}$ let $\gamma_n : \mathbb{R}/n\mathbb{Z} \to Z_n$ be the unique $n$-periodic orbit of $R_n$, parameterized so that $\gamma_n(0) \in \{0\} \times D$. Assume $H$ was chosen so that $\gamma(t) = (t, 0)$ for all $t \in \mathbb{R}/n\mathbb{Z}$ (see remark 1). Let $\alpha := \mathrm{Rot}(\varphi; H) \in \mathbb{R}$, which is necessarily irrational.*

*Then for each $n \in \mathbb{N}$ there exist two foliations $\mathcal{F}_n^+, \mathcal{F}_n^-$ of $\mathbb{R} \times Z_n$ by smoothly embedded surfaces, with the following properties:*

- ***Cylinder leaf:*** *The cylinder $C_n := \mathbb{R} \times \gamma_n(\mathbb{R}/n\mathbb{Z}) \subset \mathbb{R} \times Z_n$ is a leaf in both $\mathcal{F}_n^+$ and $\mathcal{F}_n^-$.*
- ***Pseudo-holomorphic:*** *If $F \in \mathcal{F}_n^+$ (resp. $F \in \mathcal{F}_n^-$) is not $C_n$, then $F$ is parameterized by a solution $\tilde{u}$ to (9) with $J_n$ as in (8), with $E_\lambda(\tilde{u}) + E_\omega(\tilde{u}) < \infty$ and boundary index $\lfloor n\alpha \rfloor$ (resp. $\lceil n\alpha \rceil$).*
- ***$\mathbb{R}$-invariance:*** *If $F \in \mathcal{F}_n^+$ (resp. $F \in \mathcal{F}_n^-$) is a leaf and $c \in \mathbb{R}$, the set $F + c := \{(a + c, \tau, z) \,|\, (a, \tau, z) \in F\}$ is also a leaf in $\mathcal{F}_n^+$ (resp. in $\mathcal{F}_n^-$).*
- ***Uniqueness:*** *$\mathcal{F}_n^+$ and $\mathcal{F}_n^-$ are uniquely determined by the above properties.*
- ***Smooth foliation:*** *$\mathcal{F}_n^+$ and $\mathcal{F}_n^-$ are $C^\infty$-smooth 2-dimensional foliations at each point on the complement of $C_n$.*

This a special case of a much more general result to appear in [5].

*Remark* 2. For each leaf $F \in \mathcal{F}_n^+$ (resp. $F \in \mathcal{F}_n^-$) that is not the cylinder, any parameterization $\tilde{u}$ satisfying (9) has domain $\mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}$ (resp. $\mathbb{R}^- \times \mathbb{R}/n\mathbb{Z}$). Hence the superscripts in $\mathcal{F}_n^\pm$. In either case, as the unique $n$-periodic orbit $\gamma_n$ is non-degenerate the finite energy of $\tilde{u}$ implies that the $Z_n$ component $u_\pm(s,\cdot) : \mathbb{R}/n\mathbb{Z} \to Z_n$ converges to $\gamma_n$ uniformly in $C^\infty(\mathbb{R}/n\mathbb{Z}, Z_n)$ as $s \to +\infty$ (resp. $s \to -\infty$). This can be seen in two ways. Either as a consequence of the compactness results in [18] applied to $\tilde{u}$ (as generalized in [20, 4]); or the original work of Floer [13] applied to the disk component $z$ since, by lemma 6, $z$ is a finite energy solution to Floer's equation.

The following formula will be crucial to our application. Recall that $\alpha := \mathrm{Rot}(\varphi; H) \in \mathbb{R}$. In particular $\alpha$ is irrational.

**Lemma 9.** *Let $n \in \mathbb{N}$. For any half cylinder leaf $F \in \mathcal{F}_n^+$,*

$$E_\omega(F) = \{n\alpha\}\pi$$

*where $\{\cdot\}$ denotes the fractional part of a real number, that is for $x \in \mathbb{R}$ $\{x\} := x - \lfloor x \rfloor \in [0,1)$.*

This is proven in section 5.

## 4. Proof of theorem 1

We use the finite energy foliations of theorem 8 to define new disk maps.

**Definition 10.** For each $n \in \mathbb{N}$ define $\varphi_n : D \to D$ as follows. For $\xi \in D$ there is a unique leaf $F \in \mathcal{F}_n^+$ containing $(0,0,\xi) \in \mathbb{R} \times \mathbb{R}/n\mathbb{Z} \times D$. Define $\varphi_n(\xi) = \xi'$ where $\xi' \in D$ is unique such that $(0,1,\xi') \in F$. We will refer to $\varphi_n$ as the *holonomy map* of the foliation $\mathcal{F}_n^+$.

*Remark* 3. The holonomy maps are well defined. Indeed, for each $n \in \mathbb{N}$, by lemma 6 if a leaf $F \in \mathcal{F}_n^+$ intersects the hypersurface

$$\hat{Z}_n := \{0\} \times Z_n$$

then it does so transversally, and for each $\tau \in \mathbb{R}/n\mathbb{Z}$ it will intersect the disk slice $\{\tau\} \times D \subset \hat{Z}_n$ in a unique point.

*Remark* 4. We could as easily define maps in terms of the foliations $\mathcal{F}_n^-$.

*Remark* 5. The inverse map $\varphi_n^{-1}$ exists and can be defined similarly in terms of $\mathcal{F}_n^+$.

**Lemma 11.** *Each holonomy map $\varphi_n : D \to D$ is $n$-periodic, that is $(\varphi_n)^n = \mathrm{id}_D$.*

*Proof.* We could define $n$ many disk maps using $\mathcal{F}_n^+$; say $\varphi_{n,i}$ for $i = 0, 1, \ldots, n-1$, by requiring that $\varphi_{n,i}$ takes the point $\xi \in D$ to $\xi' \in D$ if $(0,i,\xi)$ and $(0,i+1,\xi')$ lie on the same leaf in $\mathcal{F}_n^+$. Since each leaf in the foliation closes up after going once around in the $\mathbb{R}/n\mathbb{Z}$ direction, it follows that the composition $\varphi_{n,n-1} \circ, \ldots, \circ \varphi_{n,0}$ is the identity map.

We now exploit a symmetry in $\mathcal{F}_n^+$ to see that each of the maps $\varphi_{n,i}$ is equal to $\varphi_n$. Consider the $\mathbb{Z}_n$ action on $\mathbb{R} \times Z_n$ generated by the deck transformation

$$\mathcal{T} : \mathbb{R} \times Z_n \to \mathbb{R} \times Z_n$$
$$\mathcal{T}(a,\tau,z) = (a, \tau - 1, z)$$

which preserves the almost complex structure; $\mathcal{T}^* J_n = J_n$. From the uniqueness part of theorem 8 we conclude that the foliation $\mathcal{T}(\mathcal{F}_n^+) := \{\mathcal{T}(F) \,|\, F \in \mathcal{F}_n^+\}$ is equal to $\mathcal{F}_n^+$. Hence $\varphi_{n,i} = \varphi_n$ for each $i$. ∎

Note that $\varphi_n(0) = 0$ as the cylinder $C_n$ passes through the center of the disk slices $\{0\} \times D$ and $\{1\} \times D$ in $\hat{Z}_n = \{0\} \times Z_n$.

**Lemma 12.** *Each holonomy map $\varphi_n : D \to D$ is $C^\infty$-smooth on $D\backslash\{0\}$.*

*Proof.* This is basically a consequence of the fact that by theorem 8 $\mathcal{F}_n^+$ is a $C^\infty$-smooth foliation on the complement of the leaf $C_n$, not only in the local sense that a neighborhood of each point admits a foliation chart in which the leaves are "flat", but in the slightly more global sense that a neighborhood of each *leaf* admits a foliation chart.

Fix a point $\xi_0 \in D\backslash\{0\}$. Then the unique leaf $F \in \mathcal{F}_n^+$ that contains the point $p_0 = (0, 0, \xi_0) \in \mathbb{R} \times Z_n$ cannot be the cylinder $C_n$. Thus by proposition 38 an open neighborhood of $F$ is foliated by leaves of $\mathcal{F}_n^+$. More precisely, see [25] theorems 1.5 and 5.7, and their generalizations to surfaces with boundary [45], there exists a $C^\infty$-smooth map

$$\Phi : B^2_\varepsilon \times \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R} \times Z_n$$

where $\varepsilon > 0$ and $B^2_\varepsilon$ denotes the open $\varepsilon$-ball in $\mathbb{R}^2$ about the origin, having the following properties. (1) $\Phi$ is an injective immersion. (2) For each $c \in B^2_\varepsilon$ the map $\Phi(c, \cdot)$ parameterizes a leaf $F_c$ in $\mathcal{F}_n^+$ where $F_0 = F$. (3) $\Phi$ takes the following form:

$$\Phi(c, s, t) = (s - s(c), t, z(c, s, t)) \in \mathbb{R} \times \mathbb{R}/n\mathbb{Z} \times D.$$

where $s(c)$ depends smoothly on $c$, and $z(c, s, t)$ depends smoothly on $c, s, t$. Consider the two smooth immersions

$$\begin{aligned} i_0 : B^2_\varepsilon \to D \qquad & c \mapsto z(c, s(c), 0) \\ i_1 : B^2_\varepsilon \to D \qquad & c \mapsto z(c, s(c), 1) \end{aligned}$$

Recall that the holonomy map $\varphi_n : D \to D$ of $\mathcal{F}_n^+$ is defined by the property that $\varphi_n(\xi) = \xi'$ if and only if $(0, 0, \xi)$ and $(0, 1, \xi')$ lie on the same leaf in $\mathcal{F}_n^+$. Observe that for each $c \in B^2_\varepsilon$,

$$(0, 0, i_0(c)) = \Phi(c, s(c), 0) \in F_c$$

and

$$(0, 1, i_1(c)) = \Phi(c, s(c), 1) \in F_c.$$

Thus $(0, 0, i_0(c))$ and $(0, 1, i_1(c))$ lie on the same leaf $F_c \in \mathcal{F}_n^+$ and so $\varphi_n(i_0(c)) = i_1(c)$. Thus, on some neighborhood of $i_0(0) = \xi_0$ we have $\varphi_n(c) = i_1 \circ (i_0)^{-1}(c)$ and in particular $\varphi_n$ is smooth near $\xi_0$. ∎

**Lemma 13.** *Each holonomy map $\varphi_n : D \to D$ is continous at the origin.*

*Proof.* Let $\xi_j \in D\backslash\{0\}$ be a sequence of points converging to $0 \in D$, and $p_j := (0, 0, \xi_j) \in \mathbb{R} \times Z_n$. By lemma 22 there exists a sequence of parameterizations $\tilde{u}_j$ of the unique leaf $F_j \in \mathcal{F}_n^+$ containing $p_j$ that converges in the $C^\infty_{\text{loc}}$-topology to $C_n$. By lemma 6 we may choose each $\tilde{u}_j$ to take the form $\tilde{u}_j : [S_j, \infty) \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R} \times Z_n$ for some $S_j \in \mathbb{R}$, with

$$\tilde{u}_j(s, t) = (s, t, z_j(s, t))$$

for some sequence $z_j$. It follows that the sequence of points

$$F_j \cap \Big(\{(0, 1)\} \times D\Big) = (0, 1, z_j(0, 1))$$

converges to $(0, 1, 0)$. Which means that $\varphi_n(\xi_j) = z_j(0, 1) \to 0$ as $j \to \infty$. ∎

Fix a subsequence $\{n_j\}_{j\in\mathbb{N}}$ for which the sequence of fractional parts

$$\lim_{j\to\infty}\{n_j\alpha\} = 0. \tag{17}$$

From lemma (9) this implies that the $\omega$-energies of the leaves in the foliations $\mathcal{F}^+_{n_j}$ tends to zero, as $j\to\infty$, uniformly over all leaves.

For maps $f, g : D \to D$, define $d_{C^0}(f,g)$ using the linear structure and Euclidean norm on $\mathbb{R}^2$ by

$$d_{C^0}(f,g) := \sup_{\xi\in D}|f(\xi) - g(\xi)|.$$

**Proposition 14.** *The subsequence $\varphi_{n_j}$ converges to the pseudo-rotation $\varphi$ in the following sense:*

$$d_{C^0}(\varphi_{n_j},\varphi) + d_{C^0}(\varphi^{-1}_{n_j},\varphi^{-1}) \to 0$$

*as $j\to\infty$.*

*Proof.* We show that $d_{C^0}(\varphi_{n_j},\varphi)\to 0$ as $j\to\infty$, as the same argument will work for the inverses.

Arguing indirectly, there exists a sequence of points $\xi_j\in D$ and $\delta>0$ such that $|\varphi_{n_j}(\xi_j) - \varphi(\xi_j)|\geq\delta$ for all $j\in\mathbb{N}$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^2$. Restricting to a subsequence we may assume that $\xi_j\to\xi$ for some $\xi\in D$, and $\delta\leq|\varphi_{n_j}(\xi_j)-\varphi(\xi_j)|\leq|\varphi_{n_j}(\xi_j)-\varphi(\xi)| + |\varphi(\xi_j)-\varphi(\xi)|$ for all $j\in\mathbb{N}$. Therefore as $\varphi$ is continous,

$$\frac{1}{2}\delta\leq|\varphi_{n_j}(\xi_j)-\varphi(\xi)| \tag{18}$$

for all $j$ sufficiently large.

For each $j\in\mathbb{N}$ let $F_j\in\mathcal{F}^+_{n_j}$ be the unique leaf containing the point $(0,0,\xi_j)\in\mathbb{R}\times\mathbb{R}/n_j\mathbb{Z}\times D$. Let us assume that each $F_j$ is a half cylinder, otherwise the argument is even easier. There exists a solution $\tilde{u}_j$ to (9) parameterizing $F_j$. After a holomorphic reparameterization we may assume that

$$\begin{aligned}\tilde{u}_j : [S_j,\infty)\times\mathbb{R}/n_j\mathbb{Z}&\to\mathbb{R}\times Z_{n_j}\\ \tilde{u}_j(0,0) &= (0,0,\xi_j)\end{aligned}$$

for some $S_j\leq 0$. For each $j$, $E_\lambda(\tilde{u}_j)<\infty$, so by lemma 6, $\tilde{u}_j$ takes the form

$$\tilde{u}_j(s,t) = (s,t,z_j(s,t)) \tag{19}$$

some $z_j : [S_j,\infty)\times\mathbb{R}/n_j\mathbb{Z}\to D$. Moreover, the sequence $\{\tilde{u}_j\}_{j\in\mathbb{N}}$ satisfies all the criterion for the compactness result theorem 25. In particular

$$\lim_{j\to\infty}E_\omega(\tilde{u}_j) = 0$$

due to our choice of subsequence satisfying (17). We conclude that the sequence $\tilde{u}_j$ converges in the following sense: for each $j$ let $\bar{u}_j : [S_j,\infty)\times\mathbb{R}\to\mathbb{R}\times Z_\infty$ be the unique lift of $\tilde{u}_j$ to the universal covering, satisfying

$$\bar{u}_j(0,0) = (0,0,\xi_j). \tag{20}$$

After restricting to a further subsequence we can assume that $\bar{u}_j\to\bar{u}_\infty$ in the $C^\infty_{\text{loc}}(\mathbb{R}^2,\mathbb{R}\times Z_\infty)$ topology, that is, uniformly on compact sets, where $\bar{u}_\infty$ takes the form

$$\bar{u}_\infty(s,t) = (s,t,\gamma(t))$$

for some $\gamma \in C^\infty(\mathbb{R}, D)$ solving $\dot\gamma(t) = X_H(t, \gamma(t))$ for all $t \in \mathbb{R}$. From (20) we have $\bar u_\infty(0,0) = (0,0,\xi)$. We conclude that for all $j$,

$$(0, 1, \varphi_{n_j}(\xi_j)) = \bar u_j(0,1).$$

The right hand side converges to

$$\begin{aligned}\bar u_\infty(0,1) &= (0,1,\gamma(1)) \\ &= (0,1,\varphi(\gamma(0))) \\ &= (0,1,\varphi(\xi)).\end{aligned}$$

This contradicts (18) and we are done. ∎

Combining the results of this section we have proven the following statement which is almost theorem 1.

**Theorem 15.** *Suppose $\varphi \in \mathrm{Diff}^\infty(D, \omega_0)$ fixes the origin and has no other periodic points. For each $j \in \mathbb{N}$ let $\varphi_j : D \to D$ be the holonomy map of the foliation $\mathcal{F}^+_{n_j}$. Then for each $j \in \mathbb{N}$, $\varphi_j(0) = 0$, $\varphi_j \in \mathrm{Homeo}_+(D) \cap \mathrm{Diff}^\infty(D\backslash\{0\})$, $\varphi_j^{n_j} = \mathrm{id}_D$. Moreover $d_{C^0}(\varphi_j, \varphi) + d_{C^0}(\varphi_j^{-1}, \varphi^{-1}) \to 0$ as $j \to \infty$.*

There are presumably nicer ways to go from this conclusion to the final statement. For example using changes of coordinates from the pseudoholomorphic curves themselves. This will presumably follow from a more serious analysis of the asymptotic properties of the curves.

*Proof of theorem 1.* It is a classical result [7, 10, 31] that if $f \in \mathrm{Homeo}^+(D)$ satisfies $f^n = \mathrm{id}_D$ for some $n \in \mathbb{N}$, then there exists $g \in \mathrm{Homeo}^+(D)$ and $q \in \{0, 1, \dots, n-1\}$ so that

$$g \circ f \circ g^{-1} = R_{2\pi q/n}.$$

If moreover $f(0) = 0$ then $g(0) = 0$. Indeed, if $q = 0$ then $f = \mathrm{id}_D$, while if $q \neq 0$ then $R_{2\pi q/n}$ has a unique fixed point at the origin and so $g(0) = g(f(0)) = R_{2\pi q/n}(g(0))$ implies $g(0) = 0$.

Applying this result to each holonomy map $\varphi_j$ we find $g_j \in \mathrm{Homeo}^+(D)$, fixing the origin, and $p_j \in \mathbb{Z}$ such that

$$\varphi_j = g_j^{-1} \circ R_{2\pi p_j/n_j} \circ g_j.$$

Now we replace $g_j$ by a $C^0$-close smooth approximation. More precisely, let $(\hat g_j)$ be a sequence in $\mathrm{Diff}^\infty(D)$, each map fixing the origin, with $d_{C^0}(\hat g_j, g_j) + d_{C^0}(\hat g_j^{-1}, g_j^{-1}) \to 0$ as $j \to \infty$. Then the maps $\hat\varphi_j := \hat g_j^{-1} \circ R_{2\pi p_j/n_j} \circ \hat g_j$ are $C^\infty$-diffeomorphisms which converge in the $C^0$-sense to the irrational pseudo-rotation $\varphi$. The maps $\hat\varphi_j$ satisfy the conclusions of theorem 1. ∎

## 5. Calculation of the $\omega$-energy

The aim of this section is to prove lemma 9, recalled as lemma 18.

It is convenient to fix a 1-form $\lambda_0$ on the disk so that $d\lambda_0 = \omega_0 = dx \wedge dy$. For each $n \in \mathbb{N}$ define the action functional $\mathbf{A}_n : C^\infty(\mathbb{R}/n\mathbb{Z}, Z_n) \to \mathbb{R}$ (associated to $\lambda_0$) by

$$\mathbf{A}_n(\sigma) := \int_{\mathbb{R}/n\mathbb{Z}} \sigma^*\lambda_0 - \int_0^n H(\sigma(t))dt.$$

We may rewrite this as

$$\mathbf{A}_n(\sigma) := \int_{\mathbb{R}/n\mathbb{Z}} \sigma^*\eta_n$$

where $\eta_n := \lambda_0 - H d\tau$ is a primitive of $\omega_n$ the 2-form used to define the $\omega$-energy. Note that $\eta_n$ restricts to a closed 1-form on $\partial Z_n$ since $R_n$ is tangent to $\partial Z_n$ and $d\eta_n(R_n, \cdot) = \omega_n(R_n, \cdot) = 0$. Hence $\mathbf{A}_n$ restricted to $C^\infty(\mathbb{R}/n\mathbb{Z}, \partial Z_n)$ descends to a map on homology.

It is convenient to introduce the closed loops

$$\begin{aligned} 1_{\mathbb{R}/n\mathbb{Z}} : \mathbb{R}/n\mathbb{Z} \to \partial Z_n \qquad & t \mapsto (t, 1) \\ 1_{\partial D} : \mathbb{R}/n\mathbb{Z} \to \partial Z_n \qquad & t \mapsto ([0], e^{2\pi i t/n}) \end{aligned}$$

which represent homology classes that generate $H_1(\partial Z_n)$.

**Lemma 16.** *For each $n \in \mathbb{N}$*

$$\mathbf{A}_n(1_{\mathbb{R}/n\mathbb{Z}}) + \lfloor n\alpha \rfloor \mathbf{A}_n(1_{\partial D}) \leq \mathbf{A}_n(\gamma_n) \leq \mathbf{A}_n(1_{\mathbb{R}/n\mathbb{Z}}) + \lceil n\alpha \rceil \mathbf{A}_n(1_{\partial D}).$$

*Proof.* Let $\tilde{u}_\pm = (a_\pm, u_\pm) : \mathbb{R}^\pm \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R} \times Z_n$ be parameterizations of leaves $F^\pm \in \mathcal{F}_n^\pm$ respectively which satisfy (9). In either case $u_\pm(s, \cdot)$ converges uniformly in $C^\infty(\mathbb{R}/n\mathbb{Z}, Z_n)$ to a parameterization $\gamma_n(\text{const}_\pm + \cdot)$ as $s \to \pm\infty$ respectively. Applying Stokes theorem,

$$E_\omega(\tilde{u}_+) = \int_{\mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}} u_+^* \omega_n = \mathbf{A}_n(\gamma_n) - \mathbf{A}_n(u_+(0, \cdot)),$$

and

$$E_\omega(\tilde{u}_-) = \int_{\mathbb{R}^- \times \mathbb{R}/n\mathbb{Z}} u_-^* \omega_n = \mathbf{A}_n(u_-(0, \cdot)) - \mathbf{A}_n(\gamma_n).$$

Therefore, as the energies are non-negative,

$$\mathbf{A}_n(u_+(0, \cdot)) \leq \mathbf{A}_n(\gamma_n) \leq \mathbf{A}_n(u_-(0, \cdot)).$$

We observed that the action $\mathbf{A}_n$ of a closed loop in $\partial Z_n$ depends only on its homology class. From theorem 8 $u_+(0, \cdot) : \mathbb{R}/n\mathbb{Z} \to \mathbb{R}/n\mathbb{Z} \times \partial D$ and $u_-(0, \cdot) : \mathbb{R}/n\mathbb{Z} \to \mathbb{R}/n\mathbb{Z} \times \partial D$ are homologous to $\mathbb{R}/n\mathbb{Z} \ni t \mapsto (t, e^{2\pi i(\lfloor n\alpha \rfloor/n)t})$ and $\mathbb{R}/n\mathbb{Z} \ni t \mapsto (t, e^{2\pi i(\lceil n\alpha \rceil/n)t})$ respectively. This gives us the desired inequalities. ∎

**Corollary 17.** *The unique 1-periodic orbit $\gamma_1 : \mathbb{R}/\mathbb{Z} \to Z_1$ has action*

$$\mathbf{A}_1(\gamma_1) = \mathbf{A}_1(1_{\mathbb{R}/\mathbb{Z}}) + \alpha \mathbf{A}_1(1_{\partial D}). \tag{21}$$

*Proof.* From the definition of $\mathbf{A}_n$,

$$\begin{aligned} \mathbf{A}_n(\gamma_n) &= n\mathbf{A}_1(\gamma_1) \\ \mathbf{A}_n(1_{\mathbb{R}/n\mathbb{Z}}) &= n\mathbf{A}_1(1_{\mathbb{R}/\mathbb{Z}}) \\ \mathbf{A}_n(1_{\partial D}) &= \mathbf{A}_1(1_{\partial D}) \end{aligned}$$

for all $n \in \mathbb{N}$. Substituting these into the inequalities in lemma (16), and dividing through by $n$ and letting $n \to +\infty$ gives (21). ∎

**Lemma 18.** *Let $n \in \mathbb{N}$. Every leaf $F \in \mathcal{F}_n^+$ with boundary has $\omega$-energy*

$$E_\omega(F) = \{n\alpha\}\pi$$

*where $\{\cdot\}$ applied to any real number denotes its fractional part.*

*Proof.* By Stokes theorem as in the last lemma,

$$E_\omega(F) = \mathbf{A}_n(\gamma_n) - \mathbf{A}_n(u(0,\cdot))$$

where $\tilde{u} = (a,u) : \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R} \times Z_n$ is a parameterization of $F$. Using corollary 17 and that $u(0,\cdot) : \mathbb{R}/n\mathbb{Z} \to \mathbb{R}/n\mathbb{Z} \times \partial D$ is homologous to $\mathbb{R}/n\mathbb{Z} \ni t \mapsto (t, e^{2\pi i(\lfloor n\alpha \rfloor/n)t})$, this becomes

$$\begin{aligned}
E_\omega(F) &= n\Big(\mathbf{A}_1(1_{\mathbb{R}/\mathbb{Z}}) + \alpha \mathbf{A}_1(1_{\partial D})\Big) - \\
&\qquad\qquad \Big(\mathbf{A}_n(1_{\mathbb{R}/n\mathbb{Z}}) + \lfloor n\alpha \rfloor \mathbf{A}_n(1_{\partial D})\Big) \\
&= \Big(n\mathbf{A}_1(1_{\mathbb{R}/\mathbb{Z}}) + n\alpha \mathbf{A}_1(1_{\partial D})\Big) - \\
&\qquad\qquad \Big(n\mathbf{A}_1(1_{\mathbb{R}/\mathbb{Z}}) + \lfloor n\alpha \rfloor \mathbf{A}_1(1_{\partial D})\Big) \\
&= (n\alpha - \lfloor n\alpha \rfloor)\mathbf{A}_1(1_{\partial D}) \\
&= \{n\alpha\} \cdot \int_D dx \wedge dy.
\end{aligned}$$

■

## 6. Compactness

Here we prove some compactness statements that were used in the proof of theorem 1. Our tool for doing so are straightforward versions of the rescaling arguments of Gromov [16] that were inspired by Sacks-Uhlenbeck [38]. As we are dealing with non-compact domains and targets we use the concept of energy originating with Hofer [18]. We encounter one small novelty here however, which is that the sequence of target manifolds are symplectizations of a sequence of Hamiltonian energy surfaces $Z_{n_j} = \mathbb{R}/n_j\mathbb{Z} \times D$ which also lose compactness as $j \to \infty$. As a result the pseudoholomorphic curves we consider have unbounded total energy which means that the standard compactness theory does not immediately apply. The purpose of this section then is largely to handle this difficulty.

Let us briefly recall how ellipticity of the Cauchy-Riemann equations reduces compactness questions for sequences of pseudoholomorphic maps to local uniform bounds on their gradients. The fundamental elliptic estimates for the linear operator $\bar{\partial} = \partial_s + i\partial_t$ on maps from a 2-dimensional domain into $\mathbb{R}^{2d}$, $d \in \mathbb{N}$, are as follows, see [9]. For each $p \in (1,\infty)$ and $k \in \mathbb{N}$ there exists a constant $c = c(k,p) \in (0,\infty)$ such that

$$\|\nabla\varphi\|_{W^{k,p}(D)} \leq c\|\bar{\partial}\varphi\|_{W^{k-1,p}(D)}$$

for all $\varphi \in C_c^\infty(D, \mathbb{R}^{2d})$, where the latter means that $\varphi$ has compact support in the interior of the closed unit 2-disk $D$. These estimates can be used to prove the following interior estimates for the non-linear Cauchy-Riemann equation. For a self contained exposition of this see [19] Theorems 2.1 and 2.3. For $r > 0$ we denote by $D_r \subset \mathbb{C}$ the closed disk of radius $r$.

**Theorem 19.** *Let $J$ be a smooth almost complex structure on $\mathbb{R}^{2d}$, $d \in \mathbb{N}$, and let $C_0, C_1 \in (0,\infty)$ be constants. Let $\Omega(J, C_0, C_1) \subset C^\infty(D_1, \mathbb{R}^{2d})$ be the set of maps $f$ which satisfy*

$$\begin{cases} \partial_s f(s,t) + J(f(s,t))\partial_t f(s,t) = 0 \text{ for all } (s,t) \in D \\ |f(0)| \leq C_0 \\ \|\nabla f\|_{C^0(D)} \leq C_1. \end{cases}$$

*Then there exists a sequence* $(c_k) \subset (0,\infty)$ *over* $k \in \mathbb{N}$*, such that for all* $f \in \Omega(J, C_0, C_1)$*,*

$$\|\nabla f\|_{C^k(D_{1/2})} \leq c_k$$

*for all* $k \in \mathbb{N}$*.*

*Remark* 6. Replacing $D_1, D_{1/2}$ by half disks $D_1^+, D_{1/2}^+$, where $D_r^+ := \{(x,y) \in \mathbb{R}^2 | x^2+y^2 \leq r,\ y \geq 0\}$, the same statement holds for maps $f$ that take the boundary points $[-1,1] \times \{0\}$ into a smooth path of $J$-totally real subspaces in $\mathbb{R}^{2d}$. A straightforward reflection argument reduces this to the statement above.

*Remark* 7. Norms, such as $\|\nabla \tilde{u}\|_{L^\infty}, \|\nabla \tilde{u}\|_{C^r}$ etc, will be implicitely with respect to the Riemannian metric $dx^2+dy^2+d\tau^2+da^2$ on $\mathbb{R} \times Z_n$, where $(x,y)$ are the standard Euclidean coordinates on the disk, $\tau$ is the "coordinate" on $\mathbb{R}/n\mathbb{Z}$ and $a$ is the $\mathbb{R}$-coordinate. This metric is $J_n$-invariant.

We divide the rest of this section into two parts. In 6.1 we consider the situation when $n_j = n$ is fixed and the total energy of the curves is uniformly bounded. In section 6.2 we consider situations when $n_j \to +\infty$ and the total energy of the curves is also unbounded.

6.1. **Compactness when $n$ is uniformly bounded.** The aim of this section is to prove lemma 6 stated earlier. We break this into the following two lemmas.

**Lemma 20** (The Floer equation from the Cauchy-Riemann equations)**.** *Let* $\tilde{u} = (a, \tau, z) : \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R} \times \mathbb{R}/n\mathbb{Z} \times D$ *be a solution to (9) for which* $\|\nabla a\|_{L^\infty} < \infty$ *(or* $\|\nabla \tau\|_{L^\infty} < \infty$*). Then there exist* $(a_0, \tau_0) \in \mathbb{R} \times \mathbb{R}/n\mathbb{Z}$ *such that*

$$\begin{cases} a(s,t) = s + a_0 \\ \tau(s,t) = t + \tau_0 \end{cases} \tag{22}$$

*for all* $(s,t) \in \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}$*, and moreover* $z : \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z} \to D$ *satisfies the following Floer equation:*

$$\partial_s z(s,t) + i\Big(\partial_t z(s,t) - X_H(t+\tau_0, z(s,t))\Big) = 0 \tag{23}$$

*for all* $(s,t) \in \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}$*.*

*Proof.* Writing out in coordinates what it means for $\tilde{u}$ to satisfy (9) gives us

$$(a_s - \tau_t)\partial_a + (a_t + \tau_s)\partial_\tau + \Big(a_t X_H(\tau, z) + z_s + i\big(z_t - \tau_t X_H(\tau, z)\big)\Big) = 0. \tag{24}$$

The boundary condition on $\tilde{u}$ in (9) implies $a_t(0,t) = 0$ for all $t \in \mathbb{R}/n\mathbb{Z}$. From (24),

$$\begin{aligned} a_s(s,t) &= \tau_t(s,t) \\ a_t(s,t) &= -\tau_s(s,t) \end{aligned} \tag{25}$$

for all $(s,t) \in \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}$. In particular both functions $a, \tau$ lift to harmonic functions on the upper half plane with gradient bounded in $L^\infty$. The boundary conditions on $a$ allow a smooth extension by reflection to the whole plane, still with gradient in $L^\infty$, and therefore by Liouville the partial derivatives of $a$ are constant. So there exists $b, c, a_0 \in \mathbb{R}$ so that $a(s,t) = cs + bt + a_0$ for all $(s,t) \in \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}$. Putting this into (25) there exists $\tau_0 \in \mathbb{R}$ so that $\tau(s,t) = ct - bs + \tau_0$ for all $(s,t) \in \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}$. The $n$-periodicity of $a$ in the $t$ variable implies $b = 0$. By assumption the degree of the map $\tau(0,\cdot) : \mathbb{R}/n\mathbb{Z} \to \mathbb{R}/n\mathbb{Z}$ is 1, and therefore $c = 1$. This proves (22).

From (24) we also have $a_t X_H(\tau,z) + z_s + i\big(z_t - \tau_t X_H(\tau,z)\big) = 0$. But we have shown that $\tau_t \equiv 1$ and $a_t \equiv 0$. Substituting these in we obtain

$$z_s + i\big(z_t - X_H(t+\tau_0, z)\big) = 0$$

as required. ■

The next statement says that we can use the above relation between the Cauchy-Riemann and Floer equations if (and only if) the $\lambda$-energy is finite.

**Lemma 21.** *Let $\tilde{u} = (a,\tau,z)$ be a solution to (9). Then $E_\lambda(\tilde{u}) < \infty$ implies $\|\nabla a\|_{L^\infty} < \infty$ (equivalently $\|\nabla\tau\|_{L^\infty} < \infty$).*

*Proof.* The equations (25) in the last lemma are valid here as they did not require the bounds on the gradient. Therefore, bounding $\nabla a$ is equivalent to bounding $\nabla\tau$, and the map $f : \mathbb{R}^+\times\mathbb{R}/n\mathbb{Z} \to \mathbb{R}\times\mathbb{R}/n\mathbb{Z}$ given by $f(s,t) := (a(s,t),\tau(s,t))$ in terms of the $a$ and $\tau$ components of $\tilde{u}$, is holomorphic with respect to the obvious complex structure on $\mathbb{R}\times\mathbb{R}/n\mathbb{Z}$. This is useful because the $\lambda$-energy of $\tilde{u}$ depends only on $f$, indeed we may write

$$E_\lambda(\tilde{u}) = \sup_{\psi\in\mathcal{C}} \int_{\mathbb{R}^+\times\mathbb{R}/n\mathbb{Z}} f^*\Big(\psi(a)da\wedge d\tau\Big).$$

We can therefore view this as an energy of the map $f$. Recall from (11) that $\mathcal{C}$ is the set of all $\psi \in C^\infty(\mathbb{R},[0,\infty))$ for which $\int_{\mathbb{R}}\psi(x)dx = 1$.

Arguing indirectly suppose that the gradient of $a$, equivalently the gradient of $f$, is unbounded. Then there exists a sequence $(\xi_j) \subset \mathbb{R}^+\times\mathbb{R}/n\mathbb{Z}$ for which $|\nabla f(\xi_j)| \to \infty$. Clearly the points $\xi_j$ leave every compact subset of the domain, in particular they do not converge to the boundary. Therefore a standard rescaling argument applied to $f$ yields a holomorphic plane $g : \mathbb{C} \to \mathbb{R}\times\mathbb{R}/n\mathbb{Z}$ for which

$$0 < \|\nabla g\|_{L^\infty(\mathbb{C})} \leq 2 \tag{26}$$

$$\sup_{\psi\in\mathcal{C}} \int_{\mathbb{C}} g^*\Big(\psi(a)da\wedge d\tau\Big) < \infty. \tag{27}$$

The first property implies that $g$ has constant, non-zero gradient through Liouville's theorem (after for example considering a lift $\tilde{g} : \mathbb{C}\to\mathbb{C}$). It therefore follows from a direct calculation that each integral $\int_{\mathbb{C}} g^*(\psi(a)da\wedge d\tau)$, with $\psi\in\mathcal{C}$, is non-finite, contradicting the second conclusion. Thus no such sequence $(\xi_j)$ can exist and the lemma is proven.

Let us briefly elaborate on the rescaling procedure that produced $g$. Applying a well known lemma due to Hofer (see for example [36] Lemma 4.6.4) we may slightly modify the sequence $(\xi_j)$ to a sequence $(\xi_j')$ which retains the gradient blow-up property $R_j := |\nabla f(\xi_j')| \to \infty$ and aquires the additional property that there exists a sequence of positive numbers $\varepsilon_j \to 0$ such that

$$\lim_{j\to\infty} \varepsilon_j R_j = \infty$$
$$\|\nabla f\|_{L^\infty(B_{\varepsilon_j}(\xi_j'))} \leq 2R_j.$$

Define a sequence of maps

$$g_j : B_{\varepsilon_j R_j}(0) \to \mathbb{R}\times\mathbb{R}/n\mathbb{Z}$$
$$g_j(\xi) := f(\xi_j' + \xi/R_j) - (a(\xi_j'),0).$$

These "rescaled maps" are also holomorphic, and moreover satisfy $|g_j(0)| \leq n+1$ and $\|g_j\|_{L^\infty} \leq 2$ uniformly in $j \in \mathbb{N}$. Hence the sequence $(g_j)$ has uniformly bounded $C^1$-norm on each compact subset of $\mathbb{C}$. From the ellipticity of the Cauchy-Riemann equations in the form of theorem 19, we deduce uniform bounds on $(g_j)$ in the $C^r(K)$ norm for each $r \in \mathbb{N}$ and $K \subset \mathbb{C}$ compact. By the Arzela-Ascoli theorem there exists a subsequence, which we denote also by $(g_j)$, which converges in a $C^\infty_{\text{loc}}$-sense to a holomorphic map $g : \mathbb{C} \to \mathbb{R}\times\mathbb{R}/n\mathbb{Z}$ also satisfying $\|\nabla g\|_{L^\infty(\mathbb{C})} \leq 2$. Since $|\nabla g(0)| = \lim_{j\to\infty} |\nabla g_j(0)| = 1$ we have that $g$ is non-constant. Another consequence of the Cauchy-Riemann equations is that for each $\psi \in \mathcal{C}$ the integrand $g_j^*(\psi(a)da \wedge d\tau) = |\nabla g_j|^2(\psi \circ g_j)ds \wedge dt$ which we see has non-negative density. Therefore,

$$\int_{\mathbb{C}} g^*(\psi(a)da \wedge d\tau) \leq \liminf_{j\to\infty} \int_{B_{\varepsilon_j R_j}} g_j^*(\psi(a)da \wedge d\tau).$$

Each term on the right hand side is bounded above by $E_\lambda(\tilde{u}) < \infty$. We have verified that the map $g$ satisfies (26) and (27). ∎

In our proof that the maps $\varphi_n$ are continuous we used the following compactness property of each foliation.

**Lemma 22.** *For $n \in \mathbb{N}$. Suppose that $(F_j) \subset \mathcal{F}_n^+$ is a sequence of leaves, and $p_j \in F_j$ is a sequence of points, over $j \in \mathbb{N}$. Suppose that $p_j \to p$ some $p \in \mathbb{R}\times Z_n$ and let $F \in \mathcal{F}_n^+$ be the unique leaf containing $p$. Then there exists a sequence of $J_n$-holomorphic parameterizations $\phi_j$ of $F_j$ which converge in a $C^\infty_{\text{loc}}$-sense to a $J_n$-holomorphic parameterization of $F$.*

*Proof.* This is a standard property of finite energy foliations from positivity of intersections, used many times in [26]. The uniform bounds on the $\lambda$-energy and $\omega$-energy ensure existence of a convergent subsequence. Positivity of intersections ensures that the limit is a leaf in the foliation (the whole leaf as all leaves are connected and have no nodal points). ∎

6.2. **Compactness as $n \to \infty$.** In our proof of convergence of the holonomy maps $\varphi_n$ in proposition 14, we used a compactness statement for a sequence of $J_n$-holomorphic maps $\tilde{u}_n : [S_n, \infty) \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R}\times Z_n$ for which

$$E_\lambda(\tilde{u}_n) = n \to +\infty$$
$$E_\omega(\tilde{u}_n) = \{n\alpha\}\pi$$

for some irrational real number $\alpha$. Hence the total energy $E(\tilde{u}_n) = E_\lambda(\tilde{u}_n) + E_\omega(\tilde{u}_n)$ diverges to $+\infty$. In general, for a sequence of maps $\{\tilde{u}_n\}$ for which the total energy is unbounded one cannot expect uniform bounds on the gradient in $L^\infty$. However if the $\lambda$-energy grows at most linearly with $n$, and the $\omega$-energy is bounded then indeed uniform bounds on $\|\nabla\tilde{u}_n\|_{L^\infty}$ can be achieved. (Actually much weaker assumptions suffice, but we will not need to explore these here.) Our arguments will be further simplified since we restricted to a subsequence for which the $\omega$-energy of the sequence decays to zero.

Consider a sequence $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ of smooth $J_n$-holomorphic maps, for $J_n$ as in (8), with numbers $c_n, S_n \in \mathbb{R}$, $S_n \leq 0$, satisfying for each $n$,

$$\begin{cases} \tilde{u}_n = (a_n, \tau_n, z_n) : [S_n, \infty) \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R}\times Z_n, \\ \partial_s\tilde{u}_n(s,t) + J_n(\tilde{u}_n(s,t))\partial_t\tilde{u}_n(s,t) = 0 \\ \tilde{u}_n(S_n, t) \in L_{c_n} \\ \tau_n : (S_n, \cdot) : \mathbb{R}/n\mathbb{Z} \to \mathbb{R}/n\mathbb{Z} \qquad \text{has degree 1} \end{cases} \tag{28}$$

for all $(s,t) \in [S_n, \infty) \times \mathbb{R}/n\mathbb{Z}$.

**Lemma 23.** *Suppose that $E_\lambda(\tilde{u}_n) < \infty$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} E_\omega(\tilde{u}_n) = 0$. Then there exists $C \in (0,\infty)$ such that*

$$\|\nabla \tilde{u}_n\|_{L^\infty([S_n,\infty)\times\mathbb{R}/n\mathbb{Z})} \leq C$$

*for all $n \in \mathbb{N}$.*

Note that we do not assume uniform bounds on the $\lambda$-energy.

*Proof.* Since $E_\lambda(\tilde{u}_n) < \infty$ for each $n$, lemma 21 implies $\|\nabla a_n\|_{L^\infty} < \infty$ (for each $n$). Therefore, since also each $\tau_n$ has degree 1, lemma 20 applies so

$$\begin{cases} a_n(s,t) = s + a_n \\ \tau_n(s,t) = t + \tau_n \end{cases}$$

for all $(s,t) \in [S_n,\infty) \times \mathbb{R}/n\mathbb{Z}$, some constants $(a_n, \tau_n) \in [S_n,\infty) \times \mathbb{R}/n\mathbb{Z}$. Thus

$$\|\nabla a_n\|_{L^\infty} \leq 1 \qquad \text{and} \qquad \|\nabla \tau_n\|_{L^\infty} \leq 1$$

for all $n \in \mathbb{N}$.

It therefore remains to show that the gradients of the map $z_n : [S_n,\infty) \times \mathbb{R}/n\mathbb{Z} \to D$ are uniformly bounded. Arguing indirectly we find a sequence $\xi_n \in [S_n,\infty) \times \mathbb{R}/n\mathbb{Z}$ for which $|\nabla \tilde{u}_n(\xi_n)| \geq |\nabla z_n(\xi_n)| \to \infty$ as $n \to \infty$. A standard rescaling argument produces a $J_\infty$-holomorphic plane or half plane in $\mathbb{R}\times Z_\infty$. For example, suppose the sequence $\xi_n$ is uniformly bounded away from the boundary, that is, $\xi_n = (s_n, t_n)$ where $\liminf s_n - S_n > 0$. Then one produces a pseudoholomorphic plane as follows. After possibly modifying the sequence $(\xi_n)$ to $(\xi_n')$ using Hofer's lemma (see proof of lemma 21), one obtains a sequence $(\varepsilon_n) \subset (0,+\infty)$ with $\varepsilon_n \to 0$ such that $R_n := |\nabla \tilde{u}_n(\xi_n')| \to \infty$ and

$$\begin{aligned} \lim_{n\to\infty} \varepsilon_n R_n =& \infty \\ \|\nabla \tilde{u}_n\|_{L^\infty(B_{\varepsilon_n}(0))} \leq& 2R_n. \end{aligned}$$

Define "rescaled maps"

$$\begin{aligned} \tilde{v}_n : B_{\varepsilon_n R_n}(0) &\to \mathbb{R}\times\mathbb{R}/n\mathbb{Z}\times D \\ \tilde{v}_n(\xi) :=& \tilde{u}_n(\xi_n' + \xi/R_n) - (a_n(\xi_n'), \lfloor \tau_n(\xi_n') \rfloor, 0) \end{aligned}$$

for each $n \in \mathbb{N}$. Each $\tilde{v}_n$ is $J_n$-holomorphic because the almost complex structure $J_n$ on $\mathbb{R}\times\mathbb{R}/n\mathbb{Z}\times D$ is invariant under $\mathbb{R}$-translations and under the $\mathbb{Z}_n$-action on the $\mathbb{R}/n\mathbb{Z}$-coordinate generated by the deck transformation $[\tau] \mapsto [\tau+1]$. Moreover each $\tilde{v}_n$ satisfies $\|\nabla \tilde{v}_n\|_{L^\infty} \leq 2$ and $|\tilde{v}_n(0)| \leq 1$, and therefore has $C^1$-bounds on each compact $K \subset \mathbb{C}$ uniform in $n$. Ellipticity in the form of theorem 19 allow these $C^1$-bounds to be bootstrapped to $C^r(K)$-bounds for each fixed $r \in \mathbb{N}$, and $K \subset \mathbb{C}$ compact, uniform in $n$. Therefore there exists a convergent subsequence $\tilde{v}_{n_j} \to \tilde{v}$ in the $C^\infty_{\text{loc}}(\mathbb{C})$-topology to a $J_\infty$-holomorphic map

$$\tilde{v} : \mathbb{C} \to \mathbb{R}\times Z_\infty.$$

Let us write $\tilde{v} = (a,\tau,z) \in \mathbb{R}\times\mathbb{R}\times D$. Since each $|\nabla \tilde{v}_{n_j}(0)| = 1$ we have $|\nabla \tilde{v}(0)| = 1$ and so $\tilde{v}$ is non-constant. Moreover, as a result of the rescaling process the uniform bounds $\|\nabla a_{n_j}\|_{L^\infty} \leq 1$ and $\|\nabla \tau_{n_j}\|_{L^\infty} \leq 1$ are "killed" in the limit and so $\nabla a \equiv 0 \equiv \nabla\tau$. Finally, each integrand $\tilde{v}_n^*\omega = |\partial_s z_n|^2 ds\wedge dt$ has non-negative density, and so for each $\Omega \subset \mathbb{C}$ bounded,

$$0 \leq \int_\Omega \tilde{v}^*\omega \leq \liminf_{n\to\infty} \int_\Omega \tilde{v}_n^*\omega \leq \lim_{n\to\infty} E_\omega(\tilde{u}_n) = 0.$$

Thus $E_\omega(\tilde{v}) = 0$.

If the sequence $(\xi_n)$ instead converged to the boundary, and sufficiently fast, then a similar rescaling procedure would produce a $J_\infty$-holomorphic map on the upper half plane

$$\tilde{v} : \mathbb{H} \to \mathbb{R} \times Z_\infty$$

with totally real boundary conditions $\tilde{v}(\partial\mathbb{H}) \subset \{c\} \times \partial Z_\infty$ for some $c \in \mathbb{R}$, and similar features to the plane above. In either case our limiting map $\tilde{v} = (a, \tau, z)$, with domain $\mathbb{H}$ or $\mathbb{C}$, has the following properties:

$$\begin{aligned} \nabla a &\equiv 0 \\ \nabla \tau &\equiv 0 \\ |\nabla \tilde{v}(0)| &> 0 \\ E_\omega(\tilde{v}) &= 0. \end{aligned}$$

Thus there exist constants $a_0, \tau_0 \in \mathbb{R}$ such that

$$\tilde{v}(s,t) = (a_0, \tau_0, z(s,t)) \in \mathbb{R} \times \mathbb{R} \times D$$

for all $(s,t) \in \mathbb{C}$ (resp. all $(s,t) \in \mathbb{H}$). That $\tilde{v}$ is $J_\infty$-holomorphic translates into $z : \mathbb{C} \to D$ or $z : \mathbb{H} \to D$ satisfying the equation $a_t X_H(\tau, z) + z_s + i\big(z_t - \tau_t X_H(\tau, z)\big) = 0$, see (24). So $a$ and $\tau$ constant implies $z_s + i z_t = 0$. (We could alternatively have just rescaled the sequence of maps $\{z_n\}$ as in Floer theory, to get the same conclusion.) Therefore,

$$0 = E_\omega(\tilde{v}) = \int \tilde{v}^*\omega = \int \frac{1}{2}\left(|z_s|^2 + |z_t|^2\right) ds dt$$

and so $z$ is also constant. Thus we have shown that $\tilde{v}$ is constant, contradicting $|\nabla \tilde{v}(0)| > 0$. ∎

**Corollary 24.** *Suppose that $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ is a sequence of solutions to (28) such that $E_\lambda(\tilde{u}_n) < \infty$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} E_\omega(\tilde{u}_n) = 0$. Then there exists a sequence $c_k \in (0, \infty)$ over $k \in \mathbb{N}$, such that*

$$\|\nabla \tilde{u}_n\|_{C^k([S_n,\infty)\times\mathbb{R}/n\mathbb{Z})} < c_k \tag{29}$$

*for all $k \in \mathbb{N}$ and $n \in \mathbb{N}$.*

*Proof.* From lemma 23 the first bound $c_1$ exists. The remaining bounds then follow from theorem 19. Because the latter is a local result it is important to use that the almost complex structures $J_n$ satisfy: (1) they are invariant under the $\mathbb{R}$ and $\mathbb{Z}_n$ actions on $\mathbb{R} \times \mathbb{R}/n\mathbb{Z} \times D$, and (2) they each lift to the same almost complex structure $J_\infty$ on the universal covering $\mathbb{R} \times \mathbb{R} \times D$. ∎

We can now prove the goal of section 6.2.

**Theorem 25.** *Let $\bar{u}_n : [S_n, \infty) \times \mathbb{R} \to \mathbb{R} \times Z_\infty$ be a sequence of $J_\infty$-holomorphic maps over $n \in \mathbb{N}$, where each $\bar{u}_n$ is a lift of a solution $\tilde{u}_n$ to (28). Suppose that $\bar{u}_n(0,0)$ is uniformly bounded in $n$ and that $E_\lambda(\tilde{u}_n) < \infty$ for each $n \in \mathbb{N}$, and that $\lim_{n\to\infty} E_\omega(\tilde{u}_n) = 0$. Then there exists a subsequence $\{\tilde{u}_{n_j}\}_{j\in\mathbb{N}}$ such that $\bar{u}_{n_j}$ converges in $C^\infty_{\text{loc}}(\mathbb{C}, \mathbb{R}\times Z_\infty)$ to a $J_\infty$-holomorphic map $\tilde{u}_\infty$ having domain either $\Sigma = \mathbb{C}$ or $\Sigma = [S, \infty) \times \mathbb{R} \subset \mathbb{C}$ for some $S \in (-\infty, 0]$. Moreover $\tilde{u}_\infty$ takes the following form: there exist constants $a_0, \tau_0 \in \mathbb{R}$ such that*

$$\begin{aligned} \tilde{u}_\infty &: \Sigma \to \mathbb{R} \times \mathbb{R} \times D \\ \tilde{u}_\infty(s,t) &= (s + a_0, t + \tau_0, \gamma(t)) \end{aligned} \tag{30}$$

*for all* $(s,t) \in \Sigma$, *where* $\gamma \in C^\infty(\mathbb{R}, D)$ *satisfies*

$$\dot{\gamma}(t) = X_{H_{t+\tau_0}}(\gamma(t)) \tag{31}$$

*for all* $t \in \mathbb{R}$.

*Proof.* Taking a subsequence we may assume that $\tilde{u}_{n_j}(0,0)$ converges, and that $S_{n_j}$ converges to some $S \in (-\infty, 0] \cup \{-\infty\}$. The assumptions on the two energies allow us to use corollary 24 and obtain uniform bounds on $\|\nabla \bar{u}_n\|_{C^k}$ for each $k \in \mathbb{N}$, and therefore also $C^0$-bounds on $\bar{u}_n$ on compact subsets, uniform in $n$. Repeated use of the Arzela-Ascoli theorem yields a subsequence converging uniformly with all derivatives on each compact subset of $\mathbb{C}$ to a smooth map $\tilde{u}_\infty : \Sigma \to \mathbb{R} \times Z_\infty$ where $\Sigma = [S, \infty) \times \mathbb{R}$ if $S$ is finite and $\Sigma = \mathbb{C}$ otherwise. From lemma 20 each map in the sequence $\bar{u}_{n_j}$ takes the form

$$\bar{u}_{n_j}(s,t) = (s + a_j, t + \tau_j, z_j(s,t))$$

for constants $a_j, \tau_j \in \mathbb{R}$, with $z_j : [S_{n_j}, \infty) \times \mathbb{R} \to D$ satisfying

$$\partial_s z_j(s,t) + i\Big(\partial_t z_j(s,t) - X_H(t + \tau_j, z_j(s,t))\Big) = 0$$

for all $(s,t) \in [S_{n_j}, \infty) \times \mathbb{R}$. Therefore $\tilde{u}_\infty$ takes the form

$$\tilde{u}_\infty(s,t) = (s + a_\infty, t + \tau_\infty, z_\infty(s,t))$$

for constants $a_\infty, \tau_\infty \in \mathbb{R}$ and some $z_\infty : \Sigma \to D$ satisfying

$$\partial_s z_\infty + i\Big(\partial_t z_\infty - X_H(t + \tau_\infty, z_\infty)\Big) = 0.$$

Let $\omega_\infty := dx \wedge dy + d\tau \wedge dH$ on $Z_\infty$. Then

$$0 \leq \int_{\mathbb{R}^2} \tilde{u}_\infty^* \omega_\infty \leq \lim_{j \to \infty} E_\omega(\tilde{u}_{n_j}) = 0.$$

Thus

$$\frac{1}{2} \iint \left| \frac{\partial z_\infty}{\partial s}(s,t) \right|^2 + \left| \frac{\partial z_\infty}{\partial t}(s,t) - X_H(t + \tau_\infty, z_\infty(s,t)) \right|^2 ds dt = \int_{\mathbb{R}^2} \tilde{u}_\infty^* \omega_\infty = 0.$$

Hence $z_\infty(s,t) = \gamma(t)$ for some solution $\gamma : \mathbb{R} \to D$ to (31). ∎

## 7. Construction of the finite energy foliations

In this final section we give a terse proof of theorem 8. The approach is along by now fairly standard lines, the only minor complication arises from the presence of the boundary of the almost complex manifold. A more general construction will appear in [5]. We will assume more familiarity with terminology from [4] than elsewhere in this article, and with the intersection theory in [41].

To recall the statement of theorem 8 let $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R})$ be a Hamiltonian generating an irrational pseudo-rotation $\varphi$. Let $(Z, R)$ be the corresponding Hamiltonian mapping torus. That is, $Z = \mathbb{R}/\mathbb{Z} \times D$ with coordinates $(\tau, z = (x,y))$, and $R$ is the vector field

$$R(\tau, z) = \partial_\tau + X_{H_\tau}(z).$$

Denote by $\gamma : \mathbb{R}/\mathbb{Z} \to Z$ the unique 1-periodic orbit of $R$ for which $\gamma(0) \in \{0\} \times D$, and assume $H$ was chosen so that $\gamma(t) = (t, 0)$ for all $t \in \mathbb{R}/\mathbb{Z}$ (see remark 1). Let

$$\alpha := \mathrm{Rot}(\varphi; H) \in \mathbb{R}$$

denote the *real valued*, irrational, rotation number of $\varphi$ with respect to the isotopy between $\varphi$ and $\mathrm{id}_D$ induced by $H$, see definition 5. Let $J$ be the the associated $\mathbb{R}$-invariant almost complex structure on the 4-manifold $W := \mathbb{R} \times Z$ characterized by the conditions

$$\begin{cases} J\partial_{\mathbb{R}} = R \\ J|_{TD} = i. \end{cases} \tag{32}$$

We will now prove theorem 8 for $n = 1$, the general case being identical*, which states the following.

**Theorem 26.** *With the above assumptions on $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R})$ there exist two foliations $\mathcal{F}^+, \mathcal{F}^-$ of $\mathbb{R} \times Z$ by $J$-holomorphic curves with the following properties:*

- ***Cylinder leaf:*** *The cylinder $C := \mathbb{R} \times \gamma(\mathbb{R}/\mathbb{Z}) \subset \mathbb{R} \times Z$ is a leaf in both $\mathcal{F}^+$ and $\mathcal{F}^-$.*
- ***Pseudo-holomorphic:*** *If $F \in \mathcal{F}^+$ (resp. $F \in \mathcal{F}^-$) is not $C$, then $F$ is parameterized by a solution $\tilde{u}$ to (9), with $E_\lambda(\tilde{u}) + E_\omega(\tilde{u}) < \infty$ and boundary index $\lfloor \alpha \rfloor$ (resp. $\lceil \alpha \rceil$).*
- ***$\mathbb{R}$-invariance:*** *If $F \in \mathcal{F}^+$ (resp. $F \in \mathcal{F}^-$) is a leaf and $c \in \mathbb{R}$, the set $F + c := \{(a + c, \tau, z) \,|\, (a, \tau, z) \in F\}$ is also a leaf in $\mathcal{F}^+$ (resp. in $\mathcal{F}^-$).*
- ***Uniqueness:*** *$\mathcal{F}^+$ and $\mathcal{F}^-$ are uniquely determined by the above properties.*
- ***Smooth foliation:*** *$\mathcal{F}^+$ and $\mathcal{F}^-$ are $C^\infty$-smooth foliations at each point on the complement of $C$.*

In section 7.1 we prove the statement for $\mathcal{F}^+$ assuming restrictions on the boundary. Section 7.2 removes the boundary restrictions. In section 7.3 we explain how the construction of $\mathcal{F}^-$ differs from $\mathcal{F}^+$. We defer to the end two technical points: section 7.4 fills in a detail regarding compactness. In section 7.5 we explain how to find the filling of the boundary used in section 7.1.

*Remark* 8. In our proof of theorem 26 we do not actually use that $\varphi$ is a pseudo-rotation. We only use the much weaker assumption that $\varphi$ has only a single fixed point, and that it is non-degenerate. We will exploit this in section 7.2.

7.1. **Construction of $\mathcal{F}^+$ when $\varphi|_{\partial D}$ is conjugate to a rigid rotation.** In this section suppose that there exists $g \in \mathrm{Diff}^\infty_+(\partial D)$ so that

$$g \circ \varphi|_{\partial D} \circ g^{-1} = R_{2\pi\alpha} \tag{33}$$

where $R_{2\pi\alpha} : \partial D \to \partial D$ is the rigid rotation $z \mapsto ze^{2\pi i\alpha}$.

Let $H_-, H_+ \in C^\infty(Z, \mathbb{R})$ be as follows. Define $H_- = H$ where $H$ is as in the statement of theorem 26. Define $H_+$ by

$$H_+(\tau, z) := \pi\alpha|z|^2 + C \tag{34}$$

for some constant $C \leq 0$ chosen so that $\max H_+ < \min H_-$.

The function $H_+$ on the solid torus defines a closed loop of Hamiltonians on the disk which we denote by $H^t_+ := H_+(t, \cdot) : D \to \mathbb{R}$, over $t \in \mathbb{R}/\mathbb{Z}$. Similarly $H_-$ defines $H^t_- := H_-(t, \cdot) :$

*For maps more general than irrational pseudo-rotations the proof for $n > 1$ is a little more involved as the almost complex structure has additional symmetry which makes transversality less obvious. But for irrational pseudo-rotations automatic transversality suffices and the same proof works for all $n \geq 1$.

$D \to \mathbb{R}$, over $t \in \mathbb{R}/\mathbb{Z}$. Let $X_{H_+^t}(z)$ and $X_{H_-^t}(z)$ be the corresponding time-dependent Hamiltonian vector fields on the disk $D$, with respect to the symplectic form $\omega_0 = dx \wedge dy$. The time-one map generated by $X_{H_+^t}(z)$ is the rigid rotation $R_{2\pi\alpha}$, while the time-one map generated by $X_{H_-^t}(z)$ is the pseudo-rotation $\varphi$. From (34), $\mathrm{Rot}(R_{2\pi\alpha}, H_+) = \alpha$, and therefore $\mathrm{Rot}(R_{2\pi\alpha}, H_+) = \mathrm{Rot}(\varphi, H_-)$.

The two pairs of differential forms $\mathcal{H}_\pm = (\omega_\pm, \lambda_\pm)$, on $Z$, given by

$$\omega_\pm = dx \wedge dy + d\tau \wedge dH_\pm \qquad \lambda_\pm = d\tau$$

define stable Hamiltonian structures on $Z$. Their associated (stable Hamiltonian) Reeb vector fields $R_+$ and $R_-$ are defined by the conditions $\omega_\pm(R_\pm, \cdot) = 0$ and $\lambda_\pm(R_\pm) = 1$. These are calculated to be

$$R_\pm(\tau, z) = \partial_\tau + X_{H_\pm^\tau}(z).$$

Thus, the first return map of $R_+$ on the initial disk slice $\{0\} \times D \subset Z$ is the rigid rotation $R_{2\pi\alpha}$, while the first return map of $R_-$ on this disk slice is the pseudo-rotation $\varphi$. The flows generated by $R_+$ and $R_-$ each have a unique 1-periodic orbit. Let $\gamma_\pm : \mathbb{R}/\mathbb{Z} \to Z$ be the parameterization of each, uniquely satisfying $\gamma_\pm(0) \in \{0\} \times D$. These have equal Conley-Zehnder indices (using any conventions).

Along similar lines to [22], we construct a "cobordism" between the two Hamiltonian energy surfaces $(Z, X_{H_+})$ and $(Z, X_{H_-})$ on the 4-manifold $W := \mathbb{R} \times Z$. To do this we choose a function $H \in C^\infty(\mathbb{R} \times Z, \mathbb{R})$ interpolating between $H_+$ and $H_-$ in the following way:

$$\begin{cases} H(a, m) = H_+(m) & \text{for all } a \geq 1 \\ \partial_a H(a, m) < 0 & \text{for } -1 < a < 1 \\ H(a, m) = H_-(m) & \text{for all } a \leq -1. \end{cases}$$

For example, let us assume we chose $H(a, m) = \chi(a)H_+(m) + (1 - \chi(a))H_-(m)$ for some $\chi \in C^\infty(\mathbb{R}, [0, 1])$ with $\chi \equiv 0$ on $(-\infty, -1]$ and $\chi \equiv 1$ on $[1, \infty)$ and such that $\chi'(a) > 0$ for all $a \in (-1, 1)$.

Define an almost complex structure $\hat{J}$ on $W := \mathbb{R} \times Z$ by

$$\begin{cases} \hat{J}(a, \tau, z)\partial_{\mathbb{R}} = \partial_\tau + X_{H_a^\tau}(z) \\ \hat{J}|_{TD} = i \end{cases}$$

where $X_{H_a^\tau}$ is the Hamiltonian vector field of $H_a^\tau := H(a, \tau, \cdot) : D \to \mathbb{R}$. Then $(W, \hat{J})$ is an almost complex manifold with cylindrical ends $E_+ = [1, \infty) \times Z$ and $E_- = (-\infty, -1] \times Z$, adjusted to the stable Hamiltonian structures $\mathcal{H}_\pm = (\omega_\pm, \lambda_\pm)$ on these ends. In the "cobordim region" $(-1, 1) \times Z$, $\hat{J}$ tames the exact symplectic 2-form

$$\Omega := dx \wedge dy + d\tau \wedge dH.$$

Although $\hat{J}$ is not $\mathbb{R}$-invariant, the $\mathbb{R}$-invariant cylinder

$$C := \{(a, \tau, 0) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z} \times D \,|\, a \in \mathbb{R}, \tau \in \mathbb{R}/\mathbb{Z}\} \tag{35}$$

is a pseudoholomorphic curve in $(W, \hat{J})$. Its positive puncture is asymptotic to $\gamma_+$ and its negative puncture is asymptotic to $\gamma_-$. Although $C$ is not an orbit cylinder, it has the following properties in common with one:

$$C \cdot C = -1, \tag{36}$$

where $\cdot$ refers to the generalized intersection number in the sense of Siefring [41], $C$ is embedded, has Fredholm index $\mathrm{ind}(C) = 0$, and is Fredholm regular. Indeed (36) follows from the adjuntion formula, theorem 4.6, in Siefring [41].

We will show in section 7.5 that after a small perturbation of $\hat{J}$ on a neighborhood of the boundary points $[-1,1]\times\partial Z$ in $W$, that $\hat{J}$ is Levi-flat on the boundary of $W$. More precisely, $\mathbb{R}\times\partial Z$ is filled by a set $\mathcal{S}$ of immersed $\hat{J}$-holomorphic planes which in the ends $E_\pm\cap(\mathbb{R}\times\partial Z)$ coincide with the product of the $\mathbb{R}$-component and a Reeb trajectory of $R_\pm$. Let us assume that such a perturbation has been made.

We proceed to construct $\mathcal{F}^+$ in four steps.

**Step 1:** The almost complex structure $\hat{J}$ on $W$ satisfies $\hat{J}|_{E_+} = J_+|_{E_+}$ where $J_+$ is the cylindrical almost complex structure

$$\begin{cases} J_+\partial_{\mathbb{R}} = \partial_\tau + 2\pi\alpha\partial_\theta \\ J_+|_{TD} = i \end{cases} \tag{37}$$

on $\mathbb{R}\times Z$, in standard polar coordinates $(r,\theta)$ on the disk. For each $c\in\mathbb{R}$ and $z\in\partial D$, the map

$$\begin{aligned} \tilde{u}_{c,z} :&\mathbb{R}^+\times\mathbb{R}/\mathbb{Z}\to\mathbb{R}\times Z \\ \tilde{u}_{c,z}(s,t) &= (s+c, t, ze^{2\pi(\lfloor\alpha\rfloor-\alpha)s}e^{2\pi i\lfloor\alpha\rfloor t}) \end{aligned} \tag{38}$$

is $J_+$-holomorphic. The combined images of these maps along with the cylinder

$$C_+ := \{(a,\tau,0)\in\mathbb{R}\times\mathbb{R}/\mathbb{Z}\times D \,|\, a\in\mathbb{R}, \tau\in\mathbb{R}/\mathbb{Z}\} \tag{39}$$

defines an $\mathbb{R}$-invariant finite energy foliation for $(\mathbb{R}\times Z, J_+)$ with boundary index $\lfloor\alpha\rfloor$. A direct calculation shows that if $F_1$ and $F_2$ are the images of two curves in (38), then

$$F_1\cdot F_2 = 0 \qquad \text{and} \qquad F_1\cdot C_+ = 0. \tag{40}$$

See appendix A.3 for a discussion of the generalized intersection number for pairs of curves with boundary.

**Step 2:** We return to the manifold $(W=\mathbb{R}\times Z,\hat{J})$ with cylindrical ends. Let $\mathcal{M}$ denote the moduli space of all finite energy $\hat{J}$-holomorphic curves $F\subset W$ which admit a $\hat{J}$-holomorphic parameterization by a map $\tilde{u}=(a,\tau,z)\in C^\infty(\mathbb{R}^+\times\mathbb{R}/\mathbb{Z},\mathbb{R}\times Z)$ satisfying

$$\begin{cases} \tilde{u}(0,\cdot)\in L_c \\ \tau(0,\cdot):\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z} \text{ has degree } +1 \\ z(0,\cdot):\mathbb{R}/\mathbb{Z}\to\partial D \text{ has degree } \lfloor\alpha\rfloor \end{cases} \tag{41}$$

for some $c\in\mathbb{R}$. Equip $\mathcal{M}$ with the topology coming from convergence in $C^\infty_{\mathrm{loc}}\cap C^0([0,\infty)\times\mathbb{R}/\mathbb{Z},W)$. Note that $\mathcal{M}$ is non-empty as it contains the image of each curve $\tilde{u}_{c,z}$ from (38) for which $c\geq 1$, as $\hat{J}=J_+$ on the positive end $E_+=[1,\infty)\times Z$. Let

$$\mathcal{M}_0\subset\mathcal{M}$$

be the connected component containing the curves $\tilde{u}_{c,z}$ from (38) for which $c\geq 1$.

Recall that the boundary of $W$ has a filling $\mathcal{S}$ by injective immersed $\hat{J}$-holomorphic planes. Positivity of intersections between the curves in $\mathcal{M}_0$ and those in $\mathcal{S}$ imply the following.

**Lemma 27.** *Suppose $F\in\mathcal{M}_0$. Then*

(1) *$F$ meets $\partial W$ transversely along $\partial F$.*
(2) *$F \backslash \partial F$ lies in $W \backslash \partial W$.*
(3) *$\partial F$ is an embedded closed loop.*

*Remark* 9. Note that property (3) in the lemma implies that $F$ is also embedded near the boundary. That is, is parameterized by a solution $\tilde{u} : \mathbb{R}^+ \times \mathbb{R}/\mathbb{Z} \to \mathbb{R} \times Z$ to (41) which restricts to an embedding on some neighborhood $[0, \varepsilon) \times \mathbb{R}/\mathbb{Z}$. This is because the differential can only have even dimensional kernel, and by (3) $\dim \ker D\tilde{u}(0,t) \leq 1$ for all $t \in \mathbb{R}/\mathbb{Z}$.

*Proof.* The three conditions are open in the topology on $\mathcal{M}$. It suffices to show then that they are also closed. That is, suppose that $F_k \in \mathcal{M}_0$ is a sequence converging to $F \in \mathcal{M}_0$, and that each $F_k$ satisfies conditions (1), (2), and (3). We wish to show that $F$ also satisfies these properties.

Take any point $q \in \partial F$. This is a limit of points $q_k \in \partial F_k$. By assumption each $q_k$ is an isolated transverse intersection point between $F_k$ and a leaf $S_k \in \mathcal{S}$. The local intersection index at $q_k$ is $+1$ (or $+1/2$ depending on conventions) as transverse intersections immersed curves.

Let $S \in \mathcal{S}$ be the unique leaf containing $q$. It is not too hard to show from this that $q$ must also be an isolated intersection point of $F$ with $S$, and that the local intersection index is $+1$. Indeed, as each leaf in $\mathcal{S}$ is locally embedded, one can get away with using the similarity principle after viewing each $F_k$ as a graph over $S_k$ near $q_k$. The only slightly tricky point is to show that other intersections between $F_k$ and $S_k$ do not accumulate at $q$. In the target they will indeed accumulate, but in the domains of the curves they will remain isolated uniformly in $k$. This latter can be seen by considering the pairs of 1-dimensional parameterized curves $f_k$ and $s_k$ in $\partial Z$ where $F_k$ and $S_k$ intersect $L_{c_k}$ respectively. The curves $f_k$ and $s_k$ intersect transversely and all intersections have the same sign, because each $F_k$ is assumed to be in $\mathcal{M}_0$. For purely topological reasons this makes it impossible for intersections between $f_k$ and $s_k$ to accumulate, from the point of view of their domains.

The local intersection index of $+1$ at $q \in \partial F$, between $F$ and $S$, then implies that $F$, viewed locally as a graph over $S$, corresponds to a zero of the graph of order 1. This implies that $F$ is immersed at $q$ and is transverse to $S$ at $q$, and therefore also transverse to $\partial W$ at $q$.

Similar arguments, again using positivity of intersections between the curves in $\mathcal{M}_0$ and those in $\mathcal{S}$ allow to conclude that $F$ can have no boundary-boundary double points, and that $F \backslash \partial F$ is disjoint from $\partial W$. This shows that $F$ satisfies conditions (1),(2), and (3). ∎

Now that our curves have these nice properties in relation to the boundary of $W$, we can use the homotopy invariant intersection number and adjunction formula from the appendix A.3.

The explicit curves $\tilde{u}_{c,z}$ from step 1 have Fredholm index 2 and are clearly embedded. Therefore by the adjunction formula (62) $\tilde{u}_{c,z} \cdot \tilde{u}_{c,z} = 0$. For all $F \in \mathcal{M}_0$, $F$ is homotopic to the image of $\tilde{u}_{c,z}$, for any $c \geq 1$. More precisely, homotopic through curves in the space $C^\infty(\gamma_+, \partial W, A)$ described in appendix A.3 where in this case $A \in H_1(\partial Z, \mathbb{Z})$ is represented by $\mathbb{R}/\mathbb{Z} \ni t \mapsto (t, e^{2\pi i \lfloor \alpha \rfloor t})$ because for any $G \in \mathcal{M}$, $A$ is represented by $\partial G$, see (41). Thus

$$F \cdot F = 0$$

for each $F \in \mathcal{M}_0$. Hence by proposition 38 each $F \in \mathcal{M}_0$ is globally embedded.

**Lemma 28.** *The set $\mathcal{E} := \{w \in W \,|\, w \in F \text{ for some } F \in \mathcal{M}_0\}$ is a non-empty open and closed subset of $W \backslash C$.*

*Proof.* By definition $\mathcal{M}_0$ is non-empty.

Openness: each curve $F$ in $\mathcal{M}_0$ has Fredholm index $\mathrm{ind}(F) = 2$. The last lemma showed that $F$ is embedded on its boundary, see remark 9, and meets $\partial W$ transversely. Moreover we just saw that $F \cdot F = 0$. Thus proposition 38 applies, and the curves in $\mathcal{M}_0$ fill up an open neighborhood of $F$ in $W$.

Closedness: suppose that $p_k \in \mathcal{E}$ is a sequence of points converging to a point $p_\infty \in W \backslash C$. Let $F_k \in \mathcal{M}_0$ be a curve containing $p_k$, and view $p_k$ as the image of a marked point. The total energy of $F_k$ is uniformly bounded in $k$ and we find a converging subsequence in the sense of [4]. In proposition 32 the limit is found to be a height-1 building $F_\infty$ say, that is also a curve in $\mathcal{M}_0$. $F_\infty$ must contain $p_\infty$, so $p_\infty \in \mathcal{E}$. ∎

Thus each point in $W \backslash C$ contains a curve in $\mathcal{M}_0$.

**Step 3:** Let $p_k \in W \backslash C$ be a sequence of points which converge to a point $p_\infty \in C$. Let $F_k \in \mathcal{M}_0$ be a sequence of curves with $p_k \in F_k$. As in step 2 these curves have uniformly bounded total energy because the asymptotic data $\gamma_+$ is fixed, as is the homology class in $\partial W$ in which the boundaries of the curves lie in. View each $p_k$ as the image of a marked point of $F_k$. Applying the compactness theory in [4] we get a convergent subsequence to a stable nodal holomorphic building $\bar{F}$ say. By proposition 32 $\bar{F}$ is a height-2 building, with non-empty lower level a half cylinder $F_-$ in the cylindrical manifold $(\mathbb{R} \times Z, J_-)$. The half cylinder $F_-$ is asymptotic to $\gamma_-$ as a positive puncture and has boundary index $\lfloor \alpha \rfloor$. Moreover,

$$F_- \cdot F_- = 0 \qquad \text{and} \qquad F_- \cdot C_- = 0$$

where $C_-$ is the orbit cylinder in $(\mathbb{R} \times Z, J_-)$ over $\gamma_-$. From lemma 6 $F_-$ is embedded and transverse to $\mathbb{R} \times \partial Z$.

**Step 4:** We may now argue in the manner of step 2, and conclude that $F_-$ lies in a non-empty moduli space of half cylinders $\mathcal{M}_-$ which fills an open and closed subset of $(\mathbb{R} \times Z) \backslash C_-$. By lemma 6 each curve in $\mathcal{M}_-$ is embedded and meets the boundary $\mathbb{R} \times \partial Z$ transversely. By the homotopy invariance of the generalized intersection number, the properties

$$F_0 \cdot F_1 = 0 \qquad \text{and} \qquad F_0 \cdot C_- = 0 \tag{42}$$

for all $F_0, F_1 \in \mathcal{M}_-$, are inherited from $F_-$. By proposition 38 then, $\mathcal{M}_-$ is a smooth foliation of $(\mathbb{R} \times Z) \backslash C_-$ by embedded curves. Now we set $\mathcal{F}^+ := \mathcal{M}_- \cup C_-$.

It remains to show that $\mathcal{F}^+$ is $\mathbb{R}$-invariant and unique in the sense of theorem 26. Both properties follow from (42). Indeed, the moduli space $\mathcal{M}_-$ is by definition $\mathbb{R}$-invariant, and to prove uniqueness it suffices to show that if $G$ is any $J_-$-holomorphic half cylinder in $\mathbb{R} \times Z$ that is asymptotic to $\gamma_-$ as a positive puncture, and has boundary behavior as in (41), then $G$ is in $\mathcal{M}_-$. First, by lemma 6 $G$ is embedded and transverse to $\mathbb{R} \times \partial Z$. Therefore it is homotopic to any leaf $F$ in $\mathcal{M}_-$ through curves along which the intersection number remains constant. Thus,

$$F \cdot G = F \cdot F = 0$$

for all $F \in \mathcal{M}_-$. In particular choose $F \in \mathcal{M}_-$ sharing a point with $G$. Then $F$ and $G$ are not disjoint but have $F \cdot G = 0$. Therefore they are equal, and so $G \in \mathcal{F}^+$.

7.2. **Construction of $\mathcal{F}^+$ without boundary restrictions.** This completes the construction for any irrational pseudo-rotation $\varphi : D \to D$ which restricts to a circle diffeomorphism $\varphi|_{\partial D}$ on the boundary that is smoothly conjugate to a rigid rotation. A deep result of Herman implies that the set of such boundary conditions is dense, and a further limiting step can remove these restrictions. We explain how this works now.

**Theorem 29** (Herman [17])**.** *There is a subset $\mathcal{D} \subset \mathbb{R}/\mathbb{Z}$ of irrational numbers of full measure with the following property. If $f \in \mathrm{Diff}^\infty_+(\partial D)$ has rotation number in $\mathcal{D}$ then there exists $g \in \mathrm{Diff}^\infty_+(\partial D)$ such that $g^{-1}fg = R$ where $R : \partial D \to \partial D$ is a rigid rotation.*

Denote by $\mathrm{Diff}_{\mathcal{D}}(\partial D)$ those orientation preserving $C^\infty$-smooth circle diffeomorphisms that have rotation number in $\mathcal{D}$. The following would hold for any dense subset $\mathcal{D} \subset \mathbb{R}/\mathbb{Z}$ of irrational numbers.

**Lemma 30.** $\mathrm{Diff}_{\mathcal{D}}(\partial D)$ *is dense in* $\mathrm{Diff}^\infty_+(\partial D)$ *with the $C^\infty$-topology.*

*Proof.* Fix any $f \in \mathrm{Diff}_{\mathcal{D}}(\partial D)$. Consider the continuous path $f_t := R_{2\pi t} \circ f \in \mathrm{Diff}^\infty_+(\partial D)$ over $t \in [0, 1/2]$. The rotation numbers $\mathrm{Rot}(f_t)$ vary continuously with $t$. Moreover, there is a continuous family of lifts $\tilde{f}_t : \mathbb{R} \to \mathbb{R}$ with the monotonicity property that for $t > 0$, $\tilde{f}_t(x) > \tilde{f}_0(x)$ for all $x \in \mathbb{R}$. Therefore, since $\mathrm{Rot}(f_0)$ is *irrational*, it follows that for all $t \in (0, 1/2]$, $\mathrm{Rot}(f_t) > \mathrm{Rot}(f_0)$. See for example proposition 11.1.9 in [30]. As $\mathcal{D}$ is dense in $\mathbb{R}/\mathbb{Z}$, we find a sequence $t_j \in (0, 1/2]$ converging to zero such that $\mathrm{Rot}(f_{t_j}) \in \mathcal{D}$. ∎

Now suppose that $\varphi \in \mathrm{Diff}^\infty(D, \omega_0)$ is any smooth irrational pseudo-rotation. Let $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R})$ be a Hamiltonian with time-one map $\varphi$. Using the lemma one can find smooth perturbations $H_j \in C^\infty(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R})$ of $H$ near the boundary $\mathbb{R}/\mathbb{Z} \times \partial D$, so that

$$H_j \to H \text{ in } C^\infty$$
$$\mathrm{Rot}(\varphi_j|_{\partial D}) \in \mathcal{D}$$

for all $j$, where $\varphi_j$ is the time-one map of $H_j$. For $j$ sufficiently large $\varphi_j$ has no fixed points besides the origin, which of course remains non-degenerate.

*Remark* 10. We do *not* claim that we can find perturbed disk maps which are also irrational pseudo-rotations. In general a perturbation will create new periodic points with high period.

By Herman's theorem $\varphi_j|_{\partial D}$ is smoothly conjugate to a rigid rotation. Therefore, by steps 1 to 4 in section 7.1 we can find a finite energy foliation $\mathcal{F}_{H_j}$ of the cylindrical almost complex manifold $(\mathbb{R} \times Z, J_{H_j})$.

*Remark* 11. Note that we are applying remark 8 here. To know that each foliation $\mathcal{F}_{H_j}$ exists even though $\varphi_j$ is not exactly an irrational pseudo-rotation.

The almost complex structures $J_{H_j}$ converge uniformly to $J_H$ in the $C^\infty$-topology as $j \to \infty$. The energies of leaves in $\mathcal{F}_{H_j}$ will be uniformly bounded in $j$: indeed the $\lambda$-energy of each leaf will remain 1, and the $\omega$-energy will be bounded above, for example, by $\pi\{\alpha\} + \varepsilon$ for some $\varepsilon > 0$. These are sufficient conditions to obtain a limiting finite energy foliation for $(\mathbb{R} \times Z, J_H)$. For example we can take a limit of a single sequence of half cylinder leaves $F_j \in \mathcal{F}_{H_j}$ to obtain a single $J_H$-holomorphic half cylinder $F_\infty$ disjoint from the orbit cylinder $C$ and with vanishing self intersection number. The same argument as in step 4 then shows that a moduli space containing $F_\infty$ fills up the complement of $C$ by embedded curves that combine to form a smooth $\mathbb{R}$-invariant foliation. Uniqueness is also as in step 4.

7.3. **Constructing $\mathcal{F}^-$.** The construction of $\mathcal{F}^-$ is along exactly the same lines as for $\mathcal{F}^+$, but beginning with a different model foliation in step 1. Indeed, in place of the curves in (38), each of which is bounded from below, we use curves of the form

$$v_{c,z} : \mathbb{R}^- \times \mathbb{R}/\mathbb{Z} \to \mathbb{R} \times Z \tag{43}$$
$$v_{c,z}(s,t) = (s + c, t, ze^{2\pi(\lceil\alpha\rceil - \alpha)s} e^{2\pi i \lceil\alpha\rceil t})$$

over $c \in \mathbb{R}$ and $z \in \partial D$. (These maps are also pseudoholomorphic with respect to the almost complex structure in (37).) Each $v_{c,z}$ has image bounded from above, so we have to insert one of them into the negative end of our almost complex manifold $(W = \mathbb{R} \times Z, \hat{J})$. So the main difference is that from the start we reverse the roles of $H_-$ and $H_+$, this time picking a positive constant $C$ in (34) so that $\min H_- > \max H_+$ still holds. Then the remaining steps are exactly analogous, and in the final foliation the curves with boundary have boundary index $\lceil \alpha \rceil$ instead, as the curves in (43) do.

7.4. **Compactness.** In section 7.1 we applied the compactness theory in [4] in the form of proposition 32 below. We state and justify this now.

Recall that $(W = \mathbb{R} \times Z, \hat{J})$ from section 7.1 has cylindrical ends; that the Reeb flow on the positive and negative ends each has a unique simply covered periodic orbit $\gamma_+$ and $\gamma_-$ respectively. Although $\hat{J}$ is not $\mathbb{R}$-invariant, the cylinder

$$C = \{(a, \tau, 0) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z} \times D \,|\, a \in \mathbb{R},\ \tau \in \mathbb{R}/\mathbb{Z}\}$$

is a pseudoholomorphic curve in $(W, \hat{J})$. Its positive puncture is asymptotic to $\gamma_+$ and its negative puncture is asymptotic to $\gamma_-$. We observed in (36) that

$$C \cdot C = -1. \tag{44}$$

Consider a sequence $F_k$ of $\hat{J}$-holomorphic curves in $(W, \hat{J})$ with a single marked point whose image we denote by $p_k \in W$. We suppose that each $F_k$ is the image of a $\hat{J}$-holomorphic embedding $\tilde{u}_k = (a_k, \tau_k, z_k) \in C^\infty(\mathbb{R}^+ \times \mathbb{R}/\mathbb{Z}, \mathbb{R} \times Z)$ that is asymptotic to $\gamma_+$ and has the boundary conditions described by (41). Furthermore, we assume that

$$F_k \cdot F_k = 0 \qquad \text{and} \qquad F_k \cdot C = 0$$

for each $k$.

Suppose that $F_k$ converges to a generalized nodal holomorphic building $\bar{F}$ in the sense of [4].

**Lemma 31.** *The building $\bar{F}$ has no disk components.*

*Proof.* Arguing indirectly suppose that $\bar{F}$ has a disk component $D$. Suppose first that $D$ is in the middle level $(W, \hat{J})$. Then $\partial D$ must be a closed loop in one of the totally real surfaces $L_c \simeq \partial Z$ say. As $\partial D$ is contractible in $W$ it must lie in a homology class $m1_{\partial D} \in H_1(\partial Z)$ for some $m \in \mathbb{Z}$. The topological count of intersections between $D$ and the cylinder $C$ is $m$. The building is stable with at most one marked point implies that the component $D$ is not just a point. Therefore it has non-zero energy, which implies that $m \neq 0$. Therefore $D$ and $C$ are distinct holomorphic curves with interior intersections. Therefore $F_k$ and $C$ intersect for large $k$, contradicting that they are infact disjoint for all $k$.

If instead $D$ is in an upper or lower level a similar argument proves that $D$ has isolated interior intersections with the unique orbit cylinder in the relevant level. This again implies that $F_k$ and $C$ intersect for large $k$. ∎

**Proposition 32.** *Suppose the marked points $p_k$ converge to $p_\infty \in W$.*

1. *If $p_\infty \notin C$, then $\bar{F}$ is a height-1 building (i.e. there is no breaking). This building has a single component, $F$. This component $F$ is a $\hat{J}$-holomorphic half cylinder with positive puncture asymptotic to $\gamma_+$ and the same boundary index as each $F_k$.*

(2) *If $p_\infty \in C$, then $\bar{F}$ has precisely two non-empty levels. The middle level is equal to the cylinder $C$, the top levels are empty, and the lower level is a single component curve $F_-$. The component $F_-$ is a $J_-$-holomorphic half cylinder with positive puncture asymptotic to $\gamma_-$ and the same boundary index as each $F_k$. Moreover, $F_- \cdot C_- = 0$ where $C_-$ is the orbit cylinder over $\gamma_-$, and $F_- \cdot F_- = 0$.*

*Proof.* Any bubbling off on the boundary would result in $\bar{F}$ having a disk component, which is ruled out by lemma 31. Any interior bubbling would result in a component of $\bar{F}$ that is a finite energy plane, which is impossible as there are no contractible periodic orbits. Any nodes in $\bar{F}$ would imply $\bar{F}$ has a component that is a plane or a closed curve. But any closed curve would be constant as the symplectic form $\Omega$ in the cobordism region is exact. However, there are not enough marked points for $\bar{F}$ to have any constant components and yet be stable.

We are therefore reduced to the following two scenarios.

**Case 1:** Suppose no breaking occurs. Then $\bar{F}$ is a height-1 holomorphic building, with a single component, and this component must be a half cylinder asymptotic to $\gamma_+$. The boundary index must be the same for each $F_k$ as no disks bubbled off.

It follows that each $F_k$ is homotopic to the limiting curve $F$ (through curves asymptotic to $\gamma_+$). Thus $F \cdot C = F_k \cdot C = 0$. Similarly $F \cdot F = F_k \cdot F_k = 0$ for all $k$.

**Case 2:** Suppose breaking does occur. Then, as there is only one periodic orbit in the negative end, the middle level $F_0$ of $\bar{F}$ must be a single component that is a cylinder with punctures of opposite sign. The positive puncture is asymptotic to $\gamma_+$ and the negative puncture asymptotic to $\gamma_-$. Therefore $F_0$ is homotopic to $C$ through curves with fixed end points, and so $F_0 \cdot C = C \cdot C = -1$ by (44), which implies that $F_0$ is a covering of $C$ and therefore equals $C$.

The marked point must be on $F_0$. Therefore there are no orbit cylinders in the lower or upper levels. Thus, $\bar{F}$ is a height-2 holomorphic building, with middle level $F_0$ as described and lower level $F_-$ a single component that is a half cylinder asymptotic to connecting orbit $\gamma_-$ as a positive puncture. Again the boundary index of $F_-$ must be the same as for each $F_k$ as no disks bubbled off.

By assumption $F_k$ converges to the building $\bar{F}$ in the sense of [4]. The constant sequence $C_k := C$ also converges to the (unstable) building $\bar{C}$ which consists of a single middle level equal to the cylinder $C$ and a single lower level equal to the orbit cylinder $C_-$ and empty upper level. In particular both buildings have a unique connecting orbit $\gamma_-$, and this is simply covered.

The proof of Proposition 4.3, part (4), in Siefring [41] yields

$$\lim_{k\to\infty} F_k \cdot C_k = C \cdot C + F_- \cdot C_- + p(\gamma_-) \tag{45}$$

where $p(\gamma_-) \in \{0, 1\}$ denotes the parity of the Conley-Zehnder index of $\gamma_-$. Due to lemma 36, $\gamma_-$ is elliptic, so $p(\gamma_-) = 1$. By assumption $F_k \cdot C_k = 0$ for each $k$, and we observed in (44) that $C \cdot C = -1$. We conclude then from (45) that $F_- \cdot C_- = 0$.

The formula that we applied to $F_k$ and $C_k$ to obtain (44) holds equally well for self intersections. Applied to $F_k$ this yields

$$0 = \lim_{k\to\infty} F_k \cdot F_k = C \cdot C + F_- \cdot F_- + p(\gamma_-). \tag{46}$$

The left hand side vanishes, $C \cdot C = -1$ (44), and $p(\gamma_-) = 1$. So $F_- \cdot F_- = 0$. ∎

7.5. **Foliating the boundary.** Finally we prove the existence of the foliation $\mathcal{S}$ of the boundary of $\mathbb{R} \times Z$ that was used in section 7.1.

For each $a \in \mathbb{R}$, consider for a fixed value of $a$ the resulting time-dependent Hamiltonian on the disk $H_a$ given by

$$H_a^t := H(a, t, \cdot) : D \to \mathbb{R}$$

over $t \in \mathbb{R}/\mathbb{Z}$. By modifying $H_a$ on any arbitrarily small neighborhood of the boundary of the disk we may arrange that the time-one map of the path of generated Hamiltonian disk maps is any prescribed orientation preserving diffeomorphism on the boundary of the disk. By extension, given any smooth path $a \mapsto f_a \in \mathrm{Diff}_+^\infty(\partial D)$ over $a \in \mathbb{R}$, satisfying

$$f_a = \begin{cases} R_{2\pi\alpha}|_{\partial D} & \text{if } a \geq 1 \\ \varphi|_{\partial D} & \text{if } a \leq -1, \end{cases} \tag{47}$$

the function $H : \mathbb{R} \times Z \to \mathbb{R}$ may be modified on any small neighborhood of $[-1, 1] \times \partial Z$ so that for each $a \in \mathbb{R}$ the time-one map of the modified Hamiltonian $H_a$ now coincides with $f_a$ on the boundary of the disk.

Suppose that we can find a smooth path $a \mapsto f_a \in \mathrm{Diff}_+^\infty(\partial D)$ satisfying (47), and which additionally has the property that each $f_a$ is smoothly conjugate to the rigid rotation $R_{2\pi\alpha}|_{\partial D}$. More precisely, suppose that we find a smooth map $g \in C^\infty(\mathbb{R} \times \partial D, \partial D)$, so that for each $a \in \mathbb{R}$ the map $g_a := g(a, \cdot)$ is an element of $\mathrm{Diff}_+^\infty(\partial D)$, and with the property that the path $f_a := g_a R_{2\pi\alpha} g_a^{-1}$ satisfies (47). Then we may modify $H \in C^\infty(\mathbb{R} \times Z, \mathbb{R})$ near $[-1, 1] \times \partial Z$ so that for each $a \in \mathbb{R}$ the time-one map of the time-dependent Hamiltonian $H_a := H(a, \cdot)$ equals $f_a$ on the boundary of the disk. For each $a \in \mathbb{R}$ let

$$\begin{aligned} \phi_a :& \mathbb{R} \times \partial Z \to \partial Z \\ & (t, z) \mapsto \phi_a^t(z) \end{aligned}$$

denote the 1-parameter family of maps generated by $X_{H_{a,t}}$ on $\partial D$. So in particular $\phi_a^1 = f_a$ for all $a \in \mathbb{R}$. Now for each $z \in \partial D$,

$$S_z := \left\{ \big(a, t, g_a(\phi_a^t(z))\big) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z} \times \partial D \,|\, a \in \mathbb{R},\ t \in \mathbb{R} \right\}$$

is an immersed surface in $\mathbb{R} \times \partial Z$, and the union

$$\mathcal{S} := \bigcup_{z \in \partial D} S_z$$

is a foliation of $\mathbb{R} \times \partial Z$. As $\alpha$ is irrational each $S_z$ is dense in $\mathbb{R} \times \partial Z$. However, the relation

$$f_a := g_a R_{2\pi\alpha} g_a^{-1} \tag{48}$$

for all $a \in \mathbb{R}$ enables us to find a $C^\infty$-smooth almost complex structure $J'$ on $\mathbb{R} \times Z$, prescribed at points on $\mathbb{R} \times \partial Z$ so that each $S_z$ has $J'$-invariant tangent bundle. Indeed, differentiating the expression $\big(a, t, g_a(\phi_a^t(z))\big)$ in $a$ gives a vector field $V_1$ say, while differentiating it in $t$ results in a vector field $V_2$. Both are non-vanishing and transverse as we will see, so we can set $J'V_1 = V_2$. That $V_1$ is indeed a well defined vector field uses (48). Moreover, one finds that:

$$V_2 = \partial_\tau + X_{H_a^\tau},$$

while

$$V_1 = \partial_a + V_3,$$

for some $V_3$ on $\mathbb{R}\times\partial Z$ that has no $\partial_a$ component, and tends to zero in $C^0$ as $\|\partial_a f_a\|_{C^0}$ tends to zero. We can arrange that $\|\partial_a f_a\|_{C^0}$ is as small as we wish by "slowing everything down", that is, replacing the interval $[-1,1]\times Z$ by $[-N,N]\times Z$ for sufficiently large $N>0$. Then, from these expressions for $V_1, V_2$ we see that $J'$ extends to an almost complex structure $J''$ on $\mathbb{R}\times Z$ with the following properties if $\|\partial_a f_a\|_{C^0}$ is sufficiently small:

(1) $J''$ coincides with the almost complex structure $J$ outside of a small neighborhood of $[-N+1,N-1]\times\partial Z$.
(2) Each surface $L_c := \{c\}\times\partial Z$ is totally real with respect to $J''$.
(3) $J''$ is tamed by the symplectic form $\Omega$ on $(-N,N)\times Z$.

And finally of course $\mathcal{S}$ is a $J''$-holomorphic filling of the boundary $\mathbb{R}\times\partial Z$.

The only remaining question is when the relation (48) can be arranged for all $a\in\mathbb{R}$. But this holds if and only if the circle maps $\varphi|_{\partial D}$ and $R_{2\pi\alpha}|_{\partial D}$ are conjugate by an orientation preserving $C^\infty$-smooth diffeomorphism. Necessity is obvious, let us show sufficiency. Suppose that there exists $g\in \mathrm{Diff}^\infty_+(\partial D)$ such that $\varphi|_{\partial D} = gR_{2\pi\alpha}|_{\partial D}g^{-1}$. Since $g$ has degree $+1$ it is smoothly isotopic to the identity and we may find a smooth isotopy $g_a\in\mathrm{Diff}^\infty_+(\partial D)$ over $a\in\mathbb{R}$, satisfying $g_a = \mathrm{id}$ for all $a\geq 1$, and $g_a = g$ for all $a\leq -1$. Thus the smooth path $f_a\in\mathrm{Diff}^\infty_+(\partial D)$ over $a\in\mathbb{R}$ defined by $f_a := g_a R_{2\pi\alpha} g_a^{-1}$ satisfies

$$f_a = \begin{cases} R_{2\pi\alpha} & \text{if } a\geq 1 \\ \varphi|_{\partial D} & \text{if } a\leq -1 \end{cases}$$

and therefore has the properties we require.

## Appendix A.

A.1. **Proof of proposition 2.** In this appendix we prove the following statement, which implies proposition 2. The idea of the proof was explained to me by Patrice LeCalvez.

We write $R_\theta : D\to D$ to denote the rigid rotation $z\mapsto e^{i\theta}z$ through angle $\theta\in\mathbb{R}$.

**Proposition 33.** *Consider a sequence $\varphi_k\in\mathrm{Homeo}_+(D)$ converging in the $C^0$-topology to $\varphi\in\mathrm{Homeo}_+(D)$, where all maps fix $0\in D$. Under the following additional assumptions it follows that $\varphi$ has no periodic points in $D\backslash\{0\}$.*

(1) *For each $k\in\mathbb{N}$, there exists $g_k\in\mathrm{Homeo}_+(D)$, fixing the origin, such that $\varphi_k = g_k^{-1}R_{2\pi\theta_k}g_k$, some $\theta_k\in\mathbb{R}$.*
(2) *$\theta_k\to\theta$ as $k\to\infty$, where $\theta$ is irrational.*

*Remark* 12. Regarding proposition 33, note that it is sufficient for us to prove the weaker statement that under conditions (1) and (2) the limiting map $\varphi$ has no *fixed* points in $D\backslash\{0\}$. Indeed, this weaker statement will then apply to each iterate of $\varphi$ and we will be able to conclude that each iterate of $\varphi$ has no fixed points in $D\backslash\{0\}$, and therefore that $\varphi$ has no *periodic* points in $D\backslash\{0\}$.

To prove this we need to recall the notions of positively and negatively returning disks due to Franks [14]. Let $\mathring{D} := D\backslash\partial D$. Denote by $A := \mathring{D}\backslash\{0\}$ and $\tilde{\mathbb{A}} := (0,1)\times\mathbb{R}$ the open annulus and its universal covering via the covering map

$$\begin{aligned}\pi : (0,1)\times\mathbb{R} &\to \mathring{D}\backslash\{0\} \\ (x,y) &\mapsto xe^{2\pi iy}.\end{aligned}$$

Let $T : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ be the deck transformation $T(x, y) = (x, y + 1)$. By an open disk $U \subset \tilde{\mathbb{A}}$ is meant an open subset homeomorphic to $\mathring{D}$ with the subspace topology.

**Definition 34.** Let $f : \mathbb{A} \to \mathbb{A}$ be a homeomorphism homotopic to the identity, and $\tilde{f} : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ a lift via $\pi$. Consider an open disk $U \subset \tilde{\mathbb{A}}$ for which

$$\tilde{f}(U) \cap U = \emptyset. \tag{49}$$

If there exist integers $n > 0$, $p \neq 0$, such that

$$\tilde{f}^n(U) \cap T^p U \neq \emptyset \tag{50}$$

then $U$ is called a *positively*, resp. *negatively*, *returning disk* for $\tilde{f}$ if $p > 0$, resp. $p < 0$.

*Remark* 13. Consider a disk $U \subset \tilde{\mathbb{A}}$ satisfying (49) and (50). If moreover the closure of the disk satisfies (49), that is, $\tilde{f}(\bar{U}) \cap \bar{U} = \emptyset$, then for any sufficiently $C^0$-small perturbation of $\tilde{f}$, $U$ will satisfy (49) and (50) for the perturbed map also.

The key result for us is the following strong generalization of the Poincaré-Birkhoff fixed point theorem, theorem 2.1 in [14].

**Theorem 35** (Franks)**.** *Let $f : \mathbb{A} \to \mathbb{A}$ be a homeomorphism of the open annulus homotopic to the identity, and for which every point is non-wandering. If there exists a lift $\tilde{f} : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ having a positively returning disk which is a lift of a disk in $\mathbb{A}$, and a negatively returning disk that is a lift of a disk in $\mathbb{A}$, then $f$ has a fixed point.*

Recall that a point $x \in \mathbb{A}$ is *non-wandering for $f$* if for every open neighborhood $U$ of $x$ there exists $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. Using Poincaré's Recurrence Theorem every point is non-wandering for a homeomorphism $f$ if $f$ preserves a finite measure that is positive on open sets.

We now use theorem 35 to prove the proposition.

*Proof of proposition 33.* Arguing indirectly, suppose that $\varphi$ has a periodic point $z_0 \in D\backslash\{0\}$. Indeed by remark 12 we may assume that $z_0$ is a fixed point of $\varphi$. Then we will show that some iterate $\psi = \varphi^n$ has a lift via the covering map

$$\begin{aligned} \pi : (0, 1] \times \mathbb{R} &\to D\backslash\{0\} \\ (x, y) &\mapsto x e^{2\pi i y} \end{aligned}$$

to a map of the half closed infinite strip $\tilde{\psi} : (0, 1] \times \mathbb{R} \to (0, 1] \times \mathbb{R}$, having disks $U_-, U_+ \subset (0, 1) \times \mathbb{R}$ which satisfy the following:

1. $U_+$ is a positively returning disk for $\tilde{\psi}$. $U_-$ is a negatively returning disk for $\tilde{\psi}$.
2. The closures satisfy $\tilde{\psi}(\bar{U}_+) \cap \bar{U}_+ = \emptyset$ and $\tilde{\psi}(\bar{U}_-) \cap \bar{U}_- = \emptyset$.
3. $U_+$ and $U_-$ are lifts of disks in $\mathring{D}\backslash\{0\}$.

Suppose for a moment that we have established this. Let us write $\psi_k := (\varphi_k)^n : D \to D$. Then $\psi_k$ converges uniformly in the $C^0$-topology to $\psi = \varphi^n$ as $k \to \infty$. Therefore there exists a sequence of lifts

$$\tilde{\psi}_k : (0, 1] \times \mathbb{R} \to (0, 1] \times \mathbb{R}$$

of $\psi_k$ which converges uniformly in the $C^0$-topology to the given lift $\tilde{\psi}$. Hence by remark 13 there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $U_+$ is a positively returning disk for $\tilde{\psi}_k$, and $U_-$ is a negatively returning disk for $\tilde{\psi}_k$.

Recall that $U_+$ and $U_-$ lie in the interior $(0,1)\times\mathbb{R}$, and both are lifts of disks in $\mathring{D}\backslash\{0\}$. Every point is non-wandering for $\psi_k$ because it is conjugate to a rigid rotation. Therefore we may apply Franks' theorem, theorem 35, to each map of the open annulus $\psi_k : \mathring{D}\backslash\{0\} \to \mathring{D}\backslash\{0\}$ for $k \geq K$. We conclude that for all $k \geq K$, $\psi_k$ has a fixed point in $D\backslash\{0\}$. But $\psi_k = (\varphi_k)^n$ is conjugate to the rigid rotation $(R_{2\pi\theta_k})^n = R_{2\pi n\theta_k}$. It follows that for this fixed $n \in \mathbb{N}$ we have $n\theta_k \in \mathbb{Z}$ for all $k \geq K$, which contradicts the convergence of the sequence $\theta_k$ to an irrational number.

With this contradiction we will be done, and so it remains to establish that some iterate of $\varphi$ has a lift for which we can find disks $U_+, U_- \subset (0,1)\times\mathbb{R}$ satisfying conditions (1), (2), and (3) listed above.

We are assuming that $\varphi$ has a fixed point $z_0 \in D\backslash\{0\}$. From assumption (2) in Proposition 33 we know that the rotation number of $\varphi$ on the boundary is irrational, and so $\varphi$ has no periodic points on the boundary of the disk. Thus $z_0 \in \mathring{D}\backslash\{0\}$ and $\varphi$ has no fixed points on $\partial D$.

Fix a lift $\tilde{z}_0 \in (0,1)\times\mathbb{R}$, meaning that $\pi(\tilde{z}_0) = z_0$. Let $\tilde{\varphi} : (0,1]\times\mathbb{R} \to (0,1]\times\mathbb{R}$ be the unique lift of $\varphi$ satisfying $\tilde{\varphi}(\tilde{z}_0) = \tilde{z}_0$. Then for each $n \in \mathbb{N}$, $(\tilde{\varphi})^n$ is the unique lift of $\varphi^n$ that fixes $\tilde{z}_0$. Let $h : \mathbb{R} \to \mathbb{R}$ be the map characterized by

$$\tilde{\varphi}(1, y) = (1, h(y))$$

for all $y \in \mathbb{R}$. In particular $h$ is a lift of the circle map $\varphi : \partial D \to \partial D$. Since $\varphi$ has no fixed points on $\partial D$ it follows that $h$ has no fixed points. Thus there exists $\varepsilon > 0$ such that one of the following two cases occurs: (A) $h(y) > y + \varepsilon$ for all $y \in \mathbb{R}$, or (B) $h(y) < y - \varepsilon$ for all $y \in \mathbb{R}$. From here on **let us assume we are in case (A)**, as the argument for case (B) is the same with obvious modifications. It follows inductively that for each $n \in \mathbb{N}$, $h^n(y) > y + n\varepsilon$. Fix now $n_0 \in \mathbb{N}$ sufficiently large that

$$h^{n_0}(y) > y + 3 \tag{51}$$

for all $y \in \mathbb{R}$. Note that it follows that $h^n(y) > y + 3$ for all $n \geq n_0$. Consider the lift $\tilde{\varphi}^{n_0} : (0,1]\times\mathbb{R} \to (0,1]\times\mathbb{R}$. We have $\tilde{\varphi}^{n_0}(1, y) = (1, h^{n_0}(y))$ for all $y \in \mathbb{R}$. Thus, for each $y \in \mathbb{R}$ and $n \geq n_0$,

$$\tilde{\varphi}^n(1, y) = (1, y')$$

for some

$$y' > y + 3. \tag{52}$$

As $\partial D$ is a compact invariant set for $\varphi^{n_0}$ there exists a point $z_1 \in \partial D$ that is non-wandering for the circle map $\varphi^{n_0} : \partial D \to \partial D$ (e.g. take any point in the $\omega$-limit set of another point in $\partial D$). We write $d$ for the Euclidean metric on $D$ or $[0,1]\times\mathbb{R}$. Let $I \subset \partial D$ be an open neighborhood of $z_1$ sufficiently small that for some $\varepsilon_0 > 0$ the set

$$V_0 := \{z \in \mathring{D}\backslash\{0\} \,|\, d(z, I) < \varepsilon_0\}$$

is an open disk. That $z_1$ is non-wandering for $\varphi^{n_0} : \partial D \to \partial D$ means that there exists $m \in \mathbb{N}$ such that

$$\varphi^{mn_0}(I) \cap I \neq \emptyset. \tag{53}$$

Let $\tilde{z}_1 = (1, y_1) \in (0,1] \times \mathbb{R}$ be a lift of $z_1$, and $\tilde{I} \subset \{1\} \times \mathbb{R}$ the lift of $I$ that contains $\tilde{z}_1$. We chose $I$ sufficiently small above that $\tilde{I}$ will be disjoint from all of its translates. That is,

$$\tilde{I} \cap T^q(\tilde{I}) = \emptyset \tag{54}$$

for all $q \in \mathbb{Z}\backslash\{0\}$. From (53) we know that $\tilde{\varphi}^{mn_0}(\tilde{I})$ must intersect one of the translates of $\tilde{I}$. That is, there exists $p \in \mathbb{Z}$ such that

$$\tilde{\varphi}^{mn_0}(\tilde{I}) \cap T^p(\tilde{I}) \neq \emptyset. \tag{55}$$

From (52) we have $p \geq 3$.

Now set $\psi := \varphi^{n_0}$ and let $\tilde{\psi} : (0,1] \times \mathbb{R} \to (0,1] \times \mathbb{R}$ be the lift

$$\tilde{\psi} := T^{-1} \circ \tilde{\varphi}^{n_0}. \tag{56}$$

It is for this iterate $\varphi^{n_0}$, and with respect to this lift $\tilde{\psi}$, that we will construct positively and negatively returning disks $U_+, U_- \subset (0,1) \times \mathbb{R}$.

Let us begin by listing the following properties of $\tilde{\psi}$ which follow immediately from the properties of $\tilde{\varphi}^{n_0}$ we accumulated above.

(i) $\tilde{\psi}(\tilde{z}_0) = T^{-1}(\tilde{z}_0)$.
(ii) For each $y \in \mathbb{R}$, $\tilde{\psi}(1, y) = (1, y')$ where $y' > y + 2$.
(iii) There exists $m \in \mathbb{N}$ and an integer $p \geq 2$ such that

$$\tilde{\psi}^m(\tilde{I}) \cap T^p(\tilde{I}) \neq \emptyset. \tag{57}$$

Note that a consequence of the second property is that

$$\tilde{\psi}(\mathrm{cl}(\tilde{I})) \cap \mathrm{cl}(\tilde{I}) = \emptyset \tag{58}$$

where $\mathrm{cl}(\tilde{I})$ is the closure of $\tilde{I}$ in $(0,1] \times \mathbb{R}$.

First the negatively returning disk. Write $(x, y) := \tilde{z}_0 \in (0,1) \times \mathbb{R}$. Then $\tilde{\psi}(x, y) = (x, y-1)$ by (i). Thus any sufficiently small disk neighborhood $U_- \subset (0,1) \times \mathbb{R}$ of $(x, y)$ is a lift of a disk in $\mathring{D}\backslash\{0\}$ that satisfies $\tilde{\psi}(\bar{U}_-) \cap \bar{U}_- = \emptyset$. Moreover $\tilde{\psi}(U_-) \cap T^{-1}(U_-)$ is non-empty as it contains the point $(x, y-1)$. Thus $U_-$ is a negatively returning disk for $\tilde{\psi}$ satisfying conditions (1), (2), and (3) listed at the beginning.

It remains to find a suitable positively returning disk for $\tilde{\psi}$. Consider the open neighborhoods of $\tilde{I}$ in $(0,1] \times \mathbb{R}$ of the form

$$V := \{z \in (0,1] \times \mathbb{R} \,|\, d(z, I) < \varepsilon\}$$

for $\varepsilon > 0$. From (57) there exist integers $p \geq 2$ and $m \geq 1$ such that

$$\tilde{\psi}^m(V) \cap T^p(V) \neq \emptyset. \tag{59}$$

For $\varepsilon > 0$ sufficiently small we have

$$\tilde{\psi}(\bar{V}) \cap \bar{V} = \emptyset. \tag{60}$$

This is because $\tilde{\psi}$ moves points on $\{1\} \times \mathbb{R}$ in one direction a distance greater than 2, and $\tilde{I} \subset \{1\} \times \mathbb{R}$ was chosen sufficiently small.

Set $U_+ := V \cap \tilde{\mathbb{A}} = V \cap \big((0,1) \times \mathbb{R}\big)$. From (59), and baring in mind that $\tilde{\psi}^m(V) \cap T^p(V)$ is also open in $(0,1] \times \mathbb{R}$, we must have

$$\tilde{\psi}^m(U_+) \cap T^p(U_+) \neq \emptyset.$$

From (60)

$$\tilde{\psi}(\bar{U}_+) \cap \bar{U}_+ = \emptyset$$

as $\bar{U}_+ \subset \bar{V}$. Moreover, if we chose $0 < \varepsilon \leq \varepsilon_0$ then $U_+$ is a disk in $\tilde{\mathbb{A}} = (0,1) \times \mathbb{R}$ and is a lift of a disk in $\mathring{D}\backslash\{0\}$. Thus, as $p \geq 2$, $U_+$ is a positively returning disk for $\tilde{\psi}$ satisfying the required conditions (1), (2), and (3). ∎

A.2. **Nondegeneracy.** Let $\varphi : D \to D$ be an irrational pseudo-rotation as in definition 3. In this appendix we prove that the unique periodic point is non-degenerate for every iterate. More precisely:

**Lemma 36.** *For each $n \in \mathbb{N}$, the linearization $D\varphi^n(0)$ does not have eigenvalue 1.*

*Remark* 14. Infact the eigenvalues of $D\varphi^n(0)$ are $\{e^{2\pi i n\alpha}, e^{-2\pi i n\alpha}\}$, where $[\alpha] \in \mathbb{R}/\mathbb{Z}$ is the rotation number of $\varphi$. We see these are distinct from 1 for each $n$ because $\alpha$ is irrational.

*Proof.* By applying the statement to iterates it suffices to prove the statement for $n = 1$.

Arguing indirectly we find $v \in \partial D \subset \mathbb{R}^2$ such that $D\varphi(0)v = v$. Blowing up the fixed point $\varphi(0) = 0$ we obtain a map $\hat{\varphi} : [0,1] \times \partial D \to [0,1] \times \partial D$ that is an orientation preserving homeomorphism, and whose restriction to $(0,1] \times \partial D$ is a diffeomorphism that preserves the area form $dt \wedge d\theta$ and is smoothly conjugate to $\varphi : D\backslash\{0\} \to D\backslash\{0\}$, and which on the circle $\{0\} \times \partial D$ is the projection of the linear map $D\varphi(0) : \mathbb{R}^2 \to \mathbb{R}^2$, more precisely

$$\hat{\varphi}(0,x) = \left(0\,, \frac{D\varphi(0)[x]}{|D\varphi(0)[x]|}\right)$$

for each $x \in \partial D \subset \mathbb{R}^2$. For example one can take the unique continuous extension of the map $\pi^{-1} \circ \varphi \circ \pi : (0,1] \times \partial D \to (0,1] \times \partial D$ where $\pi : (0,1] \times \partial D \to D\backslash\{0\}$ is the smooth diffeomorphism $\pi(t,x) := \sqrt{t}x$.

It follows that $\hat{\varphi}(0,v) = (0,v)$ and so the rotation number of $\hat{\varphi}$ on the boundary component $\{0\}\times\partial D$ is $[0] \in \mathbb{R}/\mathbb{Z}$. The rotation number of $\hat{\varphi}$ on the other boundary component $\{1\}\times\partial D$ is the irrational number $[\alpha] \in \mathbb{R}/\mathbb{Z}$ because it must equal the rotation number of $\varphi : \partial D \to \partial D$. Thus $\hat{\varphi}$ has distinct rotation numbers on the two boundary circles. Therefore some iterate has a twist in the sense of Poincaré and Birkhoff and so their fixed point theorem applies. We conclude that $\hat{\varphi}$ has periodic points in the interior $(0,1) \times \partial D$ and so $\varphi$ has periodic points in $D\backslash(\partial D \cup \{0\})$ contradicting that $\varphi$ is an irrational pseudo-rotation. ∎

A.3. **Intersection theory.** The intersection number $F \cdot G$ between pairs of closed oriented surfaces $F$ and $G$ in a closed four manifold is of course a homological invariant. For pseudo-holomorphic curves it is especially useful since its vanishing implies the two curves are either equal or disjoint due to the positivity phenomenon in [37]. Moreover, via the so called adjunction formula, the embeddedness properties of a surface are related to its self-intersection number which is homotopy invariant.

Siefring showed that a homotopy invariant intersection number can also be associated to pseudoholomorphic curves with punctures asymptotic to nondegenerate periodic orbits [41], that retains many of the properties that hold for closed curves.

In this appendix we state without proof the properties of an intersection number that we used in section 7 for curves with a boundary component. We also state a corresponding adjunction formula. A more detailed explanation will appear in [5].

We use the notation from section 7.1. Fix an homology class $A \in H_1(\partial Z, \mathbb{Z})$. Let $C^\infty(\gamma_+, \partial W, A)$ denote the set of maps $\tilde{u} \in C^\infty(\mathbb{R}^+ \times \mathbb{R}/\mathbb{Z}, W)$ equipped with the $C^\infty_{\text{loc}} \cap C^0$ topology which satisfy:

- $\tilde{u}(0,\cdot) \in L_c$ for some $c \in \mathbb{R}$ (not fixed), and represents $A$.

- $\tilde{u}$ is asymptotically cylindrical in the sense of [41], with asymptotic orbit $\gamma_+$.
- $\tilde{u}$ takes interior points to $W \backslash \partial W$.
- $\tilde{u}$ meets $\partial W$ transversely along the loop $\tilde{u}(0, \cdot)$.

*Remark* 15. Recall that $\gamma_+$ lies in the positive end of $(W, \hat{J})$, and that it is nondegenerate and simply covered.

The homotopy invariant intersection index of Siefring can be extended to these curves with boundary. This allows us to associate to any pair of curves $\tilde{u}, \tilde{v} \in C^\infty(\gamma_+, \partial W, A)$ a half integer $\tilde{u} \cdot \tilde{v} \in \frac{1}{2}\mathbb{Z}$ with the following properties:

(1) **Symmetry:** $\tilde{u} \cdot \tilde{v} = \tilde{v} \cdot \tilde{u}$.
(2) **Homotopy invariance:** For a continous path $\tilde{u}_\tau$ in $C^\infty(\gamma_+, \partial W, A)$, $\tilde{u}_\tau \cdot \tilde{v}$ is independent of $\tau$.
(3) **Positivity:** Suppose $\tilde{u}$ and $\tilde{v}$ are also $\hat{J}$-holomorphic, embedded on their boundary circles, and do not have identical images. Then, $\tilde{u} \cdot \tilde{v} \geq 0$, and equality implies that the two curves have disjoint images.

*Remark* 16. This is more delicate for pairs of curves $\tilde{u} \in C^\infty(\gamma_+, \partial W, A)$ and $\tilde{v} \in C^\infty(\gamma_+, \partial W, B)$ for which $A \neq B$. We will not consider these cases.

Following [43, 44], which is for curves without boundary, we define the *normal Chern number* of a $\hat{J}$-holomorphic map $\tilde{u} \in C^\infty(\gamma_+, \partial W, A)$, to be the integer

$$c_N(\tilde{u}) = \mathrm{ind}(\tilde{u}) - 1 - p(\gamma_+). \tag{61}$$

Here $p(\gamma_+)$ denotes the parity of the Conley-Zehnder index of $\gamma_+$, and $\mathrm{ind}(\tilde{u})$ is the Fredholm index of the linearized Cauchy-Riemann operator at $\tilde{u}$ viewed as a solution of the free boundary problem.

In our simple situation expression (61) seems hardly to warrant a name. But it allows us to draw parallels with the general situation for curves without boundary. The terminology arises, see [43], because if $\tilde{u}$ is immersed then $c_N(\tilde{u})$ has an interpretation as a relative first Chern number of a normal bundle over $\tilde{u}$. In the case at hand this can be described as follows.

Let $\zeta \subset T(\partial W)$ denote the unique $\hat{J}$-invariant 2-plane distribution in the boundary of $W$. Each 2-torus $L \subset \partial W$ is totally real and therefore transverse to $\zeta$. To the non-degenerate periodic orbit $\gamma_+$ one can associate an operator, the so called asymptotic operator [21], with discrete real spectrum. Each eigenspace of this operator yields a homotopy class of sections of $\gamma_+^* TD \to \mathbb{R}/\mathbb{Z}$. Let $\Phi_+$ denote the class associated to the maximal negative eigenvalue of this operator. Let $\Phi_0$ denote the homotopy class of sections of $\tilde{u}(0, \cdot)^* \zeta \to \mathbb{R}/\mathbb{Z}$ that corresponds to the orientable line bundle $l := \zeta \cap TL_c$. Let $N_{\tilde{u}} \to \mathbb{R}^+ \times \mathbb{R}/\mathbb{Z}$ be a $\hat{J}$-invariant normal bundle to $\tilde{u}$ in $W$ which, outside of a compact set, is equal to the tangent planes to the disks, and which over the boundary points of $\tilde{u}$ coincides with $\zeta$. Then (61) can be interpreted as the first Chern number of the complex line bundle $(N_{\tilde{u}}, \hat{J}) \to \mathbb{R}^+ \times \mathbb{R}/\mathbb{Z}$ relative to the homotopy classes $\Phi_+$ and $\Phi_0$.

With this interpretation of the normal Chern number, the adjunction formula in our situation for curves with embedded boundary takes the following form.

**Theorem 37.** *Suppose $\tilde{u} \in C^\infty(\gamma_+, \partial W, A)$ is $\hat{J}$-holomorphic and embedded on its boundary circle. Then*

$$\tilde{u} \cdot \tilde{u} = 2\delta(\tilde{u}) + c_N(\tilde{u}) \in \mathbb{Z} \tag{62}$$

*where $\delta(\tilde{u}) \geq 0$ is an integer that vanishes if and only if $\tilde{u}$ is embedded.*

*Remark* 17. $\delta(\tilde{u})$ is the count of double and singular points of $\tilde{u}$ as defined in Siefring [41] and up to a factor of 2 is the same as in Micallef-White [37]. (There is no contribution to $\delta(\tilde{u})$ from the periodic orbit because it is simply covered.) Note that there are no additional contributions to consider from singular or double points on the boundary as we assume $\tilde{u}$ is embedded on the boundary. There are no interior-boundary double points because $\tilde{u} \in C^\infty(\gamma_+, \partial W, A)$.

*Remark* 18. For general punctured pseudoholomorphic curves the adjunction formula is due to Siefring [41]. The expression above in (62) is however closer to (A.6) in [44]. The additional terms there have vanished here because the asymptotic periodic orbit $\gamma_+$ is simply covered.

*Remark* 19. The adjunction formula usually requires the curve to be somewhere injective. This is automatically the case here.

The utility of this formula is most apparent when $c_N(\tilde{u}) \geq 0$. Then the vanishing of the left hand side guarantees that $\tilde{u}$ is embedded. This is convenient as $\tilde{u} \cdot \tilde{u}$ is a homotopy invariant.

**Proposition 38.** *Suppose that $\tilde{u} \in C^\infty(\gamma_+, \partial W, A)$ is $\hat{J}$-holomorphic, embedded on its boundary circle, that $\mathrm{ind}(\tilde{u}) = 2$ as a solution to the free boundary problem, and that $\tilde{u} \cdot \tilde{u} = 0$. Then*

1. *$\tilde{u}$ is embedded.*
2. *$\tilde{u}$ is Fredholm regular.*
3. *An open neighborhood of the image of $\tilde{u}$ in $W$ is foliated by a smooth family of embedded $\hat{J}$-holomorphic curves in $C^\infty(\gamma_+, \partial W, A)$.*

By Fredholm regular we mean that the linearized Cauchy-Riemann operator at $\tilde{u}$, viewed as a solution of the free boundary problem, is surjective.

*Proof.* It follows from (61) that $c_N(\tilde{u}) = 0$. Since $\tilde{u}$ is embedded on the boundary we can apply the adjunction formula, theorem 37, and conclude from $\tilde{u} \cdot \tilde{u} = 0$ that $\tilde{u}$ is embedded.

Automatic transversality arguments in [42, 45] then show that $\tilde{u}$ is Fredholm regular. More precisely this is because $\tilde{u}$ is immersed and $c_N(\tilde{u}) < \mathrm{ind}(\tilde{u})$.

Thus $\tilde{u}$ is embedded and Fredholm regular, has Fredholm index 2, meets the boundary transversely in an embedded totally real submanifold, and has interior disjoint from $\partial W$. These conditions suffice to apply an implicit function theorem in [42, 45] (a slight generalization of the result in [25]) to produce a local 2-dimensional family of $\hat{J}$-holomorphic curves in $C^\infty(\gamma_+, \partial W, A)$. Finally, the condition $c_N(\tilde{u}) = 0$ implies that these curves foliate an open neighborhood of the image of $\tilde{u}$. ∎

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Barney Bramham, Ruhr-Universität, Bochum, 44780 Germany
*E-mail address*: barney.bramham@rub.de